Pacific Journal of Mathematics

NON-ARCHIMEDEAN GEL'FAND THEORY

JESUS M. DOMINGUEZ

Vol. 104, No. 2 June 1983

PACIFIC JOURNAL OF MATHEMATICS
Vol. 104, No. 2, 1983

NON-ARCHIMEDEAN GELFAND THEORY

JESUS M. DOMINGUEZ

In this paper we show that if X is a Banach algebra and X_0 is its Gelfand subalgebra, then the set X_0^* of the elements in X_0 with compact spectrum is a Gelfand algebra whose maximal ideal space is compact in the Gelfand topology. We also give a representation theorem for X_0^*, which we use to derive the Van der Put characterization of C-algebras.

Introduction. Throughout all this paper we denote by F a complete field with respect to a non-trivial rank one valuation. Also X will usually denote an algebra over F. All algebras will be understood to be commutative with identity. We shall use the notation of [3], but we shall identify the ground field F with a subset of the considered algebra. Also we shall put $C(T)$, instead of $F(T)$, to denote the algebra of all F-valued continuous functions on the topological space T.

A non-archimedean Banach algebra X is called a C-algebra if there exists a compact Hausdorff space T such that X is isometrically isomorphic to $C(T)$. In [4] N. Shilkret introduces the Gelfand subalgebra; the concept of V^*-algebra is defined in [3].

1. The subalgebras X_0 and X_0^*, and their maximal ideals. Let X be an algebra over F and let X_0 be its Gelfand subalgebra. X_0 has the following properties:

1. If $x \in X_0$, then x is invertible in X_0 if and only if it is so in X; therefore $\sigma(x) = \sigma_{X_0}(x)$.

2. If M is a maximal ideal of X, then $M \cap X_0$ is a Gelfand ideal of X_0.

3. If F is not algebraically closed, then each maximal ideal of X_0 is of the form $M \cap X_0$, where M is a maximal ideal of X.

4. If X is a Banach algebra, then X_0 is a closed subalgebra of X.

The conditions 1, 2 and 4 are easy to check (cf. [3] or Shilkret [4]). To prove condition 3 it is enough to show that if m is a maximal ideal of X_0 and $x_1,\ldots,x_n \in m$ then there is a maximal ideal M of X containing all the x_i. Let $f(Z) = \lambda_0 + \lambda_1 Z + \cdots + \lambda_n Z^n$ be an irreducible polynomial with coefficients in F, of degree greater than one, and consider $a = \lambda_0 x_2^n + \lambda_1 x_1 x_2^{n-1} + \cdots + \lambda_n x_1^n$. Then a belongs to the subalgebra $F[x_1, x_2]$ generated by x_1, x_2 over F. Moreover the maximal ideals of X containing a are just those containing both x_1, x_2. Arguing by induction on n, we find an element $c \in F[x_1,\ldots,x_n]$ such that the maximal ideals of X containing c are just those containing all the x_i. Now, $c \in m$ hence, by condition 1, there is a maximal ideal M of X containing c and, therefore, all the x_i belong to M. (A more detailed proof can be found in Gommers [**1**].)

REMARK. The assumption of F being a valued field is necessary only in condition 4.

DEFINITION. We define the algebra

$$X_0^* = \{x \in X_0/\sigma(x) \text{ is precompact}\}.$$

We see that X_0^* is a subalgebra of X_0 containing the identity element.

THEOREM 1. *Let X be a Banach algebra. The subalgebra X_0^* has the following properties*:

1. *If $x \in X_0^*$, then x is invertible in X_0^* if and only if it is so in X; therefore $\sigma(x) = \sigma_{X_0^*}(x)$.*
2. *If M is a maximal ideal of X, then $M \cap X_0^*$ is a Gelfand ideal of X_0^*.*
3. *If F is not algebraically closed, then each maximal ideal of X_0^* is of the form $M \cap X_0^*$, where M is a maximal ideal of X.*
4. *X_0^* is a closed subalgebra of X.*

Proof. The conditions 1 and 2 are easily checked. To prove 3 we just repeat the above argument replacing X_0 by X_0^*. The proof of 4 is just the following: Since X_0 is a closed subalgebra of X it is enough to show that given a sequence (x_n) in X_0^* with $x_n \to x$, then $\sigma(x)$ is precompact. To see this pick $\varepsilon > 0$. Since $x_n \to x$ there exists n_0 such that $\|x - x_{n_0}\| < \varepsilon/2$. Now since $\sigma(x_{n_0})$ is precompact there exist $\mu_1, \ldots, \mu_r \in F$ such that $\sigma(x_{n_0}) \subset \bigcup_i B(\mu_i, \varepsilon/2)$. If $\lambda \in \sigma(x)$ then there is a maximal ideal M of X such that $\lambda = x(M)$. Hence $|\lambda - x_{n_0}(M)| \leq \|x - x_{n_0}\| < \varepsilon/2$ and therefore $\sigma(x) \subset \bigcup_i B(\mu_i, \varepsilon)$.

REMARK. If X is a Banach algebra and F is locally compact, then $\sigma(x)$ is compact for all $x \in X_0$, and thus $X_0^* = X_0$.

EXAMPLES. Assume that the valuation of F is non-archimedean, and that T is a 0-dimensional Hausdorff space.

EXAMPLE 1. $C(T)$ is a commutative algebra with an identity element. For all $f \in (C(T))_0$ one has that $f(T)$ is compact, hence $(C(T))_0^* = (C(T))_0$.

EXAMPLE 2. Let $BC(T)$ denote the algebra of all bounded continuous functions from T into F, and let $PC(T)$ denote the subalgebra of all functions $f \in BC(T)$ for which $f(T)$ is precompact. Then $BC(T)$ is a commutative Banach algebra with an identity element under the sup-norm, and $(BC(T))_0 = PC(T)$. Thus $(BC(T))_0^* = (BC(T))_0$.

EXAMPLE 3. Let $F\{Z\}$ denote the algebra of all formal power series, $\Sigma a_n Z^n$, in Z with coefficients in F for which $a_n \to 0$. Then $F\{Z\}$ is a commutative Banach algebra with an identity element under $\|\Sigma a_n Z^n\| = \max |a_n|$.

(a) If F is algebraically closed, then $(F\{Z\})_0 = F\{Z\}$.

(b) If F is not algebraically closed, then $(F\{Z\})_0 = F$.

For all F, $(F\{Z\})_0^* = F$. (See [7, Th.(6.38) p. 233].)

In the sequel $\mathfrak{M}$ will denote the set of maximal ideals M of X, $\mathfrak{M}_0^*$ the set of maximal ideals m of X_0^*, and $(\mathfrak{M}_0^*)'$ the set of Gelfand ideals m' of X_0^*. For any $x \in X_0^*$ we consider the function $\hat{x}$: $(\mathfrak{M}_0^*)' \to F$, $m' \mapsto x(m')$ and we endow $(\mathfrak{M}_0^*)'$ with the weakest topology for which each of the functions $\hat{x}$ is continuous.

THEOREM 2. *If X is a Banach algebra, then $(\mathfrak{M}_0^*)'$ is a compact Hausdorff space. Furthermore, if the valuation of F is non-archimedean then $(\mathfrak{M}_0^*)'$ is a 0-dimensional space.*

Proof. To prove the first part we just consider the map $(\mathfrak{M}_0^*)' \to \Pi_{x \in X_0^*} \sigma(x)$, $m' \mapsto (x(m'))_{x \in X_0^*}$ and we argue as in the case of complex Banach algebras. The second part is trivial.

THEOREM 3. *If X is a Banach algebra, then X_0^* is a Gelfand algebra.*

Proof. If F is locally compact the result follows from the Gelfand-Mazur theorem if F is algebraically closed, and from condition 3 in Theorem 1 if F is not algebraically closed. Now assume that F is not locally compact, and let m be a maximal ideal of X_0^*. If $x \in X_0^*$ let $Z(\hat{x})$ denote the set of points of $(\mathfrak{M}_0^*)'$ where $\hat{x}$ vanishes. To see that m is a Gelfand ideal we must show that $\bigcap_{x \in m} Z(\hat{x}) \neq \varnothing$. Since $(\mathfrak{M}_0^*)'$ is compact it is enough to prove that the family $\{Z(\hat{x})/x \in m\}$ has the finite intersection property. We shall prove this in two steps:

(1) Let $x_1, x_2 \in m$ and let D_1 be the set of points in $(\mathfrak{M}_0^*)'$ where $\hat{x}_1$ does not vanish. If $\hat{x}_2/\hat{x}_1$: $D_1 \to F$ is not surjective, then there exists $x \in m$ such that $Z(\hat{x}_1) \cap Z(\hat{x}_2) = Z(\hat{x})$.

Proof. Choose $x = x_2 - \lambda x_1$, where $\lambda \notin \operatorname{Im} g(\hat{x}_2/\hat{x}_1)$.

(2) If $x_1, \ldots, x_n \in m$, then $\bigcap_i Z(\hat{x}_i) \neq \varnothing$.

Proof. By induction on n. The case $n = 1$ follows from the first two conditions of Theorem 1. Assume the result true for $n - 1$. If $\hat{x}_2/\hat{x}_1\colon D_1 \to F$ is not surjective then we have just seen in (1) that there exists $x \in m$ such that $Z(\hat{x}_1) \cap Z(\hat{x}_2) = Z(\hat{x})$. The result follows from the induction hypothesis. Now assume that $\hat{x}_2/\hat{x}_1$ is surjective and $\bigcap_i Z(\hat{x}_i) = \varnothing$. Then the set $K = \{m' \in (\mathfrak{M}_0^*)'/|\hat{x}_j(m')| \leq |\hat{x}_1(m')| \text{ for } 2 \leq j \leq n\}$ is compact and it is contained in D_1. Since F is not locally compact, to get a contradiction it is enough to show that $\hat{x}_2/\hat{x}_1(K) = \{\lambda \in F/|\lambda| \leq 1\}$. In fact take $\lambda \in F, |\lambda| \leq 1$, and consider the $(n - 1)$ elements $x_2 - \lambda x_1$ and $x_j - x_1, 3 \leq j \leq n$. By the induction assumption there exists $m' \in Z(\hat{x}_2 - \lambda\hat{x}_1) \cap \bigcap_j Z(\hat{x}_j - \hat{x}_1)$. Since $\bigcap_i Z(\hat{x}_i) = \varnothing$, then m' must belong to D_1. So $\hat{x}_2/\hat{x}_1(m') = \lambda$ and $\hat{x}_j(m') = \hat{x}_1(m')$ for $3 \leq j \leq n$. Thus $m' \in K$ and $\hat{x}_2/\hat{x}_1(m') = \lambda$. The converse is trivial.

COROLLARY. *Let X be a Banach algebra. If the linear span of the idempotent elements is dense in X, then X is a Gelfand algebra and $\mathfrak{M}$ is a compact Hausdorff space in the Gelfand topology.*

2. Representation theorems.

We assume through all this section that the valuation of F is non-archimedean and that X is a non-archimedean Banach algebra.

THEOREM 4. *If X is a V^*-algebra, then X_0^* is isometrically isomorphic to $C(\mathfrak{M}_0^*)$ under the Gelfand transformation $x \mapsto \hat{x}$.*

Proof. All we need to prove is that the Gelfand transformation is an isometry ($r_\sigma(x) = \|x\|$). In this way, we further apply the Kaplansky-Stone-Weierstrass theorem to get the desired result. Now, by condition 2 in Theorem 1, X_0^* is a V^*-algebra, and by Theorems 2 and 3 above, we are in the situation of Corollary 2, page 165 of [3]. The result now follows.

DEFINITION. A family $(x_i)_{i\in I}$ of elements in X will be called an orthogonal family if $x_i x_j = 0$ for $i \neq j$.

Let E denote the idempotent elements of X having norm one.

LEMMA. *If x belongs to the linear span of E, then $r_\sigma(x) = \|x\|$.*

Proof. (1) First suppose that there exists a finite orthogonal family $e_1,\ldots,e_n$ in E and scalars $\lambda_1,\ldots,\lambda_n$ such that $x = \sum \lambda_i e_i$. We may assume $|\lambda_1| = \max|\lambda_i|$. If we show that $\lambda_1 \in \sigma(x)$, then the result will follow from: $\max|\lambda_i| = |\lambda_1| \leq r_\sigma(x) \leq \|x\| \leq \max|\lambda_i|$.

Since e_1 is a nonzero idempotent there exists a maximal ideal M of X such that $e_1 \notin M$. But $e_1(1 - e_1) = 0$ and $e_1 e_j = 0$ implies that

$(1 - e_1) \in M$ and $e_j \in M$ for $2 \leq j \leq n$. Hence $x - \lambda_1 = -\lambda_1(1 - e_1) + \Sigma_2^n \lambda_j e_j$ belongs to M, and $\lambda_1 \in \sigma(x)$.

(2) Let $x = \Sigma_1^r \mu_j u_j$, where $u_j \in E$ and $\mu_j \in F$. Now it is enough to show that there exists a finite orthogonal family $e_1, \ldots, e_n$ in E and scalars $\lambda_1, \ldots, \lambda_n$ such that $x = \Sigma \lambda_i e_i$. The proof runs by induction on r. For $r = 1$ the result is clear. Now assume the result true for $r - 1$. Then there exists a finite orthogonal family $v_1, \ldots, v_p$ in E and scalars $\alpha_1, \ldots, \alpha_p$ such that $\Sigma_2^r \mu_j u_j = \Sigma_1^p \alpha_k v_k$. Thus $x = \mu_1 u_1 + \Sigma_1^p \alpha_k v_k$. But $v_k = v_k(1 - u_1) + v_k u_1$ and $u_1 = u_1 \Pi_1^p (1 - v_k) + \Sigma_1^p u_1 v_k$. Of course, $v_k(1 - u_1)$, $v_k u_1$ and $u_1 \Pi_1^p (1 - v_k)$ are idempotents for all $1 \leq k \leq p$, those different from zero belong to E, and x may be expressed as a linear combination of them.

THEOREM (*Van der Put*). *A non-archimedean Banach algebra X is a C-algebra if and only if the linear span of E is dense in X.*

Proof. First suppose that the linear span of E is dense in X. Then X is a Gelfand algebra and $\mathfrak{M}$ is a compact Hausdorff space in the Gelfand topology. If $x \in X$, applying the lemma, we choose (x_n) in X such that $x_n \to x$ and $r_\sigma(x_n) = \|x_n\|$. The continuity of the Gelfand transformation then implies $\hat{x}_n \to \hat{x}$ in $C(\mathfrak{M})$, and so $r_\sigma(x) = \lim r_\sigma(x_n) = \lim\|x_n\| = \|x\|$. Thus X is isometrically isomorphic to $C(\mathfrak{M})$ under the Gelfand transformation. The converse is trivial.

(See Van der Put [**6**, Prop. (5.4), p. 417] or Van Rooij [**7**, Th. (6.12), p. 215], and see also [**2**] and [**5**].)

REFERENCES

1. T. Gommers, *On maximal ideals of Banach algebras over a non-archimedean valued field*, Report 7621, Mathematisch Instituut, Katholieke Universiteit, Nijmegen, (1976).
2. G. J. Murphy, *Commutative non-archimedean C*-algebras*, Pacific J. Math., **78** No. 2, (1978), 433–446.
3. L. Narici, E. Beckenstein, and G. Bachman, *Functional Analysis and Valuation Theory*, Marcel Dekker, New York (1971).
4. N. Shilkret, *Non-Archimedean Gelfand Theory*, Pacific J. Math., **32** (1970), 541–550.
5. ______, *Non-Archimedean Banach algebras*, Duke Math. J., **37** (1970), 315–322.
6. M. Van der Put, *Algèbres de fonctions continues p-adiques*, Indag. Math., **30** (1968), 401–420.
7. A. Van Rooij, *Non-Archimedean Functional Analysis*, Marcel Dekker, Inc., New York, 1978.

Received February 25, 1981 and in revised form, October 5, 1981.

UNIVERSIDAD DE VALLADOLID
DEPARTAMENTO DE ALGEBRA Y FUNDAMENTOS
VALLADOLID, SPAIN

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

DONALD BABBITT (Managing Editor)
University of California
Los Angeles, CA 90024

HUGO ROSSI
University of Utah
Salt Lake City, UT 84112

C. C. MOORE and ARTHUR OGUS
University of California
Berkeley, CA 94720

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, CA 90089-1113

R. FINN and H. SAMELSON
Stanford University
Stanford, CA 94305

ASSOCIATE EDITORS

R. ARENS　E. F. BECKENBACH (1906–1982)　B. H. NEUMANN　F. WOLF　K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF ARIZONA
UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA, RENO
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF HAWAII
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph must be capable of being used separately as a synopsis of the entire paper. In particular it should contain no bibliographic references. Please propose a heading for the odd numbered pages of less than 35 characters. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. 39. Supply name and address of author to whom proofs should be sent. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California 90024.

There are page-charges associated with articles appearing in the Pacific Journal of Mathematics. These charges are expected to be paid by the author's University, Government Agency or Company. If the author or authors do not have access to such Institutional support these charges are waived. Single authors will receive 50 free reprints; joint authors will receive a total of 100 free reprints. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: $132.00 a year (6 Vol., 12 issues). Special rate: $66.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Old back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

The Pacific Journal of Mathematics ISSN 0030-8730 is published monthly by the Pacific Journal of Mathematics at P.O. Box 969, Carmel Valley, CA 93924. Application to mail at Second-class postage rates is pending at Carmel Valley, California, and additional mailing offices. Postmaster: Send address changes to Pacific Journal of Mathematics, P. O. Box 969, Carmel Valley, CA 93924.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Copyright © 1983 by Pacific Journal of Mathematics

Pacific Journal of Mathematics
Vol. 104, No. 2 June, 1983