

# Pacific Journal of Mathematics

## **HOMOMORPHISMS OF MINIMAL FLOWS AND GENERALIZATIONS OF WEAK MIXING**

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In this paper we are concerned with generalizations of weakly mixing. Let  $\phi: (X, T) \rightarrow (Y, T)$  be a homomorphism of metric minimal flows and let  $S(\phi)$  denote the relativized equicontinuous structure relation. The main result is that if  $\phi$  has a RIM,  $\lambda$ , and  $z \in Z$  such that the support of  $\lambda_z$  equals the fiber  $X_0 = \phi^{-1}(z)$ , then:

$$oc(V_1 \times \cdots \times V_n) \supseteq S(\phi)(V_1) \times \cdots \times S(\phi)(V_n),$$

and also there exists a dense set of points  $x_1, x_2, x_3, \dots$  in  $X_0$  such that  $oc(x_1, x_2, x_3, \dots) \supseteq S(\phi)(x_1) \times S(\phi)(x_2) \times \dots$ .

**0. Introduction.** This paper is chiefly concerned with homomorphisms of minimal flows (on compact Hausdorff spaces by a discrete phase group) having relative invariant measures (RIM's). If  $\phi: X \rightarrow Z$  has a RIM,  $\lambda$ , we will frequently restrict our attention to points  $z$  in  $Z$  with the support of  $\lambda_z$  equal to  $\phi^{-1}(z)$  since otherwise the results would be substantially more difficult to state (and prove).

The major motivation for this paper is a generalization of weakly mixing — if  $(X, T)$  is a metric minimal flow having an invariant measure, then it is well known that  $Q = X \times X$  implies  $\text{cls}(x, x')T = X \times X$  for some  $x, x'$  in  $X$ ; we show that even when  $Q \neq X \times X$  a similar statement holds, that  $\text{cls}(x, x')T \supseteq Q(x) \times Q(x')$  for some  $x, x'$  in  $X$ . The main results of this paper are generalizations of this idea. Some may also be viewed as a study of the recurrence properties of various subsets of  $X$ . We will now mention some special cases of the main results.

Suppose  $\phi: X \rightarrow Z$  has a RIM,  $\lambda$ , and  $X, Z$  are minimal and metric. Then  $S(\phi) = \{(x, x'): (xu, x'u) \in Q(\phi) \text{ for some } u \in J\}$  (see 2.1). Suppose  $z \in Z$  such that the support of  $\lambda_z$  equals the fiber  $X_0 = \phi^{-1}(z)$ . If  $N = oc(V_1 \times \cdots \times V_n)$  where  $V_i$  is an open set in  $X_0$ , then  $N \supseteq S(\phi)(V_1) \times \cdots \times S(\phi)(V_n)$  (see 1.1). Also there exists a dense set of points  $x_1, x_2, x_3, \dots$  in  $X_0$  such that  $oc(x_1, x_2, x_3, \dots) \supseteq S(\phi)(x_1) \times S(\phi)(x_2) \times \dots$ , (see 1.5). If  $R$  is the smallest closed invariant equivalence relation containing  $(x_1, x_2), x_1, x_2$  as above, then  $\phi': X/R \rightarrow Z$  is almost automorphic, that is,  $Q(\phi')(y) = \{y\}$  for some  $y$  in  $X/R$  (see 1.4). If  $N = oc(\{x\} \times V_1 \times \cdots \times V_n)$  where  $V_1, \dots, V_n$  are open sets in  $X_0$ , then  $N \supseteq S(\phi)(x) \times V_1 \times \cdots \times V_n$  (see 2.9) and  $N \supseteq S(\phi)(x)V \times S(\phi)(V_1)v \times \cdots \times S(\phi)(V_n)v$  for every  $v$  in  $J$  (see 2.11). In part we showed the last statement as a possible start in determining whether or not for each  $x$  in

$X_0$  there exists  $x'$  in  $X_0$  with  $\text{cls}(x, x')T \supseteq S(\phi)(x) \times S(\phi)(x')$ . If  $x_i, y_i \in X_0$  and  $x = (x_i) \in \prod X, y = (y_i) \in \prod X$ , then  $(x, y) \in Q(\prod \phi)$  iff  $(x_i, y_i) \in Q(\phi)$  for every  $i$ , (see 2.13).

**DEFINITIONS AND NOTATION.** Let  $(X, T)$  be a flow with compact Hausdorff phase space  $X$  and discrete phase group  $T$ . We will write  $X$  for both the flow and the phase space. Suppose  $\Phi: X \rightarrow Z$  is a homomorphism of flows. We will assume  $\phi$  is onto. We denote the orbit closure of  $x$  by  $oc(x)$  ( $= \text{cls}(xT)$ ). We let  $X_m$  denote the set of transitive points (points with dense orbit),  $R_m(\phi) = \{(x, x') \in X_m \times X_m: \phi(x) = \phi(x')\}$ ,  $Q_m(\phi) = \{(x, x'): \text{there exist nets } t_n \text{ in } T \text{ and } (x_n, x'_n) \in R_m(\phi) \text{ such that } (x_n, x'_n) \rightarrow (x, x') \text{ and } (x_n, x'_n)t_n \rightarrow (x_0, x_0)\} \text{ for any } x_0 \text{ in } X_m$ ,  $S_m(\phi)$  is the smallest closed (in  $R_m(\phi)$ ) invariant equivalence relation containing  $Q_m(\phi)$ .

When  $X$  is minimal  $X_m = X$ ,  $R_m(\phi) = R(\phi)$ ,  $Q_m(\phi) = Q(\phi)$  is the relativized regionally proximal relation  $S_m(\phi) = S(\phi)$  is the relativized equicontinuous structure relation. If  $X$  is minimal and  $Z$  is the singleton flow, we denote  $Q(\phi)$  by  $Q$  and  $S(\phi)$  by  $S$ . Let  $P$  denote the proximal relation on any minimal flow.

Neighborhoods are assumed to be open, we denote the set of neighborhoods of  $x$  by  $\mathcal{U}_x$ . The Stone-Ćech compactification of  $T$  is denoted by  $\beta T$ ,  $M \subseteq \beta T$  denotes the universal minimal set (a minimal right ideal in  $\beta T$ ),  $J \subseteq M$  denotes the set of idempotents in  $M$ .

The set of closed subsets of  $X$  is denoted by  $2^X$  and is given the usual Hausdorff topology. For  $A \in 2^X, p \in \beta T$ , we denote the limit in  $2^X$  of  $At_n$  by  $A \circ p$ , where  $t_n \rightarrow p$  in  $\beta T$ ;  $Ap = \{ap: a \in A\}$ . A homomorphism of minimal flows,  $\phi: X \rightarrow Y$ , is relatively incontractible (RIC) iff for every  $p \in M$ ,  $\Phi^{-1}(yp) = (\phi^{-1}(y)u) \circ p$  where  $y \in Y, u \in J$  with  $yu = y$  (see [5<sub>b</sub>] for details).

Let  $\mathfrak{N}(X)$  be the set of Borel probability measures on  $X$ . For  $\mu$  in  $\mathfrak{N}(X)$  define  $\mu t$  by  $\mu(At^{-1})$  for every measurable set  $A$ . A RIM (relative invariant measure — also called a section)  $\lambda$  for  $\phi: X \rightarrow Z$  is a homomorphism  $\lambda: Y \rightarrow \mathfrak{N}(X)$  such that the support of  $\lambda$  is contained in the fiber  $\phi^{-1}(z)$ . If  $z$  is fixed, then for any RIM,  $\lambda, S_\lambda$  denotes the support of  $\lambda_z$ . Also we define  $\hat{\phi}: \mathfrak{N}(X) \rightarrow \mathfrak{N}(Z)$  by  $\hat{\phi}(\mu)(A) = \mu(\phi^{-1}(A))$ ,  $A$  a measurable subset of  $Z$ . For  $B \subseteq \mathfrak{N}(X)$  we denote the closed convex hull of  $B$  by  $\overline{\text{co}}(B)$ .

Given  $\varphi: X \rightarrow Z, \theta: Y \rightarrow Z, X \circ^Z Y = \{(x, y): \varphi(x) = \theta(y)\}$ . If  $N$  is a subset of  $X \times Y$ ,  $N_x = N(x) = \{y \in Y: (x, y) \in N\}$  is a set such that  $\{x\} \times N_x = N \cap (\{x\} \times Y)$ . For our purposes we will consider sets  $N$  contained in  $X \circ^Z Y$  and thus  $N_x \subseteq \theta^{-1}(\varphi(x))$ .

For the convenience of the reader we will now state some simplified results of [6] that we will be using. First we note that the assumption that  $Y$  be point-transitive in [6] was not needed.

**COROLLARY 1.4** of [6]. *Suppose  $X$  is point-transitive,  $\varphi: X \rightarrow Z$ ,  $\theta: Y \rightarrow Z$  are surjective homomorphisms, and  $\theta$  has a RIM,  $\lambda$ . If  $x_0 \in X_m$ ,  $A$  a Borel set contained in  $\theta^{-1}(\phi)(x_0)$ , and  $N = \text{cls}(\{\{x_0\} \times A\}T)$ , then for  $x \in S(\phi)(x_0)$ ,  $\lambda_{\phi(x_0)}(A \cap N_x) = \lambda_{\phi(x_0)}(A)$  (that is,  $\lambda_{\phi(x_0)}(A \setminus N_x) = 0$ ). If in addition  $A = B \cap C$  where  $B$  is a Borel set contained in the support of  $\lambda_{\phi(x_0)}$  with  $\lambda_{\phi(x_0)}(B) = 1$  and  $C$  is a non-empty open set, then  $A \subseteq N_x$ .*

Compare this with Lemma 2.6 below.

**THEOREM 1.5** of [6]. *Suppose  $X$  is minimal and  $Q: X \rightarrow Z$  has a RIM,  $\lambda$ . Then for  $x$  in the support of  $\lambda_{\phi(x)}$ ,  $Q(\phi)(x) = S(\phi)(x)$ .*

**1.** A main consequence of this section is that if  $\phi: (X, T) \rightarrow (Z, T)$  has a RIM,  $\lambda$ , then for some  $z$  in  $Z$  there exist  $x_1, x_2 \in \phi^{-1}(z)$  such that  $oc(x_1, x_2) \supseteq Q(\phi)(x_1) \times Q(\phi)(x_2)$ . This holds for all  $z$  that satisfy three types of conditions,  $z \in Z_1 \cap Z_2 \cap Z_3$ , where  $z_1, z_2, z_3$  are as follows.

First consider  $\pi: X \rightarrow X/S(\phi)$ , with  $X$  metric. Then by Lemma 3.1 of [9] there exist a residual subset  $X_1$  of  $X$  such that  $\pi$  is open at each  $x$  in  $X_1$ . By Proposition 3.1 of [10] the set  $Z_1 = \{z \in Z: X_1 \cap \phi^{-1}(z) \text{ is residual in } \phi^{-1}(z)\}$  is residual in  $Z$ . Note for  $x \in X_1$ , every neighborhood  $V$  of  $x$  has  $\pi(V)$  a neighborhood of  $\pi(x)$  and  $V' = V \cap \pi^{-1}(\text{int}(\pi V))$  a neighborhood of  $x$  with  $\pi^{-1}(\pi(V')) = \pi^{-1}(\text{int}(\pi V))$  open, that is  $S(\phi)(V') = \pi^{-1}(\pi(V'))$  open. As noted in [10],  $V \subseteq \text{cls}(V')$ .

More importantly, for fixed  $z \in Z_1$  every open set  $V^*$  in  $\phi^{-1}(z)$  contains an open set  $V^{**}$  in  $\phi^{-1}(z)$  such that  $S(\phi)(V^{**})$  is open — indeed  $V^* = V \cap \phi^{-1}(z)$  where  $V$  is open, and  $V^* \cap X_1 \neq \emptyset$  and so there is an open set  $V'$  such that  $V' \cap \phi^{-1}(z) \neq \emptyset$  and  $S(\phi)(V')$  is open, thus  $V^{**} = V' \cap \phi^{-1}(z)$  has  $S(\phi)(V^{**}) = S(\phi)(V' \cap \phi^{-1}(z)) = S(\phi)(V') \cap \phi^{-1}(z)$  open in  $\phi^{-1}(z)$ . Also  $V \subseteq \text{cls}(V')$ , so  $V^* = V \cap \phi^{-1}(z) \subseteq (\text{cls } V') \cap \phi^{-1}(z)$ .

**REMARK.** Note that in the non-metric case if  $Y$  is a singleton we have that every open set  $V$  contains an open set  $V^*$  such that  $S(V^*)$  is open,  $V \subseteq \text{cls}(V^*)$ , and  $A \cap V \subseteq V^*$  where  $A = \{x: \phi: X \rightarrow X/S \text{ is open at } x\}$ . To prove this consider  $\phi: X \rightarrow X/S$ . Then  $\phi(V)$  has non-empty interior  $W$ . Take  $V^* = V \cap \phi^{-1}(W)$ . Then  $S(V^*) = \phi^{-1}(\phi(V \cap \phi^{-1}(W))) = \phi^{-1}(W)$ . Let  $x \in V$ , then for any neighborhood  $U$  of  $x$ ,  $U \cap V \neq \emptyset$  and  $\phi(U \cap V)$  has non-empty interior. So  $\phi(U \cap V) \subseteq W$  and  $U \cap V \subseteq \phi^{-1}(W)$ . Thus  $V \subseteq \text{cls}(V \cap \phi^{-1}(W)) = \text{cls}(V^*)$ .

Recall that given a function  $f$  from a metric space  $X$  onto a metric space  $Z$ , if  $f$  is a Borel map (in particular, when it is either upper semi-continuous or lower semi-continuous), then  $f$  is continuous at a residual subset of  $X$ .

Define  $\Phi: X \rightarrow 2^X$  by  $\Phi(x) = S(\phi)(x)$ . Then  $\Phi$  is upper semi-continuous. There exists a residual set  $X_2$  of  $X$  such that  $\Phi$  is continuous at each  $x \in X_2$ . Then there exists a residual set  $Z_2$  of  $Z$  such that  $X_2 \cap \phi^{-1}(z)$  is residual in  $\phi^{-1}(z)$  for  $z \in Z_2$ .

Finally, if  $\phi: X \rightarrow Z$  admit a relative invariant measure (RIM),  $\lambda$ , then the function  $g: Z \rightarrow 2^X$  defined by  $g(z) = \text{support of } \lambda_z$  is lower semi-continuous. Then one can show that there exists a residual set  $Z_3$  of  $Z$  such that the support of  $\lambda_z$  equals  $\Phi^{-1}(z)$  for  $z$  in  $Z_3$  (Proposition 3.3 of [4]).

**1.1. PROPOSITION.** *Suppose for  $i = 1, 2, \dots, n$   $\phi_i: X_i \rightarrow Z$  has a relative invariant measure  $\lambda_i$ ,  $z \in Z$  with  $\phi_i^{-1}(z) = \text{support of } \lambda_{iz}$ , and  $V_i$  are open subsets of  $\phi_i^{-1}(z)$  with  $Q(\phi_i)(V_i)$  open in  $\phi_i^{-1}(z)$ . Then*

$$\begin{aligned} N &= oc(V_1 \times \cdots \times V_n) \supseteq Q(\phi_1)(V_1) \times \cdots \times Q(\phi_n)(V_n) \\ &= S(\phi_1)(V_1) \times \cdots \times S(\phi_n)(V_n). \end{aligned}$$

*Proof.* The last equality follows from 1.5 of [6].

Now  $N \supseteq Q(\phi_1)(V_1) \times V_2 \times \cdots \times V_n$  by Corollary 1.4 of [6] since  $V_2 \times \cdots \times V_n$  is open and  $X_2 \times \cdots \times X_n$  has a relative invariant measure. So  $N \supseteq Q(V_1) \times Q(V_2) \times V_3 \times \cdots \times V_n$  by 1.4 of [6] since  $Q(V_1) \times V_3 \times \cdots \times V_n$  is open, the proposition follows by induction.

**1.2. PROPOSITION.** *For  $i = 0, 1, \dots, n$ . Suppose  $(X_i, T)$  is a minimal flow with  $T$  abelian. Then for any  $x_0$  in  $X_0$  and for any open sets  $V_i$  in  $X_i$ ,  $i = 1, 2, \dots, n$  there exist open sets  $U_0, U_1, \dots, U_n$  such that  $x_0 \in \text{cls}(U_0)$ ;  $U_i \subseteq V_i$ ,  $i = 1, 2, \dots, n$ ; and*

$$\begin{aligned} &oc(\{x_0\} \times V_1 \times \cdots \times V_n) \\ &\supseteq oc(U_0 \times U_1 \times \cdots \times U_n) \supseteq \text{cls}[Q(U_0) \times Q(U_1) \times \cdots \times Q(U_n)] \\ &\supseteq \{x_0\} \times Q(U_1) \times \cdots \times Q(U_n). \end{aligned}$$

*If  $X_0 \rightarrow X_0/Q$  is open at  $x_0$ , then we can take  $U_0$  with  $x_0 \in U_0$ .*

*Proof.* Let  $V(t_1, t_2, \dots, t_n)$  denote  $V_1 t_1 \times V_2 t_2 \times \cdots \times V_n t_n$  where  $t_i \in T$ . Then there exist finite many  $n$ -tuples  $s_1, s_2, \dots, s_m$  in  $\prod_1^n T$  such that  $\bigcup V(s_i) = \prod_1^n X_i$ . Let  $Y = \prod_1^n X_i$ . Then  $\bigcup_1^n [\{x_0\} \times V(s_i)] = \{x_0\} \times Y$ . So  $\bigcup_1^n ([\{x_0\} \times V(s_i)]t) = (\bigcup [\{x_0\} \times V(s_i)]t) = (\{x_0\} \times Y)t = \{x_0 t\} \times Y$ . Therefore  $\bigcup_1^n \text{cls}([\{x_0\} \times V(s_i)]T) = X_0 \times Y$  and thus  $\text{cls}([\{x_0\} \times V(s_i)]T)$  has non-empty interior,  $I$ , for some  $i$ . Then for some  $t$  in  $T$ ,  $[\{x_0\} \times V(s_i)]t \cap I \neq \emptyset$ . So there exist open sets  $U'_0, U'_1, \dots, U'_n$  with  $x_0 \in U'_0$ , and  $U'_1 \times \cdots \times U'_n \subseteq V(s_i)$  such that  $(U'_0 \times U'_1 \times \cdots \times U'_n)t \subseteq I$  and so  $U'_0 \times U'_1 \times \cdots \times U'_n \subseteq \text{cls}([\{x_0\} \times V(s_i)]T)$ .

Let  $s_i = (t_1, \dots, t_n)$ , then since  $T$  is abelian,

$$U'_0 \times U'_1 t_1^{-1} \times \cdots \times U'_n t_n^{-1} \subseteq \text{cls}([\{x_0\} \times V_1 \times \cdots \times V_n]T).$$

Then by the remarks above there exist open sets  $U_i \subseteq U'_i t_i^{-1}$  such that  $Q(U_i)$  is open  $i = 0, 1, \dots, n$ , and  $x_0 \in U_0 \subseteq \text{cls } U_0$ . When  $X_0 \rightarrow X_0/Q$  is open at  $x_0$  we have  $x_0 \in U_0$ . Then the proposition follows by 1.1.

**1.3. THEOREM.** *Suppose for  $i = 1, 2, \dots, \phi_i: X_i \rightarrow Z$  has a RIM,  $\lambda_i$ , and  $X_i$  is a metric minimal flow. Let  $z_0 \in \bigcap_{i=1}^{\infty} (Z_1^i \cap Z_2^i \cap Z_3^i)$ , ( $Z_j^i = Z_j$  as above for  $\phi_j$ ). Given  $x_i$  in  $X_0^i = \phi_i^{-1}(z_0)$  and dense  $G_\delta$  subsets  $G^i$  of  $X_0^i$  there exist points  $x'_i$  in  $G^i$  such that  $x'_i \in B(x_i, 1/i)$  and  $oc(x'_1, x'_2, \dots) \supseteq Q(\phi_1)(x'_1) \times Q(\phi_2)(x'_2) \times \cdots$ . (Recall  $Q(\phi)(x) = S(\phi)(x)$  for  $x \in \phi^{-1}(Z_3)$ .)*

*Proof.* Let  $B(y_1, \dots, y_n; \varepsilon)$  denote  $B(y_1, \varepsilon) \times \cdots \times B(y_n, \varepsilon)$ . Fix  $\varepsilon > 0$ . Consider any set  $\{x_i\}$ ,  $x_i \in X_0^i \cap X_2^i$ . For each  $i$  use the continuity of  $\Phi_i$  at  $x_i$  to associate a neighborhood  $U_i^* = U_i^*(x_i, \varepsilon)$  of  $x_i$  with  $x_i$  and  $\varepsilon$  such that if  $x \in U_i^*$  and  $y_i \in Q(\phi_i)(x_i)$ , then  $Q(\phi_i)(x) \cap B(y_i, \varepsilon) \neq \emptyset$ ; and if  $\varepsilon' > \varepsilon$ , then  $U_i^*(x_i, \varepsilon') \supseteq U_i^*(x_i, \varepsilon)$ . Let  $U_i = U_i(x_i, \varepsilon) = U_i^* \cap X_0^i$  and  $y_i \in Q(\phi_i)(x_i)$ ; note  $x_i \in U_i$ . Now for each  $n$  consider the set  $W_n = W(y_1, \dots, y_n; \varepsilon) = \{w \in U_1 \times \cdots \times U_n: wt \in B(y_1, \dots, y_n; \varepsilon) \text{ for some } t \text{ in } T\}$ . Clearly  $W_n$  is open (in  $X_0^1 \times \cdots \times X_0^n$ ). Also  $W_n$  is a dense subset of  $U_1 \times \cdots \times U_n$ ; since for any basic open subset  $V = V_1 \times \cdots \times V_n$  of  $U_1 \times \cdots \times U_n$  in  $X_0^1 \times \cdots \times X_0^n$  take an open subset  $V^* = V_1^* \times \cdots \times V_n^*$  with  $Q(\phi_i)(V_i^*)$  open. Then for any point  $(x_1^*, \dots, x_n^*)$  in  $V^*$ , there exists  $y_i^* \in Q(\phi_i)(x_i^*) \cap B(y_i, \varepsilon)$  for  $i = 1, \dots, n$ , and by 1.1, we have  $oc(V^*) \supseteq Q(\phi_1)(V_1^*) \times \cdots \times Q(\phi_n)(V_n^*) \ni (y_1^*, \dots, y_n^*)$ ; so there exists  $t$  in  $T$  with  $V^*t \cap B(y_1, \dots, y_n; \varepsilon) \neq \emptyset$ , and thus  $W_n$  is dense.

Consider a cover of  $Q(\phi_1)(x_1) \times \cdots \times Q(\phi_n)(x_n)$  by sets of the form  $B(y_1, \dots, y_n; \varepsilon)$  where  $y_i \in Q(\phi_i)(x_i)$ . Take a finite subcover and the (finite) intersection  $B_n$  of the corresponding  $W_n$ 's, then  $B_n$  is open (in  $X_0^1 \times \cdots \times X_0^n$ ) and is dense in  $U_1 \times \cdots \times U_n$ . By continuity, for each  $b$  in  $B_n$ , there is a neighborhood  $E(b)$  of  $b$  contained in  $B_n$  such that any given open set in the finite subcover contains  $E(b)t$  for some  $t$  in  $T$ . From this it is clear that for any  $(y_1, \dots, y_n)$  in  $Q(\phi_1)(x_1) \times \cdots \times Q(\phi_n)(x_n)$ ,  $E(b)t \subseteq B(y_1, \dots, y_n; 2\varepsilon)$  for some  $t$  in  $T$ .

Now consider a given collection  $x_i \in X_0^i$ ,  $i = 1, 2, \dots$ . We may assume  $x_i \in X_0^i \cap X_2^i$ . Let  $H_j^i$ ,  $i, j = 1, 2, \dots$ , be dense open subsets of  $X_0^i$  such that  $H_{j+1}^i \subseteq H_j^i$  and  $\bigcap_{j=1}^{\infty} H_j^i = X_2^i \cap X_0^i \cap G^i$  for  $i = 1, 2, \dots$ . Start an induction with  $x_1, x_2$ ,  $n = 2$  and  $\varepsilon = \frac{1}{2}$ . Take  $B_2$  as above and  $b_2 \in (X_2^1 \times X_2^2) \cap (G^1 \times G^2) \cap B_2 \cap [B(x_1, 1) \times B(x_2, \frac{1}{2})]$ . Let  $E_2 = (H_1^1 \times H_2^2) \cap E(b_2) \cap B(b_2, \frac{1}{2}) \cap [B(x_1, 1) \times B(x_2, \frac{1}{2})]$ ; note it is a neighborhood of  $b_2$  in  $X_0^1 \times X_0^2$ . Now consider  $b_2 \times \{x_3\}$ ,  $n = 3$ ,  $\varepsilon = \frac{1}{3}$ , and take  $B_3$  as above

and  $b_3 \in (X_2^1 \times X_2^2 \times X_2^3) \cap (G^1 \times G^2 \times G^3) \cap B_3 \cap [E_2 \times B(x_3, \frac{1}{3})]$ . Take a neighborhood  $E_3$  of  $b_3$  with  $\text{cls}(E_3) \subseteq (H_2^1 \times H_2^2 \times H_2^3) \cap E(b_3) \cap B(b_3, \frac{1}{3}) \cap [E_2 \times B(x_3, \frac{1}{3})]$ . Consider  $b_3 \times \{x_4\}$ ,  $n = 4$ ,  $\varepsilon = \frac{1}{4}$ , take  $B_4$  as above and  $b_4 \in (X_2^1 \times X_2^2 \times X_2^3 \times X_2^4) \cap (G^1 \times G^2 \times G^3 \times G^4) \cap [E_3 \times B(x_4, \frac{1}{4})] \cap B_4$ . Continue in this way.

Note  $\bigcap_{n=b_2}^{\infty} (E_n \times \prod_{n+1}^{\infty} X_i)$  is a singleton, say  $\{(x'_1, x'_2, \dots)\}$ , and note  $(x'_1, x'_2, \dots) \in [(X_2^1 \cap X_0^1 \cap G^1) \times (X_2^2 \cap X_0^2 \cap G^2) \times (X_2^3 \cap X_0^3 \cap G^3) \times \dots] \cap [B_1(x_1, 1) \times B(x_2, \frac{1}{2}) \times B(x_3, \frac{1}{3}) \times \dots]$ . We claim  $oc(x'_1, x'_2, \dots) \supseteq Q(\phi_1)(x'_1) \times Q(\phi_2)(x'_2) \times \dots$ . For any  $(y_1, y_2, \dots)$  in  $Q(\phi_1)(x'_1) \times Q(\phi_2)(x'_2) \times \dots$ , a basic neighborhood of it is of the form  $B(y_1, \dots, y_n; \lambda) \times \prod_{n+1}^{\infty} X_i$  for some  $n$  and  $\lambda > 0$ . Let  $U'_i = U(x'_i, \lambda)$  for  $i = 1, 2, \dots, n$ . Take  $j$  such that  $b_j \in U'_1 \times \dots \times U'_n \times \prod_{n+1}^j X_i$  and  $1/(j+1) < \lambda$ . Then

$$[Q(\phi_1)(b_{j1}) \times \dots \times Q(\phi_n)(b_{jn})] \cap B(y_1, \dots, y_n; \lambda) \neq \emptyset,$$

where  $b_j = (b_{j1}, \dots, b_{jn})$ , (since  $b_{ji} \in U'_i$ ). Let  $(y_1^*, \dots, y_n^*)$  be a point in this intersection. Then there exists  $t$  in  $T$  such that  $(x'_1, x'_2, \dots)t \in E_{j+1}t \subseteq B(y_1^*, \dots, y_n^*; 2/(j+1)) \subseteq B(y_1, \dots, y_n; 3\lambda)$ . Thus  $(y_1, y_2, \dots) \in oc(x'_1, x'_2, \dots)$ .

**1.4. COROLLARY.** *Suppose  $X$  is metric, minimal flow and  $\phi: X \rightarrow Z$  has a RIM. Then there exists  $(x_0, x_1) \in X \times X$  such that  $\phi': Y = X/R(x_0, x_1) \rightarrow Z$  is an almost automorphic extension of  $Z$  (i.e., there is a point  $y$  in  $Y$  with  $Q(\phi')(y) = \{y\}$ ) where  $R(x_0, x_1)$  is the smallest closed invariant equivalence relation containing  $(x_0, x_1)$ .*

*Proof.* This is clearly the case if we take  $(x_0, x_1)$  such that  $oc(x_0, x_1) \supseteq Q(\phi)(x_0) \times Q(\phi)(x_1)$ .

**2.** In this section we develop some connections of a RIM on  $\phi: X \rightarrow Y$  to the relativized equicontinuous structure relation,  $S(\phi)$ , and apply them to study the orbit closures of sets of the form  $\{x\} \times A^2 \times \dots \times A^n$  in a product space and to give a special characterization  $S(\phi)$  in the case when  $(R(\phi), T)$  has a dense set of almost periodic points.

Suppose  $\phi: X \rightarrow Y$  has a RIM,  $\lambda$ ,  $X$  is minimal and  $N$  is a closed invariant set in  $R(\phi)$ . Then  $\phi_N: R(\phi) \rightarrow [0, 1]$  defined by  $\phi_N(x, x') = \lambda_{\phi(x)}(N(x) \Delta N(x')) = 2\lambda_{\phi(x)}(N(x) \setminus N(x'))$  is continuous, [6] where  $\{x\} \times N(x) = N \cap (\{x\} \times X)$  and  $\Delta$  is the symmetric difference. So for each  $N$ ,  $\phi_N(x, x')$  is a pseudo-metric on each fiber that is invariant,  $\phi_N(xt, x't) = \phi_N(x, x')$ . Defining  $R_N = \{(x, x') \in R(\phi): \phi_N(x, x') = 0\}$ , we have  $X \rightarrow X/R_N \xrightarrow{\psi_N} Y$  and  $\psi_N$  is an isometric homomorphism (and thus almost periodic).

Consider  $S^*(\phi) = \{(x, x') : \phi_N(x, x') = 0 \text{ for all closed invariant subsets } N \text{ of } R(\phi)\}$ . Then by 1.2 of [6]  $S(\phi) \subseteq S^*(\phi)$ . We wish to show that  $S^*(\phi) \subseteq S(\phi)$ . Note by 1.2 of [6]  $S^*(\phi)$  is closed and invariant. Suppose  $(x, x') \in S^*(\phi)$ . Let  $\phi(x) = z_0$ , let  $x_1 \in S_\lambda = \text{support of } \lambda_{z_0}$ , and let  $pu \in M$  such that  $xpu = x_1$ . Note  $(xpu, x'pu) \in S^*(\phi)$ . For any  $V \in \mathcal{N}_{x_1}$  consider  $N = oc(\{x'pu\} \times V \cap S_\lambda)$ . By 1.4 of [6]  $N \supseteq \{x_1\} \times (V \cap S_\lambda)$ , so  $N \cap V \times (V \cap S_\lambda) \neq \emptyset$ , so there exists  $t_V$  in  $T$  and  $x_V$  in  $V \cap S_\lambda$ , such that  $x_V t_V \in V$  and  $x'put_V \in V$ . Thus  $x' = xpu \in Q(\phi)(x'pu)$  and so  $(xu, x'u) \in Q(\phi)$  and  $(x, x') \in S(\phi)$ . Thus we have the following proposition.

**2.1. PROPOSITION.** *If  $\phi: X \rightarrow Z$  has a RIM,  $\lambda$ , then  $\{(x, x') \in R(\phi) : \lambda_{\phi(x)}(N(x)\Delta N(x')) = 0 \text{ for all closed invariant sets } N \text{ in } R(\phi)\} = S(\phi) = \{(x, x') \in R(\phi) : (xu, x'u) \in Q(\phi) \text{ for some (and thus every) } u \in J\}$ .*

**2.2. PROPOSITION.** *Suppose  $\Phi: X \rightarrow Y$  has a RIM,  $\lambda$ , and  $X$  and  $Y$  are minimal. If  $\phi$  is open and  $S(\phi) = R(\phi)$ , then  $Q(\phi) = S(\phi)$ .*

*Proof.* Let  $(x, x') \in R(\phi) = S(\phi)$ , we will show  $(x, x') \in \overline{Q(\phi)} = Q(\phi)$ . Let  $U$  and  $V$  be open neighborhoods of  $x$  and  $x'$  respectively. Let  $x_0$  be any point in the support of  $\lambda_{\phi(x)}$ . Since  $\phi$  is an open map,  $\phi(V) \cap \phi(U)$  is an open neighborhood of  $\phi(x)$ . There exist  $t_0$  in  $T$  with  $x_0 t_0 \in V$  and  $\phi(x_0 t_0) \in \phi(V) \cap \phi(U)$ . So there is  $x_1 \in U$  with  $\phi(x_1) = \phi(x_0 t_0)$ ; then  $(x_1 t_0^{-1}, x_0) \in R(\phi) = S(\phi)$  and by 1.5 of [6],  $x_1 t_0^{-1} \in S(\phi)(x_0) = \overline{Q(\phi)}(x_0)$ . Therefore  $(x_1, x_0 t) = (x_1 t_0^{-1}, x_0) t_0 \in Q(\phi)$  and  $(x, x') \in \overline{Q(\phi)} = Q(\phi)$ .

**2.3. LEMMA.** *Given  $\phi: X \rightarrow Y$ ,  $\theta: Y \rightarrow Z$ ,  $X$  minimal. Let  $x \in X$  and  $y = \phi(x)$ . Then for any  $y' \in S(\theta)(y)$  there exists  $x' \in S(\theta \circ \phi)(x)$  with  $y' = \phi(x')$ . (Note this is somewhat stronger than the statement  $\phi \times \phi(S(\theta \circ \phi)) = S(\theta)$ .)*

*Proof.* By 14.2 of [2<sub>b</sub>],  $\phi \times \phi(Q(\theta \circ \phi)) = Q(\theta)$ . Consider  $M \xrightarrow{\psi} X$  with  $\psi(m) = xm$ . Then  $\phi \times \phi(\psi \times \psi(Q(\theta \circ \phi \circ \psi))) = Q(\theta)$ . Let  $u \in J$  with  $xu = x$ . Note  $Q(\theta \circ \phi \circ \psi)$  is left invariant under  $G = M_0 u$ ,  $M_0 = (\theta \circ \phi \circ \psi)^{-1}(y)$ ; and so  $S(\theta \circ \phi \circ \psi)$  is also, since  $g \times g(S(\theta \circ \phi \circ \psi))$  is a closed invariant equivalence relation containing  $g \times g(Q(\theta \circ \phi \circ \psi)) = Q(\theta \circ \phi \circ \psi)$ , for  $g \in G$ . Let  $R$  denote  $\phi \times \phi(\psi \times \psi(S(\theta \circ \phi \circ \psi)))$ . Also  $S(\theta) \supseteq \phi \times \phi(S(\theta \circ \phi)) \supseteq R$ . To show the reverse inclusion first note  $Q(\theta) = \phi \times \phi(\psi \times \psi(Q(\theta \circ \phi \circ \psi))) \subseteq R$ . Also  $R$  is closed and invariant; we will now show that  $R$  is an equivalence relation and thus  $S(\theta) \subseteq R$  and the lemma will follow. We only need to show that if  $(y_1, y_2) \in R$  and



$(y_2, y_3) \in R$ , then  $(y_1, y_3) \in R$ . Let  $m_1, m_2, m'_2, m_3 \in M$  with  $(m_1, m_2), (m'_2, m_3) \in S(\theta \circ \phi \circ \psi)$  and  $\phi \circ \psi(m_i) = y_i, i = 1, 2, 3$   $\phi \circ \psi(m'_2) = y_2$ . Choose  $m \in M$  so that  $mm'_2 \in m_2J$ . Then  $(mm'_2, mm_3) \in S(\theta \circ \phi \circ \psi)$  and  $(m_2, mm'_2) \in S(\theta \circ \phi \circ \psi)$ , and so  $(m_1, mm_3) \in S(\theta \circ \phi \circ \psi)$ . Also  $\phi \circ \psi(mm_3) = y_3$  since  $\phi \circ \psi(mm'_2) = \phi \circ \psi(m'_2)$ ; so  $(y_1, y_3) \in R$ . Thus we have that  $S(\theta) = \phi \times \phi(S(\theta \circ \phi)) = R$ .

Now suppose  $y' \in S(\phi)(y)$ , and  $(m, m') \in S(\theta \circ \phi \circ \psi)$  with  $\phi \circ \psi(m) = y, \phi \circ \psi(m') = y'$ . We may assume  $m = mu$  since  $S(\theta \circ \phi \circ \psi)$  is an equivalence relation. Then  $(u, m^{-1}m') \in S(\theta \circ \phi \circ \psi)$  and  $\psi(u) = x$ . Let  $x' = \psi(m^{-1}m')$ . Then  $(x, x') \in S(\theta \circ \phi)$  and  $\phi(x') = y'$ . Thus the lemma is proved.

**2.4. LEMMA.** *Let  $M$  be the universal minimal set,  $Z$  a minimal flow,  $z$  a fixed element of  $Z$ ,  $u \in J$  with  $zu = z$ , and  $\psi: M \rightarrow Z$  be defined by  $\psi(p) = zp, p \in M$ .*

*If  $p \in S(\psi)(u)$  and  $pv = p, v \in J$ , then  $[S(\psi)(u)]v = [S(\psi)(u)]p$ .*

*Proof.* If  $m \in S(\psi)(u)$ , then  $mp \in S(\psi)(up) = S(\psi)(u)$  since  $up = p$  and  $S(\psi)$  is a closed invariant equivalence relation. So  $S(\psi)(u)p \subseteq S(\psi)(u)$  and so  $S(\psi)(u)p \subseteq S(\psi)(u)v$ .

Let  $p^{-1}$  be the inverse of  $p$  in the group  $Mv$ . Then  $S(\psi)(u)p^{-1} \in S(\psi)(u)$  and  $S(\psi)(u)v = S(\psi)(u)p^{-1}p \subseteq S(\psi)(u)p$ .

**2.5. COROLLARY.** *Using the same notation as in Lemma 2.4 and  $v \in J$ ; if  $p \in S(\psi)(u)$ ,  $pv = p$ , and  $\phi: X \rightarrow Z$ , then  $S(\phi)(x)p = S(\phi)(x)v$  for all  $x$  in  $X$  with  $\phi(x) = z$  and  $xu = x$ .*

*Proof.* Straightforward.

The following lemma is a variation of Corollary 1.4 of [6].

**2.6. LEMMA.** *Suppose  $\phi: X \rightarrow Z, \theta: Y \rightarrow Z, Z$  minimal and  $\theta$  has a RIM (section),  $\lambda$ . Let  $r \in X$  and  $z = \phi(r)$ , let  $V$  be an open set in the support of  $\lambda_z$ , and let  $N = oc(\{r\} \times V)$  and  $v \in J$ , with  $zv = z$ . Then  $N \supseteq \{rv\} \times v$ . (Note  $X$  and  $Y$  are not required to be minimal, otherwise it would be trivial in view of 1.4 of [6] since  $rv$  and  $r$  are proximal and so  $(rv, r)$  would be in  $S(\phi)$ .)*

*Proof.* We will assume the reader is familiar with the notation and definitions in [6]. Let  $W \in \mathcal{U}(N_{rv})$  with  $\lambda_z(W) < \lambda_z(N_{rv}) + \varepsilon$ . Then there exists  $t$  in  $T$  for which  $N_r t \subseteq W$  and  $N_{rv} t \subseteq W$  and  $|\lambda_{z_t}(W) - \lambda_z(W)| < \varepsilon$ .

Then

$$\begin{aligned}
 \lambda_z(N_r \setminus N_{rv}) &= \lambda_{zt}(N_r t \setminus N_{rv} t) \leq \lambda_{zt}(W \setminus N_{rv} t) \\
 &= \lambda_{zt}(W) - \lambda_{zt}(N_{rv} t) \\
 &= \lambda_{zt}(W) - \lambda_z(N_{rv}) \\
 &\leq |\lambda_{zt}(W) - \lambda_z(W)| + |\lambda_z(W) - \lambda_z(N_{rv})| \\
 &< 2\varepsilon.
 \end{aligned}$$

Thus  $\lambda_z(N_r \setminus N_{rv}) = 0$ . Now  $\lambda_z(V \setminus N_{rv}) \leq \lambda(N_r \setminus N_{rv}) = 0$ , and so  $V \setminus N_{rv} = \emptyset$  since  $V \setminus N_{rv}$  is open in the support of  $\lambda_z$ . Thus  $V \subseteq N_{rv}$ .

**2.7 LEMMA.** *Suppose  $\phi: X \rightarrow Z$ ,  $\theta: Y \rightarrow Z$ ,  $Z$  minimal, and  $\theta$  has a RIM. Let  $x_0 \in X$ ,  $z_0 = \phi(x_0)$ , and let  $\Lambda$  be a non-empty subset of the set  $\{\lambda: \lambda \text{ is a RIM for } \theta\}$ . Let  $S_\lambda$  be the support of  $\lambda_{z_0}$  and  $S = \text{cls}(\bigcup_{\lambda \in \Lambda} S_\lambda)$ . Let  $C$  be an open set in  $\theta^{-1}(z_0)$  and  $A = S \cap C$ . Consider  $N = \text{oc}(\{x_0\} \times A)$ . Then  $N_x \supseteq A$  for all  $x \in S_m(\phi)(x_0)$ . Note if  $X$  is minimal,  $S_m(\phi) = S(\phi)$ . In addition if  $v \in J$  with  $z_0 v = z_0$ , then  $N \supseteq \{x_0 v\} \times A$ .*

*Proof.* By 1.4 of [6],  $A \cap S_\lambda = C \cap S_\lambda \subseteq N_x$  for every  $\lambda$  in  $\Lambda$ . So  $A \cap (\bigcup S_\lambda) = C \cap (\bigcup S_\lambda) \subseteq N_x$  and  $\text{cls}(C \cap (\bigcup S_\lambda)) \subseteq N_x$ . If  $y \in C \cap S$ , then for every open neighborhood  $V$  of  $y$  in  $\theta^{-1}(z_0)$  with  $V \subseteq C$ , there exists  $y_\lambda$  in  $S_\lambda$  for some  $\lambda$  in  $\Lambda$  with  $y_\lambda \in V \subseteq C$ ; thus  $A = C \cap S \subseteq \text{cls}(C \cap (\bigcup S_\lambda)) \subseteq N_x$ . The additional statement follows similarly from 2.6.

**2.8. PROPOSITION.** *Given homomorphisms  $\alpha: W \rightarrow X$ ,  $\phi: X \rightarrow Z$  where  $\phi$  has RIM  $\lambda$  and  $W$  is minimal, let  $\psi = \phi \circ \alpha$ . Then there exists a strongly proximal extension  $\theta: Z^\sim \rightarrow Z$  such that the following diagram commutes*

$$\begin{array}{ccc}
 W & \xleftarrow{\quad} & W^\sim \subseteq W \circ^Z Z^\sim \\
 \alpha \downarrow & & \downarrow \alpha^\sim \\
 X & \xleftarrow{\psi^\sim} & X^\sim \subseteq X \circ^Z Z^\sim \\
 \phi \downarrow & & \downarrow \phi^\sim \\
 Z & \xleftarrow{\theta} & Z^\sim
 \end{array}$$

(Curved arrows on the left and right indicate commutativity:  $\psi = \phi \circ \alpha$  on the left,  $\psi^\sim = \phi^\sim \circ \alpha^\sim$  on the right.)

where  $W^\sim$  and  $X^\sim$  are the unique minimal sets in  $W \circ^Z Z^\sim$  and  $X \circ^Z Z^\sim$  respectively. And  $\psi^\sim$  has a RIM  $\gamma$ , that induces the RIM  $\lambda_{\theta(v)} \times \delta_v$ ,  $v \in Z^\sim$  on  $\phi^\sim$ . ( $\delta_v$  is the point mass at  $v \in Z^\sim$ ).

REMARK. (a) Compare this with 5 of [7].

(b) When  $W$  is the universal minimal set  $M$ , we have  $\tilde{M}$  which is in fact isomorphic with  $M$  through the map  $(p, \gamma) \rightarrow p$ .

(c) If  $Z$  is a universal strongly proximal flow, then  $\theta: \tilde{Z} \rightarrow Z$  is an isomorphism and any RIM on  $\phi$  can be lifted to a RIM on  $\psi$ .

*Proof.* We assume the reader is familiar with the contents of [4]. Let  $x_0 \in X_0$ ,  $w_0 \in W$  with  $\alpha(w_0) = x_0$ , and  $u \in J$  with  $\overline{w_0}u = w_0$ . Consider  $\hat{\psi}: \mathfrak{M}(W) \rightarrow \mathfrak{M}(Z)$ ,  $\hat{\alpha}: \mathfrak{M}(W) \rightarrow \mathfrak{M}(X)$ . Let  $P = \text{co}(oc(\lambda_{z_0}))$  and note  $\hat{\phi}: P \rightarrow \mathfrak{M}(Z)$  is  $\mathfrak{M}(Z)$  irreducible since  $\lambda$  is a RIM. Let  $Q$  be a  $P$ -irreducible subset of  $\hat{\alpha}^{-1}(P)$  and note that  $Q$  is also  $\mathfrak{M}(Z)$  irreducible. Let  $\tilde{Z} = \overline{\text{ex}(Q)}$  and  $\theta = \hat{\psi}|_{\tilde{Z}}$  — the restriction of  $\hat{\psi}$  to  $\tilde{Z}$ ; we identify  $z$  in  $Z$  with  $\delta_z$  and consider  $\theta: \tilde{Z} \rightarrow Z$ . Let  $\tilde{X}$ ,  $\tilde{W}$  be the unique minimal sets in  $\{(x, \nu) \in X \times \tilde{Z}: \phi(x) = \theta(\nu)\}$ ,  $\{(p, \nu) \in W \times \tilde{Z}: \phi \circ \alpha(p) = \theta(\nu)\}$  respectively. Let  $\tilde{\phi}$  be the projection of  $\tilde{X}$  onto  $\tilde{Z}$  and  $\tilde{\psi}$  be the projection of  $\tilde{W}$  onto  $\tilde{Z}$ . For each  $\nu$  in  $\tilde{Z}$  the measure  $\nu \times \delta_\nu$  on  $W \times \tilde{Z}$  is supported in  $\tilde{W}$  and the map  $\gamma: \tilde{Z} \rightarrow \mathfrak{M}(\tilde{W})$ ,  $\gamma_\nu = \nu \times \delta_\nu$  is a RIM for  $\tilde{\psi}: \tilde{W} \rightarrow \tilde{Z}$ . Also  $(\hat{\alpha} \times \text{id}): \tilde{W} \rightarrow \tilde{X}$  induces a RIM  $\beta = (\hat{\alpha} \times \text{id})(\gamma)$  on  $\tilde{\phi}: \tilde{X} \rightarrow \tilde{Z}$  by  $\beta_\nu = (\hat{\alpha} \times \text{id})(\gamma_\nu) = \hat{\alpha}(\nu) \times \delta_\nu$ ; note  $\hat{\alpha}(\nu_0) = \lambda_{z_0}$  for some  $\nu_0$  in  $\tilde{Z}$  and so  $\theta(\nu_0) = z_0$  and  $\beta_\nu = \lambda_{\theta(\nu)} \times \delta_\nu$  for  $\nu$  in  $\tilde{Z}$ .

2.9. THEOREM. Suppose for  $i = 1, 2, 3, \dots, n$ ,  $\phi_i: X^i \rightarrow Z$  are homomorphisms and  $X^i$  is a minimal flow. Suppose  $\phi_1$  has a RIM,  $\lambda$ . Let  $z_0 \in Z$ ,  $X_0^i = \phi_i^{-1}(z_0)$ . Then, there exist non-empty subsets  $J^* \subseteq J'$  of  $J$  such that  $X_0^i J^*$ ,  $X_0^i J'$  are compact subsets of  $X_0^i$  and such that given  $A^i = (X_0^i J^*) \cap V^i$  where  $V^i$  is an open subset of  $X_0^i$ ,  $x, x' \in X_0^1 J'$  with  $(x, x') \in S(\phi_1)$ , and  $N = oc(\{x\} \times A^2 \times \dots \times A^n)$  we have  $N \supseteq \{x'\} \times A^2 \times \dots \times A^n$ .

REMARK (a) If  $u, v \in J'$ ,  $x_1 u \in X_0^1$ ,  $x_2 v \in X_0^2$ , and

$$N = oc(\{(x_1 u, x_2 v)\}) \times A^3 \times \dots \times A^n;$$

then  $N \supseteq \{(x_1 u, x_2 u)\} \times A^3 \times \dots \times A^n$ .

(b)  $X_0^1 J^* \supseteq \bigcup \{X_0^1 u: u \in J \text{ for which } x' u \in S_\mu \text{ for some } x' \text{ in } X_0^i \text{ and some RIM, } \mu, \text{ for } \phi_i\}$  where  $S_\mu$  is the support of  $\mu_{z_0}$ .

(c)  $J^*$  and  $J$  depend on  $Z$  but not on the  $\phi_i$ 's.

(d) For  $n = 2$  compare this with 1.4 of [6], where  $\phi_2$  has a RIM and  $\phi_1$  is not required to have a RIM.

2.10. COROLLARY. If  $X = X^i$ ,  $i = 1, 2$ ,  $x \in X_0 J^*$ ,  $x' \in X_0 J'$ , and  $(x, x') \in S(\phi)$ , then there exist  $x_n$  in  $X_0 J^*$  and  $t_n$  in  $T$  with  $x_n \rightarrow x$ ,  $x_n t_n \rightarrow x$ ,  $x' t_n \rightarrow x$ ; in particular  $(x, x') \in Q(\phi)$ .

*Proof of 2.9.* Let  $u \in J$  with  $z_0 u = z_0$ . Define  $\psi: M \rightarrow Z$  by  $p \rightarrow z_0 p$ . Let  $M_0 = \psi^{-1}(z_0)$ . Fix  $x_0^i \in \phi_i^{-1}(z_0) = X_0^i$  with  $x_0^i u = x_0^i$  and define  $\alpha_i: M \rightarrow X^i$  by  $\alpha_i(p) = x_0^i p$ . Note  $\psi = \alpha_i \circ \phi_i$ . By 4.1 of [4], there is a

strongly proximal extension  $Z^\sim$  of  $Z$ ,  $Z^\sim$  is minimal,  $\theta: Z^\sim \rightarrow Z$  is strongly proximal, such that the projection  $\psi^\sim$  of  $M^\sim$  onto  $Z^\sim$  has a RIM where  $M^\sim$  is the unique minimal set in  $M \circ^Z Z^\sim = \{(m, z) \in M \times Z: \psi(m) = \theta(z)\}$ . By 2.7 we see that we are interested that the union of the supports of the RIM's on  $\psi^\sim$  be as large as possible. We will now determine one aspect of the size of this union by "translating" measures. Given a RIM  $\gamma$  on  $\psi^\sim$  define the translation  $p\gamma$  by  $p\gamma_\nu(A) = \gamma_\nu(pA)$  for  $p \in M_0^\sim = \psi^{\sim -1}(\psi(u))$  and  $\nu \in Z^\sim$ . It is easy to see that  $p\gamma$  is again a RIM on  $\psi^\sim$ . Let  $\gamma_0 = \psi^\sim(u)$ ,  $\Gamma$  be the set of all RIM's on  $\psi^\sim$ , and  $S_\gamma, \gamma \in \Gamma$ , be the support of  $\gamma_{\nu_0}$ . From the above it is easy to see that  $\text{cls}\{\bigcup S_\gamma: \gamma \in \Gamma\}$  is of the form  $M_0^\sim J^* \subseteq M_0^\sim \subseteq M_0 \times \{\nu_0\}$  for some subset  $J^*$  of  $J$ .

Now to prove the theorem we first show a similar result for  $\phi_1^\sim$  and then reduce it to  $\phi_1$ . Suppose  $A^i = (X_0^i J^*) \cap V^i$  where  $V^i$  is open in  $X^i$  and  $((x, \nu_0), (x', \nu_0)) \in S(\phi_1^\sim)$ . Let  $N^\sim = oc(\{x_0, \nu_0\}) \times (A^2 \times \{\nu_0\}) \times \cdots \times (A^n \times \{\nu_0\})$ . Then if  $(p, \nu_0) \in M_0^\sim$  with  $\alpha_1(p) = x$ , there exist  $(p', \nu_0) \in M_0^\sim$  with  $\alpha_1(p') = x'$  and  $((p, \nu_0), (p', \nu_0)) \in S(\psi^\sim)$ . Consider

$$N^* = oc\{((p, \nu_0)) \times (\alpha_2^{-1}(V^2) \times \{\nu_0\}) \cap M_0^\sim J^*\} \\ \times \cdots \times (\alpha_n^{-1}(V^n) \times \{\nu_0\}) \cap M_0^\sim J^*\}.$$

For  $i = 2, 3, \dots, n$ , let  $\gamma_i \in \Gamma$ , then  $\prod \gamma_i$  is a RIM and  $S_{\prod \gamma_i} = \prod S_{\gamma_i}$  so  $\text{cls}(\bigcup \{S_{\prod \gamma_i}: \gamma_i \in \Gamma, i = 2, \dots, n\}) = \prod_2^n M_0^\sim J^*$ . So by 2.7

$$N^* \supseteq \{((p', \nu_0)) \times [\alpha_2^{-1}(V^2) \times \{\nu_0\}) \cap M_0^\sim J^*\} \\ \times \cdots \times [a_n^{-1}(V^n) \times \{\nu_0\}) \cap M_0^\sim J^*],$$

and  $N^\sim \supseteq \{(x', \nu_0)\} \times (A^2 \times \{\nu_0\}) \times \cdots \times (A^n \times \{\nu_0\})$ , since if  $\alpha_i(p_i) = a_i \in A^i$  then  $a_i u_i^* = a_i$  for some  $u_i^* \in J^*$  and  $(p_i u_i^*, \nu_0) \in (\alpha_i^{-1}(V^i) \times \{\nu_0\}) \cap M_0^\sim J^*$ . Thus

$$N = oc(\{x\} \times A^2 \times \cdots \times A^n) \supseteq \{x'\} \times A^2 \times \cdots \times A^n.$$

We will now show that if  $(x, x') \in S(\phi_1)$  and  $(x, \nu_0), (x', \nu_0) \in X^{1^\sim}$ , then  $((x, \nu_0), (x', \nu_0)) \in S(\phi_1^\sim)$ , where  $X^{1^\sim}$  is the unique minimal set in  $X^1 \circ^Z Z^\sim$ . (We let  $J' = \{v \in J: \nu_0 v = \nu_0\}$  and note for  $x \in X_0^1$ ,  $(x, \nu_0) \in X^{1^\sim}$  iff  $x \in X_0^1 J'$ .) First suppose  $x \in S_\lambda$ , then there exist  $x_n$  in  $S_\lambda$  and  $t_n$  in  $T$  with  $x_n \rightarrow x$ ,  $x_n t_n \rightarrow x$ ,  $x' t_n \rightarrow x$  by 1.5 of [6]. Now  $S_\lambda \times \{\nu_0\} \subseteq X^{1^\sim}$  since  $\lambda_{\theta(\nu)} \times \delta_\nu, \nu \in Z^\sim$  is a RIM on  $\phi_1^\sim$  by 5 of [7]. So  $(x_n, \nu_0) \in X^{1^\sim}$  and we have  $((x, \nu_0), (x', \nu_0)) \in S(\phi_1^\sim)$ . Now suppose  $x \notin S_\lambda$ , let  $x_1 \in S_\lambda$  and  $w \in J^*$  such that  $(x_1, \nu_0)w = (x_1, \nu_0)$ . Let  $pw \in M$  with  $xpw = x_1$ ; then  $(x' pw, xpw) \in S(\phi_1)$  and  $((x' pw, \nu_0), (xpw, \nu_0)) \in S(\phi_1^\sim)$ . Multiplying on the right by  $(pw)^{-1} \in Mw$ , we get  $((x' w, \nu_0), (xw, \nu_0)) \in S(\phi_1^\sim)$  and therefore  $((x', \nu_0), (x, \nu_0)) \in S(\phi_1^\sim)$ . Remark (a) is easily proved as above applying Lemma 2.7 to  $\phi^*$ :  $oc(x_1 u, x_2 v) \rightarrow Z$ . Remark (b) follows from 5 to [7] and 2.8.

2.11. THEOREM. Suppose for  $i = 1, 2, \dots, n$  that  $\phi_i: X_i \rightarrow Z$  has a RIM,  $\mu_i$ , and  $X_i$  is minimal flow. Suppose  $z \in Z$  such that the support  $S_{\mu_i}$  of  $\mu_i$  equals the fiber  $\phi_i^{-1}(z)$ , for  $i = 1, \dots, n$ . Suppose  $X_0$  is a minimal flow and  $\phi_0: X_0 \rightarrow Z$  is a homomorphism. Given  $x$  in  $\phi_0^{-1}(z) \subseteq X_0$  and open sets  $V_i$  in  $\phi_i^{-1}(z) \subseteq X_i$ , the set  $N = oc(\{x\} \times V_1 \times \dots \times V_n) \supseteq S(\phi_0)(x)v \times Q(\phi_1)(V_1)v \times \dots \times Q(\phi_n)(V_n)v$  for every  $v$  in  $J$ , and thus  $N \supseteq [S(\phi_0)(x)v] \circ v \times [Q(\phi_1)(V_1)v] \circ v \times \dots \times [Q(\phi_n)(V_n)v] \circ v$ .

*Proof.* Let  $v \in J$  with  $zv = z$ . We will show  $N \supseteq S(\phi_0)(x)v \times Q(\phi_1)(V_1)v \times \dots \times Q(\phi_n)(V_n)v$  by induction. But first some preliminaries. Let  $(x_0, x_1, \dots, x_n)$  be an element in the right hand side. Then we have  $x_1v = x_1$  and for some  $r_1$  in  $V_1$ ,  $x_1 \in Q(\phi_1)(r_1)$ . Suppose  $v_1 \in J$  with  $r_1v_1 = r_1$ ; define  $\alpha: M \rightarrow X_1$  by  $\alpha(p) = r_1p$ , then  $\phi_1 \circ \alpha = \psi: M \rightarrow Z$  where  $\Psi(p) = zp$ . So by 2.3 we see that there exists  $p_1$  in  $S(\psi)(v) = S(\psi)(v_1)$  such that  $r_1p = \alpha(p_1) = x_1$  and  $p_1v = p_1$  since  $x_1v = x_1$ . By 2.5  $S(\phi_0)(x)p_1 = S(\phi_0)(x)v$  and since  $x_0 \in S(\phi_0)(x)v$  there exists  $r_0$  in  $S(\phi_0)(x)$  with  $r_0p_1 = x_0$ , and we may assume  $r_0v_1 = r_0$ . Now (a)  $N \supseteq S(\phi_0)(x) \times V_1 \times \dots \times V_n$  by 1.4 of [6] and so  $N \supseteq \{(r_0, r_1)\} \times V_2 \times \dots \times V_n$ . Now consider the flow  $oc(r_0v_1, r_1v_1)$ . It is minimal and has an induced map  $\hat{\phi}: oc(r_0v_1, r_1v_1) \rightarrow Z$ . Thus by 1.4 of [6]  $N \supseteq S(\hat{\phi})(r_0v_1, r_1v_1) \times V_2 \times \dots \times V_n$  which equals  $S(\hat{\phi})(x_0, x_1) \times V_2 \times \dots \times V_n$  since  $(r_0v_1, r_1v_1)p_1 = (x_0, x_1)$  and  $p_1 \in S(\psi)(v)$ .

Now we note that when  $n = 1$  we have for any  $x_0 \in S(\phi_0)(x)v$  and  $x_1 \in Q(\phi_1)(V_1)v$ , (b)  $oc(\{x\} \times V_1) \supseteq S(\hat{\phi})(x_0, x_1) \ni (x_0, x_1)$  and so  $oc(\{x\} \times V_1) \supseteq S(\phi_0)(x)v \times Q(\phi_1)(V_1)v$ .

Proceeding by induction, assume that the theorem is true for  $n = k - 1$  and prove it for  $n = k$ . With  $n = k$ , we have for any  $x_0 \in S(\phi_0)(x)v$  and  $x_1 \in Q(\phi_1)(V_1)v$ , (c)  $oc(\{x\} \times V_1 \times \dots \times V_k) \supseteq oc(S(\phi)(x_0, x_1) \times V_2 \times \dots \times V_k) \supseteq oc(\{x_0, x_1\} \times V_2 \times \dots \times V_k) \supseteq S(\hat{\phi})(x_0, x_1)v \times Q(\phi_2)(V_2)v \times \dots \times Q(\phi_k)(V_k)v \supseteq \{x_0\} \times \{x_1\} \times Q(\phi_2)(V_2)v \times \dots \times Q(\phi_k)(V_k)v$  by induction. And so  $oc(\{x\} \times V_1 \times \dots \times V_k) \supseteq S(\phi_0)(x)v \times Q(\phi_1)(V_1)v \times Q(\phi_2)(V_2)v \times \dots \times Q(\phi_k)(V_k)v$ ; thus the theorem is proved for every  $v \in J$  with  $zv = z$  and thus for every  $v \in J$ .

2.12. THEOREM. Suppose for  $i = 0, 1, \dots, n$  that  $\phi_i: X^i \rightarrow Z$  has a RIM,  $\mu_i$ , and  $X^i$  is minimal. Suppose  $z \in Z$  and  $X_0^i = \phi_i^{-1}(z)$ . Let  $J^*$  and  $J'$  be as in 2.9. Let  $V^i$  be open in  $X_0^i$ ,  $A^i = V^i \cap X_0^i J^*$ , and  $x \in X_0^i$ . Then

$$\begin{aligned} N &= oc(\{x\} \times A^1 \times \dots \times A^n) \\ &\supseteq Q(\phi_0)(x)v \times Q(\phi_1)(A^1)v \times \dots \times Q(\phi_n)(A^n)v \end{aligned}$$

for every  $v$  in  $J$ .

*Proof.* We indicate where the proof differs from the above. Of course  $V_i$  is replaced by  $A^i$  and  $J$  by  $J^*$ . Statement (a) would read “Now  $N \supseteq S(\phi_0)(x) \cap X_0^0 J^* \times A^1 \times \cdots \times A^n$  by Proposition 2.9.” Note  $S(\phi_0)(x) \cap X_0^0 J^* = S(\phi_0)(x) J^*$ . Statement (b) would read “ $oc(\{x\} \times A^1) \supseteq S(\phi)(x_0, x_1) J^* \ni (x_0, x_1)$ .” Statement (c) would read

$$“oc(\{x\} \times A^1 \times \cdots \times A^n) \supseteq oc(S(\tilde{\phi})(x_0, x_1) J^* \times A^2 \times \cdots \times A^n).”$$

**2.13. COROLLARY.** *Suppose  $\Gamma$  is an index set and for  $i \in \Gamma$ ,  $\phi_i: X_i \rightarrow Z$  has a RIM and  $X_i$  is minimal. Suppose  $z \in Z$  and  $x_i, y_i \in X_i$  with  $x_i, y_i \in X_0^i J^* = \phi_i^{-1}(z) J^*$ ,  $x = (x_i) \in \prod_{i \in \Gamma} X_i$ ,  $y = (y_i) \in \prod_{i \in \Gamma} X_i$  where  $J^*$  is taken as in 2.9. Then  $(x, y) \in Q(\prod \phi_i)$  iff  $(x_i, y_i) \in Q(\phi_i)$  for every  $i$  in  $\Gamma$ .*

*Proof.* ( $\Rightarrow$ ) Clear.

( $\Leftarrow$ ) Suppose  $u \in J^*$ . Let  $\prod_{i \in F} U_i \times \prod_{i \notin F} X_i$  and  $\prod_{i \in F} V_i \times \prod_{i \notin F} X_i$  be neighborhoods of  $x$  and  $y$  respectively, where  $F$  is a finite subset of  $\Gamma$ . Let  $A_i = U_i \cap X_0^i J^*$  and  $B_i = V_i \cap X_0^i J^*$ . Then  $N = oc(\prod A_i \times \prod B_i) \supseteq \prod Q(A_i)u \times \prod Q(B_i)u \supseteq \prod Q(x_i)u \times \prod Q(y_i)u \ni \prod \{x_i u\} \times \prod \{y_i u\}$ ; and the corollary clearly follows.

**REMARK.** The above was known under various more specialized conditions, see [1, 3].

**2.14. COROLLARY.** *Suppose  $\phi: X \rightarrow Z$  has a RIM,  $\lambda$ , let  $z \in Z$ ,  $x_0 \in X_0 J^* = \phi^{-1}(z) J^*$ ,  $\Gamma$  an index set, and  $x_i \in S(\phi)(x_0) J^*$ ,  $i \in \Gamma$ . Then there exist nets  $t_n$  in  $T$  and  $x_i^n$  in  $X_0 J^*$  with  $x_i^n \xrightarrow{n} x_i$ ,  $x_i^n t_n \xrightarrow{n} x_0$  for  $i \in \Gamma$ , and  $x_0 t_n \rightarrow x_0$ .*

*Proof.* Let  $u \in J^*$  with  $x_0 u = x_0$ . For any neighborhood  $V_i$  of  $x_i$  in  $X_0$  let  $A_i = V_i \cap X_0 J^*$ . Then

$$\begin{aligned} oc(\{x_0\} \times \prod A_i) &\supseteq Q(\phi(x_0)u) \times (\prod Q(\phi)(A_i)u) \\ &\supseteq Q(\phi)(x_0) \times \prod Q(\phi)(x_i)u \ni \{x_0\} \times \prod \{x_0\}; \end{aligned}$$

and the corollary clearly follows.

**2.15. LEMMA.** *Suppose  $Z$  is a minimal flow and  $z_0 \in Z$ . Define  $\psi: M \rightarrow Z$  by  $p \rightarrow z_0 p$  and let  $M_0 = \psi^{-1}(z_0)$ . Suppose  $\psi$  is RIC and has a RIM,  $\lambda$ . Then there exist  $w \in M_0 \cap J$  such that for  $p$  in  $\text{cls}(M_0 w)$  and  $q$  in  $Q(\psi)(p)$  there exist nets  $p_n$  in  $M_0 w$  and  $t_n$  in  $T$  such that  $p_n \rightarrow p$ ,  $p_n t_n \rightarrow p$ ,  $q t_n \rightarrow p$ . In particular for  $p$  in  $\text{cls}(M_0 w)$ ,  $Q(\psi)(p) = \{q: \text{there exist nets } p_n \text{ in } M_0 w \text{ and } t_n \text{ in } T \text{ with } p_n \rightarrow p, p_n t_n \rightarrow p, q t_n \rightarrow p\} = \cap \{\text{cls}(\beta T(p) \cap \text{cls}(M_0 w)): \beta = V \times V, V \text{ an open set in } M\}$ . (Recall that  $S(\psi) = Q(\psi)$  if  $\psi$  in RIC, see [9].)*

*Proof.* Let  $S_\lambda$  be the support of  $\lambda_{z_0}$  and  $p \in S_\lambda \subseteq M_0$ . Suppose  $u \in J$  with  $pu = p$  and  $q \in M_0u$ , then  $\mu$  defined by  $\mu_z(A) = \lambda_z(qp^{-1}A)$  is a RIM and  $q \in S_\mu$ . So if  $S = \text{cls}(\bigcup \{S_\mu: \mu \text{ is a RIM for } \psi\})$ , then  $S = M_0J_1$  for some subset  $J_1$  of  $J \cap M_0$ . Now consider the left flow  $(M_0u, S)$  with the action being multiplication on the left and  $M_0u$  is a group given the discrete topology. Then it contains a minimal set  $(M_0u, \text{cls}(M_0w))$  for some  $w$  in  $J_1$ .

Suppose  $V$  is an open subset of  $M_0$  and  $V \cap M_0w \neq \emptyset$ . Then there exists a finite set  $F$  of  $f$ 's in  $M_0w$  such that  $\bigcup_{f \in F} F(V \cap \overline{M_0w}) \supseteq \overline{M_0w}$ . Let  $B = B_V = \text{cls}(V \cap \overline{M_0w}) = \text{cls}(V \cap M_0w)$ . Then  $\bigcup_{f \in F} F(B \circ w) = [\bigcup_{f \in F} fB] \circ w \supseteq \overline{M_0w} \circ w = M_0$  since  $\psi$  is RIC. So  $\bigcup_{f \in F} (S \cap f(B \circ w)) \supseteq S$ . So  $\text{int}(S \cap f(B \circ w)) \neq \emptyset$  for some  $f$  in  $F$  where the interior is with respect to  $S$ , and thus  $\text{int}(S \cap (B \circ w)) \neq \emptyset$ . Let  $p \in \overline{M_0w}$  and  $p^* \in \bigcap_{V \in \mathcal{U}_p} \text{cls int}(S \cap (B_V \circ w))$ .

Suppose  $q^* \in Q(\psi)(p^*)$  and consider

$$N_V = oc(\{q^*w\} \times \text{int}[S \cap (B_V \circ w)])$$

then by Lemma 2.7

$$N_V \supseteq \{p^*\} \times \text{cls}(\text{int}[S \cap (B_V \circ w)]) \ni (p^*, p^*).$$

Let  $U \in \mathcal{U}_{p^*}$ . Then there exist  $t = t_{VU}$  in  $T$  and  $r = r_{VU}$  in  $S \cap (B_V \circ w)$  such that  $q^*wt \in U$  and  $rt \in U$ . Then there exist  $s = s_{VU}$  and  $m = m_{VU}$  in  $V \cap M_0w$  such that  $q^*s$  is near  $q^*w$  and  $ms$  is near  $r$ ; that is,  $q^*s \in Ut^{-1}$  and  $ms \in Ut^{-1}$ . Thus we have nets  $m_{VU}$  in  $M_0w$  and  $s_{VU}t_{VU}$  in  $T$  with  $m_{VU} \rightarrow p$ ,  $m_{VU}s_{VU}t_{VU} \rightarrow p^*$  and  $q^*s_{VU}t_{VU} \rightarrow p^*$  thus  $(q^*, p) \in Q(\psi)$ .

So we have assumed  $(p^*, q^*) \in Q(\psi)$  and shown  $(q^*, p) \in Q(\psi)$ . Now suppose  $(p, q) \in Q(\psi)$ ; we can repeat the preceeding paragraph with  $q$  in place of  $q^*$  to obtain the lemma.

**2.16. PROPOSITION.** Suppose  $\phi: X \rightarrow Z$  is a homomorphism of minimal flows such that the set  $D(\phi)$  of almost periodic points in  $R(\phi)$  is dense. Let  $x_0 \in X$ ,  $\phi(x_0) = z_0$ , and  $X_0 = \phi^{-1}(z_0)$ . Then there exists  $w \in J$  with  $z_0w = z_0$  such that for  $x, y$  in  $\text{cl}(X_0w)$  with  $y$  in  $Q(\phi)(x)$  and for  $p \in \text{cls}(Mw)$  with  $x_0p = x$ , there exist  $q$  in  $M$  and nets  $p_n$  in  $Mw$  and  $t_n$  in  $T$  such that  $x_0q = y$  and  $p_n \rightarrow p$ ,  $qt_n \rightarrow p$ ,  $p_nt_n \rightarrow p$ .

*Proof.* Let  $X_0 \in X_0 = \phi^{-1}(z_0)$ . Define  $\beta: M \rightarrow X$  by  $\beta(p) = x_0p$ . Let  $\psi = \phi \circ \beta: M \rightarrow Z$ ,  $M_0 = \psi^{-1}(z_0)$ . Take a proximal extension  $Z^*$  of  $Z$ ,  $\theta: Z^* \rightarrow Z$  such that  $\psi^*: M^* \subseteq M \circ^Z Z^* \rightarrow Z^*$  is RIC and has a RIM. Let  $z_0^* \in \theta^{-1}(z_0)$ ,  $M_0^* = \psi^{*-1}(z_0^*)$ , and let  $w \in J \cap M_0^*$  as in Lemma 2.15. If  $x \in \text{cls}(X_0w)$  and  $y \in Q(\phi)(x)$ , then by 2.1.4 of [6],  $((x, z), (y, z)) \in Q(\phi^*)$  for some  $z$  in  $Z^*$ , and thus  $((xw, z_0), (yw, z_0)) = ((xw, zw), (yw, zw)) \in Q(\phi^*)$ . Since  $x, y \in \text{cls}(X_0w)$ ,  $(x, z_0), (y, z_0) \in X^*$  and so

$((x, z_0), (xw, z_0)) \in P$ ,  $((y, z_0), (yw, z_0)) \in P$  and  $((x, z_0), (y, z_0)) \in Q(\phi^*)$ . Let  $p \in \text{cls}(M_0w)$  with  $(x_0, z_0)p = (x, z_0)$ . By 14.2 of [2<sub>b</sub>] we can take  $q$  in  $Q(\psi)(p)$  with  $(x_0, z_0)q = (y, z_0)$ . The proposition clearly follows from Lemma 2.15.

A stronger result can be obtained if we assume  $Z$  is a singleton. Fix  $x_0$  in  $X$  and define  $\psi: M \rightarrow X$  by  $p \rightarrow x_0p$ . Let  $u \in J$ . Then  $Mu$  is a group. Give it the discrete topology and consider the (left) flow  $(Mu, M)$  with the action being multiplication on the left. Then it contains a minimal set  $(Mu, \overline{Mw})$  for some  $w$  in  $J \subseteq M$ . Note  $(Mw, \overline{Mw})$  is also minimal. See 2.10 of [8] for related results.

**2.17. THEOREM.** *Suppose  $X$  is a minimal flow and has an invariant measure. Let  $w \in J$  such that  $(Mw, \overline{Mw})$  is a minimal (left) flow as above. Let  $x \in X$ . Suppose  $x_0\overline{Mw} \circ w = X$ , (that is,  $X$  is incompressible). Then for each  $x$  in  $\overline{Xw} = x_0\overline{Mw}$ ,  $p$  in  $\psi^{-1}(x) \cap Mw$  and  $x'$  in  $Q(x)$ , there exist nets  $m_n$  in  $Mw$  and  $t_n$  in  $T$  with  $m_n \rightarrow p$ ,  $x_0m_n \rightarrow x_0p = x$ ,  $x't_n \rightarrow x^*$ ,  $x_0m_nt_n \rightarrow x^*$  for any  $x^*$  in  $X$ . In particular, for  $x$  in  $\overline{Xw}$ ,*

$$\begin{aligned} Q(x) &= \{x': \text{there exist nets } x_n \text{ in } Xw \text{ and } t_n \text{ in } T \\ &\quad \text{with } x_n \rightarrow x, x_nt_n \rightarrow x, x't_n \rightarrow x^*\} \\ &= \cap \{\text{cls}(\alpha T(x) \cap Xw): \alpha = V \times V, V \text{ an open set in } X\} \\ &= \cap \{\text{cls}(\alpha T(x) \cap \overline{Xw}): \alpha = V \times V, V \text{ an open set in } X\}. \end{aligned}$$

*Proof.* Suppose  $x \in x_0\overline{Mw}$ ,  $p \in \psi^{-1}(x) \cap \overline{Mw}$  and  $V \in \mathcal{N}_p$ . Then  $V \cap \overline{Mw} \neq \emptyset$  and is open in  $Mw$ . Then since  $(Mw, \overline{Mw})$  is minimal, there exists a finite set  $F$  of  $f$ 's in  $Mw$  such that  $\bigcup_{f \in F} f(V \cap \overline{Mw}) \supseteq \overline{Mw}$ . Let  $B = B_v = \overline{V \cap \overline{Mw}}$ . Then  $\bigcup_{f \in F} x_0fB \circ w = x_0 \bigcup_{f \in F} fB \circ w = x_0[\bigcup_{f \in F} fB] \circ w \supseteq x_0\overline{Mw} \circ w = X$ . So  $\text{int}(x_0fB \circ w) \neq \emptyset$  for some  $f$  in  $F$ . Then  $\text{int}(B \circ w) \neq \emptyset$ . Therefore  $\text{int}(x_0B \circ w) \neq \emptyset$ . Let  $x^* \in \bigcap_{V \in \mathcal{N}_p} \text{cls int}(x_0B_V \circ w)$ .

Suppose  $x^\# \in Q(x^*)$  and consider  $N_V = \text{oc}(\{x^\#w\} \times \text{int}(x_0B_v \circ w))$ . Then by 1.4 of [6],

$$N_V \supseteq \{x^*\} \times \text{cls int}[x_0B_v \circ w] \ni (x^*, x^*).$$

Let  $U \in \mathcal{N}_{x^*}$ . Then there exists  $t = t_{V,U}$  in  $T$  and  $y = y_{V,U}$  in  $X_0B_V \circ w$  such that  $x^\#wt \in U$  and  $yt \in U$ . Then there exists  $s = s_{V,U}$  in  $T$  and  $m = m_{V,U}$  in  $V \cap Mw$  such that  $x^\#s \in Ut^{-1}$  and  $x_0ms \in Ut^{-1}$ . Thus we have nets  $m_{V,U}$  in  $Mw$  and  $s_{V,U}t_{V,U}$  in  $T$  with  $m_{V,U} \rightarrow p$ ,  $x_0m_{V,U} \rightarrow x_0p$ ,  $x_0m_{V,U}s_{V,U}t_{V,U} \rightarrow x^*$ ,  $x^\#s_{V,U}t_{V,U} \rightarrow x^*$ . Thus  $(x^\#, x) = (x^\#, x_0p) \in Q$ .



So we have assumed  $(x^*, x^\#) \in Q$  and shown  $(x^\#, x) \in Q$ . Now suppose  $(x, x') \in Q$ ; then  $(x^*, x') \in Q$  and we can repeat the preceding paragraph with  $x'$  in place of  $x^\#$  to obtain the theorem (note the  $x^*$  can be replaced as the limit by any point in  $X$  since  $X$  is a minimal flow).

#### REFERENCES

1. Jesse P. Clay, *Proximity relations in transformation groups*, Trans. Amer. Math. Soc., **108** (1963), 88–96.
2. Robert Ellis, *Lectures on Topological Dynamics*, Benjamin, New York, 1969.
3. R. Ellis and H. B. Keynes, *A characterization of the equicontinuous structure relation*, Trans. Amer. Math. Soc., **161** (1971), 171–183.
4. Samuel Glasner, *Relatively invariant measure*, Pacific J. Math., **58** (1975), 393–417.
5. ———, *Proximal flows*, Lecture Notes in Math., Vol. **517**, Springer-Verlag, Berlin and New York, 1976.
6. Doug McMahon, *Relativized weak disjointness and relatively invariant measures*, Trans. Amer. Math. Soc., **236** (1978), 225–237.
7. ———, *Relativized weak disjointness of uncountable order*, Canad. J. Math., **32** (1980), 559–566.
8. D. McMahon and T. S. Wu, *On proximal and distal extensions of minimal sets*, Bull. Inst. Math. Acad. Sinica, **2** (1974), no. 1, 93–107.
9. W. A. Veech, *Topological dynamics*, Bull. Amer. Math. Soc., **83** (1977), 775–830.
10. ———, *Point-distal flows*, Amer. J. Math., **92** (1970), 205–242.

Received April 9, 1981.

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The Pacific Journal of Mathematics ISSN 0030-8730 is published monthly by the Pacific Journal of Mathematics at P.O. Box 969, Carmel Valley, CA 93924. Application to mail at Second-class postage rates is pending at Carmel Valley, California, and additional mailing offices. Postmaster: Send address changes to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924.

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