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**COMPACT OPERATORS AND DERIVATIONS INDUCED BY
WEIGHTED SHIFTS**

C. RAY ROSENTRATER

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In this paper we study the question: which compact operators are contained in $\mathfrak{R}(\delta_S)^-$, the norm closure of the range of the derivation $\delta_S(X) = SX - XS$ induced by a weighted shift S ? We find that $\mathfrak{R}(\delta_S)^-$ always contains the lower triangular (with respect to the basis (e_i) on which S is a shift) compact operators. Further, $\mathfrak{R}(\delta_S)^-$ contains the n -lower triangular (operators T satisfying $(Te_i, e_j) = 0$ for $i - j > n$) compact operators if and only if $e_1 \otimes e_{n+1} \in \mathfrak{R}(\delta_S)^-$. We also find necessary and sufficient conditions on the weights of S in order that $e_1 \otimes e_{n+1} \in \mathfrak{R}(\delta_S)^-$ and that \mathfrak{K} , the algebra of compact operators, be contained in $\mathfrak{R}(\delta_S)^-$. These results completely answer the question: which essentially normal weighted shifts are d -symmetric?

Let $T \in \mathfrak{B}(\mathfrak{H})$, the algebra of bounded linear operators on a complex Hilbert space \mathfrak{H} . The derivation induced by T is the map $\delta_T(X) = TX - XT$ from $\mathfrak{B}(\mathfrak{H})$ to itself. Let $(e_n)_{n=1}^\infty$ (respectively $(e_n)_{n=-\infty}^\infty$) be an orthonormal basis for \mathfrak{H} and let S be the unilateral (respectively bilateral) weighted shift $Se_n = w_n e_{n+1}$, $n \in \mathbf{N}$ (respectively $n \in \mathbf{Z}$) with nonzero weights w_n . By taking a unitarily equivalent weighted shift, we may assume that $w_n = |w_n| > 0$.

Recall that for $f, g \in \mathfrak{K}$, the operator $f \otimes g \in \mathfrak{B}(\mathfrak{H})$ is defined by $(f \otimes g)h = (h, g)f$ for $h \in \mathfrak{H}$. In particular, $(e_i \otimes e_j)e_n = e_i$ if $n = j$ and $(e_i \otimes e_j)e_n = 0$ otherwise. In Theorem 2 we show that $e_1 \otimes e_{n+1} \in \mathfrak{R}(\delta_S)^-$ if and only if $\sum_k w_k \cdot w_{k+1} \cdot \cdots \cdot w_{n+k-1} = \infty$. In Corollary 2, we find that this is also equivalent to $\mathfrak{R}(\delta_S)^-$ containing all the n -lower triangular compact operators.

The above results enable us to characterize those essentially normal weighted shifts that are d -symmetric (i.e., satisfy $\mathfrak{R}(\delta_S)^- = \mathfrak{R}(\delta_S)^{-*}$). Namely, an essentially normal weighted shift is d -symmetric if and only if S satisfies the total products condition $\sum_k w_k \cdot w_{k+1} \cdot \cdots \cdot w_{k+n} = \infty$ for all $n \in \mathbf{N}$. This yields another proof of the fact proved in Corollary 4 of [8] that all hyponormal (and hence all subnormal) weighted shifts are all d -symmetric.

THEOREM 1. *Let S be the unilateral (bilateral) weighted shift $Se_n = w_n e_{n+1}$ $n \in \mathbf{N}$ (\mathbf{Z}). Then $e_i \otimes e_j \in \mathfrak{R}(\delta_S)$ for all $i, j \in \mathbf{N}$ (\mathbf{Z}) with $i > j$.*

Proof. Write $i = j + n$ with $n > 0$. Let $a_0 = 1/w_j$, $a_k = w_{j+n} \cdot \cdots \cdot w_{j+n+k-1}/w_j \cdot \cdots \cdot w_{j+k}$ for $k \geq 1$, and $a_k = 0$ for $k < 0$. Then

for $k > n$, cancellation is possible and

$$a_k = w_{j+k+1} \cdots w_{j+n+k-1} / w_j \cdots w_{j+n-1} \leq \|S\|^{n-1} / w_j \cdots w_{j+n-1}.$$

Thus the a_k 's are uniformly bounded by some constant B_n . Also note that $a_k w_{j+n+k} = a_{k+1} w_{j+k+1}$ for $k \neq 1$ so $w_{m+n-1} a_{m-j-1} = a_{m-j} w_m$ for $m - j - 1 \neq -1$.

Now define $T = \sum_{k=0}^{\infty} a_k e_{j+n+k} \otimes e_{j+k+1}$. Then $\|T\| = \sup_k a_k \leq B_n$ so $T \in \mathfrak{B}(\mathcal{H})$. Further,

$$\begin{aligned} (ST - TS)(e_m) &= Sa_{(m-j-1)} e_{j+n+(m-j-1)} \otimes e_{j+(m-j-1)+1}(e_m) \\ &\quad - a_{(m-j)} e_{j+n+(m-j)} \otimes e_{j+(m-j)+1}(w_m e_{m+1}) \\ &= Sa_{m-j-1} e_{m+n-1} - a_{m-j} w_m e_{m+n} \\ &= (w_{m+n-1} a_{m-j-1} - a_{m-j} w_m) e_{m+n} \\ &= \begin{cases} 0 & m - j - 1 \neq -1 \\ 0 - a_0 w_j e_{j+n} & m - j = 0 \end{cases} \\ &= \begin{cases} 0 & m \neq j \\ -e_i & m = j. \end{cases} \end{aligned}$$

Thus $ST - TS = -e_i \otimes e_j$ and $\delta_S(-T) = e_i \otimes e_j$. □

LEMMA 1. *If $Se_n = w_n e_{n+1}$, $n \in \mathbf{N}(\mathbf{Z})$ is a unilateral (bilateral) weighted shift and $f \in \mathfrak{B}(\mathcal{H})^*$ is in the annihilator of $\mathfrak{R}(\delta_S)$, then*

$$f(e_{i+k} \otimes e_{j+k}) = \frac{w_j \cdot w_{j+1} \cdots w_{j+k-1}}{w_i \cdot w_{i+1} \cdots w_{i+k-1}} f(e_i \otimes e_j)$$

for $i, j \in \mathbf{N}(\mathbf{Z})$ and $k \in \mathbf{N}$.

Proof. Since f annihilates $\mathfrak{R}(\delta_S)$,

$$0 = f(S(e_i \otimes e_{j+1}) - (e_i \otimes e_{j+1})S) = w_i f(e_{i+1} \otimes e_{j+1}) - w_j f(e_i \otimes e_j).$$

Thus $f(e_{i+1} \otimes e_{j+1}) = (w_j/w_i) f(e_i \otimes e_j)$ for all i, j and the lemma follows by induction. □

COROLLARY 1. *If $Se_n = w_n e_{n+1}$, $n \in \mathbf{N}(\mathbf{Z})$ is a unilateral (bilateral) weighted shift and $e_n \otimes e_m \in \mathfrak{R}(\delta_S)^-$, then $e_i \otimes e_j \in \mathfrak{R}(\delta_S)^-$ for all $i, j \in \mathbf{N}(\mathbf{Z})$ satisfying the condition $m - n = j - i$. □*

THEOREM 2. *Let S be the unilateral (bilateral) weighted shift $Se_n = w_n e_{n+1}$, $n \in \mathbf{N}(\mathbf{Z})$. For $i \in \mathbf{N}(\mathbf{Z})$ and $n \in \mathbf{N}$, we have $e_i \otimes e_{i+n} \in \mathfrak{R}(\delta_S)^-$ if and only if $\sum_k w_k \cdot w_{k+1} \cdot \dots \cdot w_{K+n-1} = \infty$ where the sum is taken over \mathbf{N} or \mathbf{Z} as S is unilateral or bilateral.*

Proof. By Corollary 1, it suffices to consider $e_1 \otimes e_{n+1}$.

Suppose that $e_1 \otimes e_{n+1} \in \mathfrak{R}(\delta_S)^-$. If J is a trace class operator that commutes with S , the equation

$$\begin{aligned} \text{trace}((SA - AS)J) &= \text{trace}(SAJ - AJS) \\ &= \text{trace}(SAJ) - \text{trace}(SAJ) = 0 \end{aligned}$$

shows that $\text{trace}(\cdot J)$ annihilates $\mathfrak{R}(\delta_S)^-$. Since S^n commutes with S and $\text{trace}(S^n(e_1 \otimes e_{n+1})) = \text{trace}(w_1 \cdot w_2 \cdot \dots \cdot w_n e_{n+1} \otimes e_{n+1}) = w_1 \cdot w_2 \cdot \dots \cdot w_n \neq 0$, it follows that S^n cannot be of trace class. Hence $\infty = \sum_k (|S^n| e_k, e_k) = \sum_k w_k \cdot w_{k+1} \cdot \dots \cdot w_{k+n-1}$.

Conversely, suppose that $\sum_k w_k w_{k+1} \dots w_{k+n-1} = \infty$ and that $f \in \mathfrak{B}(\mathcal{H})^*$ annihilates $\mathfrak{R}(\delta_S)^-$. Then $\sum_{k=1}^\infty w_k \cdot w_{k+1} \cdot \dots \cdot w_{k+n-1} = \infty$ or (in the bilateral case) $\sum_{k=0}^\infty w_k \cdot w_{k+1} \cdot \dots \cdot w_{k+n-1} = \infty$. In the first case define $T_N = \sum_{k=n}^{N+n} e_k \otimes e_{n+k}$. Then $\|T_N\| = 1$ and using Lemma 1,

$$\begin{aligned} \|f\| &\geq |f(T_N)| = \left| \sum_{k=n}^{N+n} \frac{w_{n+1} \cdot w_{n+2} \cdot \dots \cdot w_{n+k-1}}{w_1 \cdot w_2 \cdot \dots \cdot w_{k-1}} f(e_1 \otimes e_{n+1}) \right| \\ &= \left| \sum_{k=n}^{N+n} \frac{w_k \cdot \dots \cdot w_{n+k-1}}{w_1 \cdot \dots \cdot w_n} f(e_1 \otimes e_{n+1}) \right| \\ &= \frac{|f(e_1 \otimes e_{n+1})|}{w_1 \cdot \dots \cdot w_n} \sum_{k=n}^{N+n} w_k w_{k+1} \cdot \dots \cdot w_{k+n-1}. \end{aligned}$$

Since $\sum_{k=n}^{N+n} w_k \cdot w_{k+1} \cdot \dots \cdot w_{k+n-1} \rightarrow \infty$ as $N \rightarrow \infty$, we see that $f(e_1 \otimes e_{n+1}) = 0$ and $e_1 \otimes e_{n+1} \in \mathfrak{R}(\delta_S)^-$.

Now suppose that $\sum_{k=0}^\infty w_k \cdot w_{k+1} \cdot \dots \cdot w_{k+n-1} = \infty$. If $l < 0$, we can apply Lemma 1 to $k = -l + 1$ to show that

$$f(e_1 \otimes e_{n+1}) = \frac{w_{n+l} \cdot \dots \cdot w_n}{w_l \cdot \dots \cdot w_0} f(e_l \otimes e_{n+1})$$

or

$$f(e_l \otimes e_{n+1}) = \frac{w_l \cdot \dots \cdot w_0}{w_{n+l} \cdot \dots \cdot w_n} f(e_1 \otimes e_{n+1}).$$

Defining $R_N = \sum_{l=-n}^{-N-n} e_l \otimes e_{n+l}$, we see that

$$\begin{aligned} \|f\| &\geq |f(R_N)| = \left| \sum_{l=-n}^{-N-n} \frac{w_l \cdots w_0}{w_{n+l} \cdots w_n} f(e_l \otimes e_{n+l}) \right| \\ &= \left| \sum_{l=-n}^{-N-n} \frac{w_l \cdots w_{n+l-1}}{w_1 \cdots w_n} f(e_l \otimes e_{n+l}) \right| \\ &= \frac{|f(e_1 \otimes e_{n+1})|}{w_1 \cdots w_n} \sum_{l=-n}^{-N-n} w_l \cdots w_{n+l-1}. \end{aligned}$$

As before, the fact that $\sum_{l=-n}^{-N-n} w_l \cdots w_{n+l-1} \rightarrow \infty$ implies that $f(e_1 \otimes e_{n+1}) = 0$ and $e_1 \otimes e_{n+1} \in \mathfrak{R}(\delta_S)^-$. \square

REMARK. Note that if we take $n = 0$ in the proof of Theorem 1 then the a_n become $1/w_n$. Thus $e_i \otimes e_i \in \mathfrak{R}(\delta_S)$ if the w_n are bounded away from zero. If the weights are not bounded away from zero, then taking $n = 0$ in the proof of Theorem 2 we find that $\|f\| \geq \sum_{k=0}^N |f(e_1 \otimes e_1)|$ and thus $e_i \otimes e_i \in \mathfrak{R}(\delta_S)^-$.

COROLLARY 2. Let S be the unilateral (bilateral) weighted shift $Se_n = w_n e_{n+1}$, $n \in \mathbf{N}(\mathbf{Z})$. Then the following are equivalent.

- (a) $\mathfrak{R}(\delta_S)^-$ contains the n -lower triangular compact operators.
- (b) $e_1 \otimes e_{1+n} \in \mathfrak{R}(\delta_S)^-$
- (c) $e_i \otimes e_{i+n} \in \mathfrak{R}(\delta_S)^-$ for some $i \in \mathbf{N}(\mathbf{Z})$.
- (d) $\sum_k w_k \cdot w_{k+1} \cdots w_{k+n-1} = \infty$.

Proof. The equivalence of (b), (c) and (d) has already been established and (b) follows from (a) since $e_1 \otimes e_{1+n}$ is compact and n -lower triangular. It remains to be shown that (b) implies (a). From the proof of Theorem 2, we see that if $e_1 \otimes e_{n+1} \in \mathfrak{R}(\delta_S)^-$, then S^n is not trace class. Hence S^m is not trace class for $0 \leq m < n$. Thus $\sum_k w_k \cdot w_{k+1} \cdots w_{k+m-1} = \infty$ and all operators of the form $e_i \otimes e_{i+m}$ are elements of $\mathfrak{R}(\delta_S)^-$. Since by Theorem 1, and the above remark, $e_i \otimes e_{i+m} \in \mathfrak{R}(\delta_S)^-$ for $m \leq 0$, it follows that $\mathfrak{R}(\delta_S)^-$ contains the closed linear span of $\{e_i \otimes e_{i+m} : m \leq n\}$ (i.e., the n -lower triangular compact operators). \square

REMARK. It is not true that if $\mathfrak{R}(\delta_S)^-$ contains an n -lower triangular compact operator which is not $(n - 1)$ -lower triangular then $\mathfrak{R}(\delta_S)^-$ contains all n -lower triangular compact operators. In fact $\mathfrak{R}(\delta_S)^-$ will always contain such an operator; namely $\delta_S(e_1 \otimes e_{n+2}) = w_1 e_2 \otimes e_{n+2} - w_{n+1} e_1 \otimes e_{n+1}$.

DEFINITION. A weighted shift satisfies the *total products condition* if $\sum_k w_k \cdot w_{k+1} \cdot \dots \cdot w_{k+n} = \infty$ for all $n \in \mathbf{N}$.

COROLLARY 3. Let $Se_n = w_n w_{n+1}$, $n \in \mathbf{N}$ (\mathbf{Z}) be a unilateral (bilateral) weighted shift. Then $\mathcal{K} \subseteq \mathfrak{R}(\delta_S)^-$ if and only if S satisfies the total products condition. □

We now make application to the question: which weighted shifts are d -symmetric? Recall that an operator T is d -symmetric if $\mathfrak{R}(\delta_T)^- = \mathfrak{R}(\delta_T)^- *$. In [2] it is proved that an operator T is d -symmetric if and only if $TT^* - T^*T \in \mathcal{C}(T) = \{C \in \mathfrak{B}(\mathcal{H}) : C\mathfrak{B}(\mathcal{H}) + \mathfrak{B}(\mathcal{H})C \subseteq \mathfrak{R}(\delta_T)^-\}$.

THEOREM 3. The weights of a d -symmetric weighted shift S satisfy the total products condition.

Proof. By Theorem 1, $e_i \otimes e_j \in \mathfrak{R}(\delta_S)^-$ for $i \geq j$. By the d -symmetry of S , we see that $e_j \otimes e_i = (e_i \otimes e_j)^* \in \mathfrak{R}(\delta_S)^-$ for $j \leq i$. Thus \mathcal{K} , the linear span of all $e_i \otimes e_j$, is contained in $\mathfrak{R}(\delta_S)^-$ and so by Corollary 3, the weights of S satisfy the total products condition. □

The total products condition is not sufficient for d -symmetry else any weighted shift with weights bounded away from zero would be d -symmetric. However the weighted shift with weights alternating between 1 and 2 has an irreducible representation as the operator $\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$ on \mathbf{C}^2 , while in [2] it is shown that any irreducible representation of a d -symmetric operator must be over a Hilbert space of dimension 1 or \aleph_0 . There are, however, natural conditions under which the total products condition is sufficient.

THEOREM 4. An essentially normal weighted shift S is d -symmetric if and only if it satisfies the total products condition.

Proof. The necessity of the total products condition follows from Theorem 3 and sufficiency follows from the facts that $SS^* - S^*S$ is compact and that $\mathcal{K} \subseteq \mathfrak{R}(\delta_S)^-$ implies $\mathcal{K} \subseteq \mathcal{C}(S)^-$. □

COROLLARY 4. A hyponormal (in particular subnormal) weighted shift $Se_n = w_n e_{n+1}$ is d -symmetric.

Proof. If S is hyponormal, then its weights are increasing and bounded. Thus

$$SS^* - S^*S = \text{diag}(w_{i-1}^2 - w_i^2)$$

is compact and $\sum_{k=1}^\infty w_k \cdot w_{k+1} \cdot \dots \cdot w_{k+n-1} \geq \sum_{k=1}^\infty w_1^n = \infty$ for all $n \in \mathbf{N}$. □

REFERENCES

1. J. Anderson, *On normal derivations*, Proc. Amer. Math. Soc., **38** (1973), 135–140.
2. J. Anderson, J. Bunce, J. Deddens, and J. P. Williams, *C^* -algebras and derivation ranges*, Acta. Sci. Math., **40** (1978), 211–227.
3. C. Apostol and J. G. Stampfli, *On derivation ranges*, Indiana University Math. J., **21** (1976), 857–865.
4. J. Bunce and J. Deddens, *C^* -algebras generated by weighted shifts*, Indiana University Math. J., **23** (1973), 257–271.
5. B. E. Johnson and J. P. Williams, *The range of a normal derivation*, Pacific J. Math., **58** (1975), 105–122.
6. C. R. Rosentrater, *Not every d -symmetric operator is GCR*, Proc. Amer. Math. Soc., **81** (1981), 443–446.
7. J. G. Stampfli, *Derivations on $\mathfrak{B}(\mathfrak{H})$: The range*, Illinois J. Math., **17** (1973), 518–524.
8. ———, *On self-adjoint derivation ranges*, Pacific J. Math., **82** (1979), 257–278.
9. J. P. Williams, *On the range of a derivation*, Pacific J. Math., **38** (1971), 273–279.
10. ———, *On the range of a derivation II*, Proc. Roy. Irish. Acad. Sect. A, **74** (1974), 299–310.

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Leo James Alex, Simple groups and a Diophantine equation	257
Herbert James Alexander and John Wermer, On the approximation of singularity sets by analytic varieties	263
Waleed A. Al-Salam and Mourad Ismail, Orthogonal polynomials associated with the Rogers-Ramanujan continued fraction	269
J. L. Brenner and Roger Conant Lyndon, Permutations and cubic graphs	285
Ian George Craw and Susan Ross, Separable algebras over a commutative Banach algebra	317
Jesus M. Dominguez, Non-Archimedean Gel'fand theory	337
David Downing and Barry Turett, Some properties of the characteristic of convexity relating to fixed point theory	343
James Arthur Gerhard and Mario Petrich, Word problems for free objects in certain varieties of completely regular semigroups	351
Moses Glasner and Mitsuru Nakai, Surjective extension of the reduction operator	361
Takesi Isiwata, Ultrafilters and mappings	371
Lowell Duane Loveland, Double tangent ball embeddings of curves in E^3	391
Douglas C. McMahon and Ta-Sun Wu, Homomorphisms of minimal flows and generalizations of weak mixing	401
P. H. Maserick, Applications of differentiation of \mathcal{L}_p -functions to semilattices	417
Wayne Bruce Powell and Constantine Tsinakis, Free products in the class of abelian l -groups	429
Bruce Reznick, Some inequalities for products of power sums	443
C. Ray Rosentrater, Compact operators and derivations induced by weighted shifts	465
Edward Silverman, Basic calculus of variations	471
Charles Andrew Swanson, Criteria for oscillatory sublinear Schrödinger equations	483
David J. Winter, The Jacobson descent theorem	495