Pacific Journal of Mathematics

LATTICE VERTEX POLYTOPES WITH INTERIOR LATTICE POINTS

DOUGLAS AUSTIN HENSLEY

Vol. 105, No. 1 September 1983

PACIFIC JOURNAL OF MATHEMATICS
Vol. 105, No. 1, 1983

LATTICE VERTEX POLYTOPES WITH INTERIOR LATTICE POINTS

DOUGLAS HENSLEY

Consider a convex polytope with lattice vertices and at least one interior lattice point. We prove that the number of boundary lattice points is bounded above by a function of the dimension and the number of interior lattice points. This extends to arbitrary dimension a result of Scott for the two dimensional case.

Introduction. In real Euclidean space $\mathbf{R}^D$ of dimension D there is the lattice $\mathbf{Z}^D$ of points with integer coordinates. Unless a different lattice is specified, a *lattice point* will mean a point of $\mathbf{Z}^D$, and a *lattice simplex* or *lattice convex polytope* will mean a simplex or convex polytope whose vertices are integer points, that is, elements of $\mathbf{Z}^D$. The interior in $\mathbf{R}^D$ of a set S is denoted by S°; if the affine span of S has dimension less than D, we denote the relative interior of S by S'.

Consider a lattice convex polytope $P \subseteq \mathbf{R}^D$ with the number $K = \#(P^\circ \cap \mathbf{Z}^D)$ of interior lattice points non-zero, and with a total of $J = \#(P \cap \mathbf{Z}^D)$ lattice points. *Our principal result is that J is bounded above by a function $B(K, D)$ of K and D alone.*

For the case of zero symmetric convex polytopes P there is no need to assume that the vertices are lattice points. By Van der Corput's generalization of Minkowski's theorem $\mathrm{vol}(P) \leq K \cdot 2^D$ **[4]**^{40}.† By a theorem of Blichfeldt, if the lattice points of P span $\mathbf{R}^D$, $J \leq D + D!\,\mathrm{vol}(P)$ **[1]**^{55}. Otherwise we can consider a subspace of $\mathbf{R}^D$ and get the same inequality $J \leq D + D!K \cdot 2^D$. On the other hand if P need not be symmetric or have lattice point vertices then even for $D = 2$ and $K = 1$, J can be arbitrarily large. For instance, P might be the convex hull of $(-n, 0)$, $(0, 1 + 1/n^2)$, $(n, 0)$. With the restriction to lattice point vertices and $D = 2$ we have Scott's result that $J \leq 3K + 7$ ($3K + 6$ for $K > 1$), and of course when $D = 1$ we have trivially $J \leq K + 2$. These three bounds are best possible. Our results are far from best possible, but in any case the largest possible J grows rapidly with D, even for $K = 1$. Zaks, Perles and Wills have given examples of lattice simplices in $\mathbf{R}^D$ for which $K = 1$ and $J > 2^{2^{D-1}}$ **[11]**. There are some grounds for the belief that these examples are best possible. (See §4.) The existence of $B(K, D)$ will follow from some facts about Diophantine approximation which we now establish.

†Here the number above the brackets gives the page number on which this result is found in Lekkerkerker [7].

2. Number theory. We start with a well-known approximation lemma.

LEMMA 2.1. *Given a vector* $\vec{v} = (v_1, v_2 \cdots v_D) \in \mathbf{R}^D$ *and an integer* $T > 0$ *there exist integers* $a_1, a_2 \cdots a_D, b$ *such that* $1 \le b \le T^D$ *and* $|bv_i - a_i| \le 1/T$ *for* $1 \le i \le D$.

Proof. Consider the $T^D + 1$ points $k\vec{v}$, $0 \le k \le T^D$ reduced modulo 1 in each coordinate. Partitioning the unit cube $\{\vec{x}\colon 0 \le x_i \le 1 \text{ for } 1 \le i \le D\}$ into T^D cubes of side $1/T$, we conclude from the Dirichlet box principle that some two of them, say k_1 and k_2 with $k_1 > k_2$, lie in the same small cube. Let $b = k_1 - k_2$ and let a_i be the integer nearest bv_i for $1 \le i \le D$. □

LEMMA 2.2. *Let* $\vec{w} = (w_1, w_2 \cdots w_D)$ *such that* $\Sigma_1^D w_i = 1$ *and each* $w_i > 0$, *and let* $T > D$. *Then there exist integers* $P_1, P_2 \cdots P_D$, $Q = \Sigma_1^D P_i$ *such that* $1 \le Q \le T^{D-1}$, $P_i \ge 0$ *for* $1 \le i \le D$, $|Qw_1 - P_1| \le D/T$ *and* $|Qw_i - P_i| \le 1/T$ *for* $2 \le i \le D$.

Proof. We write $\vec{w} = \vec{e}_1 + \Sigma_2^D w_i(\vec{e}_i - \vec{e}_1)$. By Lemma 1 there exists Q, $1 \le Q \le T^D$, and $P_2, P_3 \cdots P_D$ such that $|Qw_i - P_i| \le 1/T$ $(2 \le i \le D)$. Since each $w_i > 0$, $Qw_i > 0$ so $P_i \ge 0$ for $i \ge 2$.

Let $P_1 = Q - \Sigma_2^D P_i$. Then $|P_1 - Qw_1| = |\Sigma_2^D P_i - Q\Sigma_2^D w_i| < D/T < 1$ so that also $P_1 \ge 0$. □

LEMMA 2.3. *For each integer* $D \ge 1$ *there exists* $\varepsilon(D) > 0$ *such that if* $\vec{\alpha} = (\alpha_1, \alpha_2 \cdots \alpha_D)$, *each* $\alpha_i > 0$ *and* $1 > \Sigma_1^D \alpha_i > 1 - \varepsilon(D)$ *then there exist integers* $Q \ge 1$ *and* $P_1, P_2 \cdots P_D \ge 0$ *such that* $\Sigma_1^D P_i = Q$ *and* $(Q+1)\alpha_i > P_i$ *for each* i, $1 \le i \le D$.

Proof. For $D = 1$ thus just says that there is an integer Q such that $(Q+1)\alpha_1 > Q$, so that we may take $\varepsilon(1) = 1/2$. Now suppose $D > 1$ and the lemma holds for $D - 1$. Let $\vec{\alpha} = (\alpha_1, \ldots, \alpha_D)$ and without loss of generality assume $\alpha_1 \ge \alpha_2 \cdots \ge \alpha_D > 0$. We want to choose $\varepsilon(D)$ in terms of $\varepsilon(D-1)$ so that if $1 > \Sigma_1^D \alpha_i > 1 - \varepsilon(D)$ then the $P_1, \ldots, P_D$ and Q of Lemma 2.3 exist. We choose it this way: Let

$$T = \max\left\{1 + \left[4(\varepsilon_{D-1})^{-1}\right], 4D^2 + 4D + 1\right\}.$$

Let $\varepsilon(D)$ (>0) be $\min\{\frac{1}{2}\varepsilon(D-1), (D-1)^{-1}, \frac{1}{4}T^{1-D}\}$. Let $w_i = \alpha_i(1-\varepsilon)^{-1}$ where $\varepsilon = 1 - \Sigma_1^D \alpha_i < \varepsilon(D)$.

By Lemma 2.2 there exist $P_1, P_2, \ldots, P_D \ge 0$ and $Q = \Sigma_1^D P_i$ such that $1 \le Q \le T^{D-1}$ and $|Qw_1 - P_1| \le D/T$, $|Qw_i - P_i| \le 1/T$ for $2 \le i \le D$.

Now for $2 \leq i \leq D$,

$$\begin{aligned}(Q+1)\alpha_i - P_i = \alpha_i + Q\alpha_i - P_i &= \alpha_i + Q\alpha_i - Qw_i + Qw_i - P_i \\ &\geq \alpha_i - Q\alpha_i(1/(1-\varepsilon) - 1) - 1/T \\ &> \alpha_i - Q\alpha_i(1/(1-\varepsilon(D)) - 1) - 1/T \\ &\geq \alpha_i(1 - 2Q\varepsilon(D)) - 1/T \geq \alpha_i(1 - 2T^{D-1}\varepsilon(D)) - 1/T.\end{aligned}$$

If now $\alpha_i \geq \frac{1}{2}\varepsilon(D-1)$ this last is positive, from the definitions of T and $\varepsilon(D)$. If $\alpha_i < \frac{1}{2}\varepsilon(D-1)$ then $\alpha_D < \frac{1}{2}\varepsilon(D-1)$ so that $\Sigma_1^{D-1}\alpha_i > 1 - \varepsilon(D) - \frac{1}{2}\varepsilon(D-1) \geq 1 - \varepsilon(D-1)$. In this case the $P_1, \ldots, P_{D-1}$, Q guaranteed by Lemma 2.2 (assumed true for $D-1$) can be extended with $P_D = 0$.

The case $i = 1$ is a little different. Here we have $\alpha_1 \geq 1/(D+1)$ since $\varepsilon < \varepsilon(D) \leq 1/(D+1)$, and we need $\frac{1}{2}\alpha_1(1 - 2T^{D-1}\varepsilon(D)) > D/T$, which follows from $T > 4D(D+1)$. □

We can determine the best constants $\varepsilon(D)$ in Lemma 2.3 for $D = 1, 2$ or 3. As noted, we can take $\varepsilon(1) = 1/2$. No larger choice is possible because if $\alpha_1 = 1/2$, $(Q+1)\alpha_1 > Q$ has no positive integer solution.

For $D = 2$ and $\alpha_1 \geq \alpha_2$ if $\alpha_1 > 1/2$ we take $Q = 1$, $P_1 = 1$ and $P_2 = 0$, while if $\alpha_2 > 1/3$, $Q = 2$, $P_1 = P_2 = 1$. Thus we may take $\varepsilon(2) = 1 - 1/2 - 1/3 = 1/6$. For $D = 3$ we can prove by such considerations that $\varepsilon(3)$ can be taken $= 1/42$. For if $\alpha_1 + \alpha_2 + \alpha_3 > 41/42$ while $\alpha_1 \leq 1/2$ and $\alpha_2 \leq 1/3$ then $\alpha_3 > 1/7$. Now if $7(\alpha_1, \alpha_2, \alpha_3) \not> (3, 2, 1)$ (coordinatewise), then either $\alpha_1 \leq 3/7$ or $\alpha_2 \leq 2/7$. Either way, $\alpha_3 > 1/7 + 1/21 = 4/21$. Eventually one arrives at $\alpha_3 > 1/4$, and then $4(\alpha_1, \alpha_2, \alpha_3) > (1, 1, 1)$.

For $D = 1$, 2 or 3 these $\varepsilon(D)$ are best possible (consider $\alpha_1 = 1/2$, $\alpha_2 = 1/3$ and $\alpha_3 = 1/7$). For $D \geq 4$ this approach seems to break down.

In the next lemma we treat the case $K > 1$.

LEMMA 2.4. *For integers $K \geq 2$, $D \geq 1$ there exists $\varepsilon(K, D) > 0$ such that if $1 > \Sigma_1^D \alpha_i > 1 - \varepsilon(K, D)$ and each $\alpha_i > 0$ then there exist integers $P_1, P_2 \cdots P_D \geq 0$ and $Q = \Sigma_1^D P_i \geq 1$ such that $(KQ+1)\alpha_i > KP_i$ for $1 \leq i \leq D$.*

Proof. For $D = 1$ this says simply that if $\alpha < 1$ is sufficiently large then there exists $Q \geq 1$ such that $(KQ+1)\alpha_1 > KQ$, and we take $\varepsilon(K, 1) = 1/(K+1)$. We now prove Lemma 2.4 for fixed K by induction on D. Suppose it holds for $D-1$. Let $\vec{\alpha} = (\alpha_1, \alpha_2 \cdots \alpha_D)$ with each $\alpha_i > 0$ and $\Sigma_1^D \alpha_i = 1 - \varepsilon$, $\varepsilon > 0$. If $\alpha_D < \varepsilon(K, D-1) - \varepsilon$ then $\Sigma_1^{D-1}\alpha_i > 1 - \varepsilon(K, D-1)$ so we can use $P_1, P_2 \cdots P_{D-1}$, 0 and Q as in Lemma 2.3.

Otherwise we use Lemma 2.2. Let

$$T = \max\left\{1 + \left[4K(\varepsilon(K, D-1))^{-1}\right], 4D^2 + 4D + 1\right\}.$$

Let

$$\varepsilon(K, D) = \min\left\{1/4D^2, \frac{1}{4}\varepsilon(K, D-1), \varepsilon(1, D), (4K)^{-1}T^{1-D}\right\}.$$

For $2 \le i \le D$,

$$(KQ+1)\alpha_i - KP_i = \alpha_i + K(Q\alpha_i - P_i) \ge \alpha_i(1 - 2KQ\varepsilon) - K/T,$$

with $Q \le T^{D-1}$. This then is $> \frac{1}{2}\varepsilon(K, D-1)(1 - 2KT^{D-1}\varepsilon(K, D)) - K/T$. By the choice of $\varepsilon(K, D)$, $(1 - 2KT^{D-1}\varepsilon(K, D)) < 1/2$, and by the choice of T, $\frac{1}{4}\varepsilon(K, D-1) > K/T$.

For $i = 1$ we have $\alpha_1 \ge (D+1)^{-1}$ so we need $\frac{1}{2}(D+1)^{-1}(\frac{1}{2}) > KD/T$, which still follows from $T > 4D(D+1)$. □

REMARK. The growth of $(\varepsilon(D))^{-1}$ is about like $2^{(D!)}$. The example of [**11**] has a simple variant with ε like 2^{2^D}. So bound and example have asymptotic log log log's.

3. Geometry. Suppose now that S is a simplex with vertices $0, X_1, X_2 \cdots X_D \in \mathbf{Z}^D$ and an interior lattice point $Y = \sum_1^D \alpha_i X_i$.

LEMMA 3.1. *If* $\sum_1^D \alpha_i > 1 - \varepsilon(K, D)$ *then there are at least* $K+1$ *integer lattice points in* S°.

Proof. Apply Lemma 2.3 or 2.4. The points $Z_k = (kQ+1)Y - k\sum_{i=1}^D P_i X_i$ are lattice points, distinct, and interior to S, for $0 \le k \le K$.

By translation we can make any vertex of a simplex be zero. This, with Lemma 3.1, gives

THEOREM 3.1. *Suppose* S *is simplex in* $\mathbf{R}^D$ *with integer lattice vertices* $X_0, X_1 \cdots X_D$ *and exactly* K *interior lattice points* Y_j, $1 \le j \le K$, $Y_j = \sum_{i=0}^D \alpha_{ij} X_i$ *with* $\alpha_{ij} > 0$, $\sum_{i=1}^D \alpha_{ij} = 1$. *Then for each* i *and* j, $\varepsilon(K, D) \le \alpha_{ij} \le 1 - D\varepsilon(K, D)$.

COROLLARY 3.2. *Suppose* F *is a lattice convex polytope in* $\mathbf{R}^D$ *of spanning dimension* $D-1$, *and lattice vertices* $X_1, X_2 \cdots X_M$. *Let* X_0 *be a lattice point not in the span of* F, *and let* P *be the conical polytope with* X_0 *the tip and* F *the opposite face. If* $\#(P^\circ \cap \mathbf{Z}^D) = K \ge 1$ *then in any barycentric representation* $Y = \sum_0^M \alpha_i X_i$ *of an interior point of* P *we have* $\alpha_0 \ge \varepsilon(K, D)$.

Proof. By Caratheodory's theorem [3] there are $E \le D$ vertices of F, say $V_1, V_2 \cdots V_E$ such that Y is in the relative interior of the simplex S with vertices $X_0, V_1 \cdots V_E$. Every lattice point in S' is also in P° (proof follows), so there are no more than K in S'. By Theorem 1, if $Y = \beta_0 X_0 + \Sigma_1^E \beta_i V_i$ then $\beta_0 \ge \varepsilon(K, D)$. But $\beta_0 = \alpha_0$, since it is the ratio of the length of $\overline{YZ}$ to $\overline{X_0 Z}$, where Z is the intersection of the line through X_0 and Y with F.

We now prove that $S' \subseteq P^\circ$.

LEMMA 3.3. *If C is a convex set in $\mathbf{R}^D$, $Y \in C^\circ$ and $W_0 \cdots W_E$ form the vertices of a simplex W in C, with $E \le D$ and $Y \in W$, then $W' \subseteq C^\circ$.*

Proof. Since $Y \in C^\circ$ there exists $\varepsilon > 0$ such that if $\|\vec{U}\| \le 1$ and $|\theta| \le \varepsilon$ then $Y + \theta U \in C$. Write Y as $\Sigma_0^E \alpha_i W_i$, $\alpha_i > 0$, $\Sigma_0^E \alpha_i = 1$. If $Z \in W' = \Sigma_0^E \beta_i W_i$ with $\beta_i > 0$ and $\Sigma_0^E \beta_i = 1$ then there exists $\delta > 0$ such that $\beta_i > \delta\alpha_i$ for $0 \le i \le E$. Now $Z + \theta\delta U = \Sigma_0^E (\beta_i - \delta\alpha_i) W_i + \delta(Y + \theta U)$ is a convex positive combination of elements of C, so it is in C. □

Until now it has been convenient to have the fixed lattice $\mathbf{Z}^D$ in mind, but all the results are equally true for any full lattice L in $\mathbf{R}^D$, as there is a nonsingular linear transformation Φ: $\mathbf{R}^D \to \mathbf{R}^D$ which maps $\mathbf{Z}^D$ onto L while preserving barycentric coordinates, interiors and relative interiors, etc. We use this device to give an upper bound for the volume of an integer lattice simplex S with $\#(\mathbf{Z}^D \cap S^\circ) = K \ge 1$. Without loss of generality take 0 as one vertex of S, and let Φ be a linear transformation which takes S onto the "standard simplex" H with vertices $0, \vec{e}_1, \ldots, \vec{e}_D$, where $\vec{e}_i$ is the ith unit coordinate vector in $\mathbf{R}^D$. Then Φ takes the lattice $\mathbf{Z}^D$ to a new lattice L, and the norm of L, $|L|$ is $|\det \Phi|$, and $\operatorname{vol}(S) = 1/D!|\det \Phi^{-1}|$. Thus any *lower* bound for $|L|$ gives an upper bound for $\operatorname{vol}(S)$. Suppose $Y_1 \in S^\circ \cap \mathbf{Z}^D$, $Y_1 = \Sigma_1^D \alpha_i X_i$. Let $V_1 = \Phi Y_1 = \Sigma_1^D \alpha_i \vec{e}_i$. Given $U = \Sigma_1^D u_i \vec{e}_i$ with $|u_i| < \alpha_i$, either $V_1 + U \in H^\circ$ or $V_1 - U \in H^\circ$, since $\alpha_i \pm u_i > 0$ and one of $\Sigma_1^D(\alpha_i + u_i)$, $\Sigma_1^D(\alpha_i - u_i)$ is less than 1.

By Van der Corput's theorem the region $\{V_1 + U\colon |u_i| < \alpha_i, 1 \le i \le D\}$ contains at least $(\Pi_1^D \alpha_i)|\det \Phi^{-1}|$ pairs of points $V_1 \pm U \in L$. Of each pair at least one is in H°. Thus $K = \#(S^\circ \cap \mathbf{Z}^D) = \#(H^\circ \cap L) \ge (\Pi_1^D \alpha_i)|\det \Phi^{-1}|, \ge (\varepsilon(K, D))^D |\det \Phi^{-1}|$ by Theorem 3.1. So $|\det \Phi| \ge (\varepsilon(K, D))^D K^{-1}$. Since $|\det \Phi| = \operatorname{vol} H/\operatorname{vol} S$, we have $\operatorname{vol} S \le (D!)^{-1} K(\varepsilon(K, D))^{-D}$. We summarize this in

THEOREM 3.4. *Suppose S is a simplex in $\mathbf{R}^D$ with vertices in $\mathbf{Z}^D$, and let $K = \#(S^\circ \cap \mathbf{Z}^D)$. If $K \ge 1$ then* $\operatorname{vol} S \le (D!)^{-1} K(\varepsilon(K, D))^{-D}$.

REMARK. We could get a better lower bound for $\prod_1^D \alpha_i$ by using the fact that not only is each $\alpha_i \geq \varepsilon(K, D)$, but (perhaps renaming some vertices) $\sum_1^D \alpha_i \approx 1$ yet $\sum_1^E \alpha_i \leq 1 - \varepsilon(K, E)$ for $E < D$. With such a weak bound for $\varepsilon(K, D)$, though, this seems pointless.

A theorem of Blichfeldt says that if a convex body P in $\mathbf{R}^D$ has $J = \sharp(\mathbf{Z}^D \cap P) > D$ lattice points, spanning $\mathbf{R}^D$, then $\operatorname{vol}(P) \geq (J - D)/D!$ [**1**], or equivalently $J \leq D + D! \operatorname{vol}(P)$. Thus we get the

COROLLARY 3.5. *Under the hypotheses of Theorem* 3.4, $\sharp(S \cap \mathbf{Z}^D) \leq D + K(\varepsilon(K, D))^{-D}$.

For a general convex polytope P with vertices in $\mathbf{Z}^D$ and $K \geq 1$ lattice points in P°, from Corollary 3.2 we have that the coefficient σ of asymmetry about any of the interior lattice points is $\leq (1 - \varepsilon(K, D))/\varepsilon(K, D)$. When $K = 1$ we have by a theorem of Mahler (Sawyer gives a little sharper version) [**8, 9**][45] that $V(P) \leq (\varepsilon(D))^{-D}$. The proof of Mahler's theorem given in [**7**][45] uses Blichfeldt's theorem [**2**][35] that a region of volume > 1 contains two points x, y congruent modulo $\mathbf{Z}^D$. Van der Corput [**4**][40] generalized this to say that a region of volume $> K$ contains $K + 1$ points congruent modulo $\mathbf{Z}^D$. If we use this in place of Blichfeldt's result we get an analogous generalization of Mahler's theorem. From it we conclude that for arbitrary $K \geq 1$,

$$\operatorname{vol}(P) \leq K(\varepsilon(K, D))^{-D}.$$

This and a corollary complete the story.

THEOREM 3.6. *Let P be a convex polytope in* $\mathbf{R}^D$ *with vertices in* $\mathbf{Z}^D$ *and with* $K = \sharp(P^\circ \cap \mathbf{Z}^D) \geq 1$. *Then* $\operatorname{vol}(P) \leq K(\varepsilon(K, D))^{-D}$.

COROLLARY 3.7. *If* $J = \sharp(P \cap \mathbf{Z}^D)$ *then* $J \leq D + K(D!)(\varepsilon(K, D))^{-D}$.

4. Toward best possible results. Here we indicate some reasons for our belief that the examples of [**11**] with $K = 1$ and $D \geq 3$ are best possible. Suppose S is a lattice simplex with lone interior point $Y = \sum_0^D \alpha_i X_i$, where $X_0, \ldots, X_D$ are the vertices of S and $\alpha_1 \geq \cdots \geq \alpha_D \geq \alpha_0$. We proved in §2 that for arbitrary D, $\alpha_1 + \alpha_2 \leq 5/6$, and $\alpha_1 + \alpha_2 + \alpha_3 \leq 41/42$. For $D = 4$, if $\sum_1^4 \alpha_i > 1805/1806$ then $\alpha_4 > 1/43$. The minimum of $\alpha_1\alpha_2\alpha_3\alpha_4$ subject to $\sum_1^4 \alpha_i \geq 1805/1806$, $\sum_1^3 \alpha_i \leq 41/42$, $\sum_1^2 \alpha_i \leq 5/6$ and $\alpha_1 \leq 1/2$, $0 < \alpha_4 \leq \alpha_3 \leq \alpha_2 \leq \alpha_1$ is $1/1806$, by elementary calculus. Since $\operatorname{Norm}(L) \geq 1/1806$ and $\operatorname{vol}(\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4, \Phi\vec{Y})$ {simplex} is $\frac{1}{4!}(1 - \sum_1^4 \alpha_i) \geq \frac{1}{4!}\operatorname{Norm}(L)$, $\sum_1^4 \alpha_i \leq 1805/1806$. This proves that for $D = 3$, (4) the simplex with vertices 0, $2\vec{e}_1$, $3\vec{e}_2$, $7\vec{e}_3$, $(43\vec{e}_4)$ has maximal coefficient σ of asymmetry about Y. Unfortunately it does not show that for arbitrary D, $\Sigma_1^4 \alpha_i \le 1805/1806$.

For any D, the α_i must be rational. For let Λ' be the lattice generated by $\{X_i - X_0,\ 1 \le i \le D\}$. If some α_i were irrational there would be infinitely many points of Λ in a fundamental cell of Λ' since no two $n(Y - X_0)$, $n \ge 1$, would be congruent mod Λ'. But Λ is discrete so this is impossible. So let $\alpha_i = v_i/x_i$, $0 \le i \le D$, with $v_i, x_i > 0$ and $\gcd(v_i, x_i) = 1$ for $0 \le i \le D$).

The numbers 2, 3, 7, 43 in the simplex examples for $D = 3$ or 4 are the start of a well-known sequence given recursively by $y_1 = 2$, $y_{n+1} = y_n^2 - y_n + 1$ for $n \ge 1$. The y_i's are pairwise relatively prime, and $\Sigma_1^D y_i^{-1} = 1 - (y_{D+1} - 1)^{-1} < 1$. Thus the lattice simplex S_D with vertices 0 and $y_i\vec{e}_i$, $1 \le i \le D$ has the single interior lattice point $Y_D = \Sigma_1^D \vec{e}_i$. This example (here slightly modified) is first given in **[11]** and has at least $2^{2^{D-1}}$ boundary lattice points. We believe it to be best possible in the sense that the coefficient σ_D of asymmetry for S_D about $Y_D \ge \sigma$ for any other lattice simplex S with lone interior lattice Y, about Y.

Let S be such a simplex, and $Y = \Sigma_0^D \alpha_i X_i = \Sigma_0^D (v_i/x_i) X_i$ as before, with $\alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_D \ge \alpha_0 > 0$. With the additional assumption that $(x_1, x_2, \ldots, x_D)$ are pairwise relatively prime we can prove this conjecture, or what is the same, the following theorem.

THEOREM 4.1. *Suppose* $(x_1, x_2, \ldots, x_D)$ *are pairwise relatively prime. Then* $\Sigma_1^D v_i/x_i \le \Sigma_1^D 1/y_i$.

Conjecture. This holds whether or not the x_i's are pairwise relatively prime. (We have seen so for $1 \le D \le 4$.)

We begin the proof of Theorem 4.1 with an old Egyptian fractions result.

LEMMA 4.1. (*Curtis* **[5]**, *Erdös* **[6]**.) *Let* $x_1, x_2 \cdots x_D$ *be positive integers. If* $\Sigma_1^D (1/x_i) < 1$ *then* $\Sigma_1^D (1/x_i) \le \Sigma_1^D (1/y_i) = 1 - \Pi_1^D y_i^{-1} = 1 - (y_{D+1} - 1)^{-1}$.

Let $\varepsilon_k = (y_{k+1} - 1)^{-1}$.

LEMMA 4.2. *For every* K, $D \ge 1$ *if* (v_i, x_i), $1 \le i \le D$ *are* D *pairs of relatively prime positive integers, and if* $1 - \varepsilon_{D+K-1} < \Sigma_1^D (v_i/x_i) < 1$ *then* $\Sigma_1^D v_i \ge D + K$.

Proof. (I. Borosh, private communication.) If each v_i/x_i is replaced with v_i copies of $1/x_i$ there are then at least $D + K$ Egyptian fractions in the sum, by Lemma 4.1.

LEMMA 4.3. *Let $D \geq 2$, K, $v_1 \cdots v_D$, $x_1 \cdots x_D$ be positive integers such that $\gcd(v_i, x_i) = 1$ for $1 \leq i \leq D$ and $\gcd(x_i, x_j) = 1$ for $1 \leq i < j \leq D$. Let $M = \prod_1^D x_i$ and $A_i = Mv_i/x_i$, $1 \leq i \leq D$. Let $\alpha_i = v_i/x_i = A_i/M$ and suppose $\gcd(A_D, M) \leq \gcd(A_i, M)$, $1 \leq i < D$, or equivalently $x_D \geq x_i$. Let $\theta_2, \theta_3 \cdots \theta_K$ be any $K - 1$ rational numbers $0 < \theta_i < 1$. If*

$$1 - \varepsilon_{D+K-1} < \sum_1^D \alpha_i < 1$$

then there exist positive integers $a_1, a_2 \cdots a_D$, m such that

(i) $a_i/m < \alpha_i$ *for* $1 \leq i \leq D$

(ii) $m\alpha_D - a_D \neq \theta_j$ *for* $2 \leq j \leq K$, *and* $m\alpha_D - a_D \neq \alpha_D$, *and*

(iii) $\sum_1^D (mA_i - Ma_i) < M$.

REMARK. For Theorem 4.1 we only need the case $K = 1$.

Proof. By Lemma 4.2, $\sum_1^D (v_i - 1) \geq K$. Since $\gcd(A_D, M) \leq \gcd(A_i, M)$ for $i \neq D$, $x_D \geq x_i$ for $i \neq D$. Since $\prod_1^D (1/x_i) \leq 1 - \sum_1^D v_i/x_i < \varepsilon_{D+K-1}$, $x_D^D \geq (\varepsilon_{D+K-1})^{-1}$ and $x_D > K + 1$. For it is readily seen that $\varepsilon_i^{-1} \geq 2^{2^{i-1}}$ for $i \geq 1$, and $D - \log_2 D \geq 1$, $K - (\log\log)_2 K \geq 2$ so that $D + K - 2 \geq 1 + \log_2 D + (\log\log)_2 K$ and $2^{2^{D+K-2}} \geq K^{2D} > K + 1$ for $K > 1$, while for $K = 1$, we have directly $\varepsilon_D^{-1} > 2$ since already $\varepsilon_2^{-1} = 6$. Now by the Chinese remainder theorem, for each integer r, $1 \leq r \leq K + 1$ there exists an $m > 1$ such that $mv_i \equiv 1 \bmod x_i$ for $1 \leq i < D$ and $mv_D \equiv r \bmod x_D$. (This is why we had to assume the x_i relatively prime). Since $x_D > K + 1$ these $K + 1$ possibilities are distinct. Choose r so that $r/x_D \neq \alpha_D$, $\theta_2, \theta_3 \cdots \theta_K$. Let $a_i = (mv_i - 1)/x_i$ for $1 \leq i < D$, and $a_D = (mv_D - r)/x_D$. These are integers because of the congruence conditions, and clearly (i) and (ii) are satisfied. Now since $x_D \geq x_i$ for $1 \leq i < D$, and since $\sum_1^D v_i \geq D + K$,

$$(K + 1)/x_D + \sum_{i=1}^{D-1} (1/x_i) \leq \sum_1^D (v_i/x_i) < 1$$

implies that

$$\sum_1^D \left(mv_i x_i^{-1} - a_i\right) = \left\{\sum_1^{D-1} 1/x_i\right\} + r/x_D < 1,$$

which is equivalent to (iii).

Suppose $0, X_1 \cdots X_D$ are the vertices of S, and are in $\mathbf{Z}^D$. If $Y_1, Y_2 \cdots Y_K$ are lattice points of S° and $Y_1 = \Sigma_1^D \alpha_i X_i$ with relatively prime x_i, and if $\Sigma_1^D \alpha_i > 1 - \varepsilon_{D+K-1}$ then let θ_j, $2 \leq j \leq K$ be the X_D coefficient of Y_j. Apply Lemma 4.3 and let $Y_{K+1} = mY_1 - \Sigma_1^D a_i X_i$. Then $Y_{K+1} \in S^\circ$ and different from $Y_1 \cdots Y_K$ by Lemma 4.3. The case $K = 1$ of these conclusions is Theorem 4.1.

Remark. The estimate due to Borosh is not best possible. It would be interesting to know the maximum value of $\Sigma_1^D v_i/x_i$ subject to $0 < v_i/x_i$, $\Sigma_1^D v_i/x_i < 1$ and $\Sigma_1^D v_i = D + K - 1$.

References

1. H. F. Blichfeldt, *A new principle in the geometry of numbers, with some applications*, Trans. Amer. Math. Soc., **15** (1914), 227–235.

2. ______, *Notes on geometry of numbers*, Bull. Amer. Math. Soc., **27** (1921), 150–153.

3. C. Caratheodory. (See H. G. Eggleston, Convexity, Cambridge University Press (1958), p. 35.)

4. J. G. van der Corput, *Verallgemeinerung einer Mordellschen Beweismethode in der Geometrie der Zahlen II*, Acta Arith., **2** (1936), 145–146.

5. Curtis, *On Kellogg's Diophantine Problem*, Amer. Math. Monthly, **29** (1922), 380–387.

6. P. Erdös, *Az* $\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} = \frac{a}{b}$ *egyenletegész számú megoldásairól,* Math. Lapok, (1950), 192–210.

7. C. G. Lekkerkerker, *Geometry of Numbers*, p. 35, 40, 45, and 55, Wolters-Noordhoff, Groningen, 1969.

8. K. Mahler, *Ein Übertragungsprinzip für konvexe Körper*, Časopis Pěst. Mat. Fyz., **68** (1939), 93–102.

9. D. B. Sawyer, *The lattice determinants of asymmetric convex regions*, J. London Math. Soc., **29** (1954), 251–254.

10. P. R. Scott, *On convex lattice polygons*, Bull. Austral. Math. Soc., **15** (1976), 395–399.

11. J. Zaks, M. A. Perles and J. M. Wills, *On the lattice polytopes having interior lattice points*, Elemente der Math., **37**, No. 2 (1982), 44–46.

Received May 28, 1981 and in revised form November 2, 1981.

Texas A & M University
College Station, TX 77843

Pacific Journal of Mathematics

Vol. 105, No. 1 September, 1983