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## **SOME POINCARÉ SERIES RELATED TO IDENTITIES OF $2 \times 2$ MATRICES**

PATRICK RONALD HALPIN

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A partial solution to a problem of Procesi has recently been given by Formanek, Halpin, Li by determining the Poincaré series of the ideal of two variable identities of  $M_2(k)$ . Two related results are obtained in this article.

A weak identity of  $M_n(k)$  is a polynomial which vanishes identically on  $sl_n$ , the subspace of  $M_n(k)$  of matrices of trace zero. We show that the Poincaré series of the ideal of two variable weak identities of  $M_2(k)$  is rational. In addition it is shown that the ideal of identities of upper triangular  $2 \times 2$  matrices in an arbitrary finite number of variables has a rational Poincaré series. As an application we are able to determine this ideal precisely.

**Introduction.** Let  $S = K \langle x_1, \dots, x_n \rangle$  be the free associative algebra over  $k$  where  $k$  is any field of characteristic zero.  $S$  is naturally graded by giving  $x_1$  degree  $(1, 0, \dots, 0)$ ,  $x_2$  degree  $(0, 1, \dots, 0)$ , etc. Denote by  $S_{(i_1, \dots, i_n)}$  the subspace of  $S$  generated by monomials of degree  $(i_1, \dots, i_n)$ . If  $A$  is a homogeneously generated ideal of  $S$  then we associate a series to  $A$ , called the Poincaré series of  $A$ , via

$$P(A) = \sum_{i_1, \dots, i_n \geq 0} a(i_1, \dots, i_n) s_1^{i_1} s_2^{i_2} \cdots s_n^{i_n}$$

where  $a(i_1, \dots, i_n) = \dim_k(A \cap S_{(i_1, \dots, i_n)})$ . In [1] Formanek, Halpin, Li showed that the Poincaré series of the ideal of two variables identities of  $M_2(k)$  is a rational function in  $s_1$  and  $s_2$ . In this article we obtain two related results.

A weak identity of  $M_n(k)$  is a polynomial which vanishes upon substitution of elements of  $sl_n(k)$ , where  $sl_n(k)$  denotes the subspace of  $M_n(k)$  of matrices of trace zero. The notion of a weak identity was introduced by Razmyslov [2] in connection with the study of central polynomials. Let  $T_2^W(x_1, x_2)$  denote the ideal of  $k \langle x_1, x_2 \rangle$  of weak identities of  $M_2(k)$ . In Section 1 we determine  $P(T_2^W(x_1, x_2))$  and find that it is again a rational function in  $s_1$  and  $s_2$ .

In §2 we consider the identities of the subalgebra of  $M_2(k)$  consisting of upper triangular matrices. By restricting to upper triangular matrices we are able to obtain results more complete than those obtained in [1]. We

calculate the Poincaré series of the ideal of identities of upper triangular  $2 \times 2$  matrices in an arbitrary finite number of variables. As an application the ideal of identities of upper triangular  $2 \times 2$  matrices is determined explicitly.

**1. Weak identities of  $M_2(k)$ .** Let  $T_2^W(x_1, x_2)$  denote the collection of two variable weak identities of  $M_2(k)$  where  $k$  is a field of characteristic zero. It is easy to see that  $T_2^W(x_1, x_2)$  is an ideal of  $k\langle x_1, x_2 \rangle$ , although it is not a  $T$ -ideal in the usual sense. As in the case of the identities of  $M_n(k)$ , the ideal of weak identities  $M_n(k)$  is homogeneously generated. The goal of this section is to determine  $P(T_2^W(x_1, x_2))$ .

Let

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{22} & -X_{11} \end{pmatrix}, \quad Y = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & -Y_{11} \end{pmatrix}$$

be  $2 \times 2$  generic matrices of trace zero. The  $x_{ij}, y_{ij}$  are commuting indeterminates. Define  $R = k[X, Y]$  as the algebra generated over  $k$  by  $X$  and  $Y$ .  $R$  may be graded by assigning  $X$  degree  $(1, 0)$  and  $Y$  degree  $(0, 1)$ . Let  $A = k[x_{ij}, y_{ij}]$  be the commutative polynomial ring generated over  $k$  by the six indeterminates  $x_{ij}, y_{ij}$ .  $A$  may be graded by assigning each  $x_{ij}$  degree  $(1, 0)$  and each  $y_{ij}$  degree  $(0, 1)$ .

The following lemma, which is analogous to a well known result on identities of  $M_n(k)$ , is clear.

LEMMA 1. *The sequence*

$$0 \rightarrow T_2^W(x_1, x_2) \rightarrow k\langle x_1, x_2 \rangle \xrightarrow{\pi} k[X, Y] \rightarrow 0,$$

where  $\pi(x_1) = X$  and  $\pi(x_2) = Y$ , is an exact sequence of graded  $k$ -modules.

By  $D, T$  we denote determinant, trace respectively. We define

$$\begin{aligned} B &= k[D(X), D(Y), T(XY)] \\ &= k[x_{11}^2 + x_{12}x_{21}, y_{11}^2 + y_{12}y_{21}, x_{12}y_{21} + x_{21}y_{12} + 2x_{11}y_{11}] \end{aligned}$$

$B$  inherits a grading as a homogeneously generated submodule of  $A$ .

LEMMA 2.  *$B$  is a commutative polynomial ring over  $k$  in  $D(X), D(Y), T(XY)$ .*

*Proof.* This is easily seen by specializing  $x_{12} = x_{21} = 0$ .

The proof of the following lemma is routine and is therefore omitted.

LEMMA 3.  $I, X, Y, XY$  are linearly independent over  $A$  and so are linearly independent over  $B$ .

THEOREM 4.  $R = BI \oplus BX \oplus BY \oplus BXY$ , a direct sum of  $k$ -spaces.

*Proof.* The following relations are easily verified and show that  $BI \oplus BX \oplus BY \oplus BYX \subseteq R$ :

$$X^2 = -D(X)I,$$

$$Y^2 = -D(Y)I,$$

$$XY + YX = T(XY)I.$$

For the other inclusion note that  $B$  is the ring generated by  $D(X), D(Y), T(XY)$ . Therefore the three relations above show that  $BI \oplus BX \oplus BY \oplus BXY$  is a ring containing  $X, Y$  and hence  $R \subseteq BI \oplus BX \oplus BY \oplus BXY$ .

The following easy lemma, used in [1], will be used extensively in the article.

LEMMA 5. Let  $M$  and  $N$  be homogeneous  $k$ -submodules of  $M_2(k[x_{ij}, y_{ij}])$ .

(1) If  $M \oplus N$  is a direct sum then  $P(M \oplus N) = P(M) + P(N)$ .

(2) If  $U \in M_2(k[x_{ij}, y_{ij}])$  is a homogeneous nonzero divisor of degree  $(p, q)$  then  $P(MU) = s_1^p s_2^q P(M)$ .

THEOREM 6. We have

$$(1) \quad P(R) = \frac{1}{(1 - s_1)(1 - s_2)(1 - s_1 s_2)}$$

and

$$(2) \quad P(T_2^w(x_1, x_2)) = \frac{s_1 s_2 (s_1 + s_2 - s_1 s_2)}{(1 - s_1)(1 - s_2)(1 - s_2 s_2)(1 - s_1 - ns_2)}.$$

*Proof.* By Lemma 2  $B$  is a commutative polynomial ring in  $D(X), D(Y), T(XY)$  of degrees  $(2, 0), (0, 2), (1, 1)$  respectively. Therefore

$$\begin{aligned} P(B) &= P(k[D(X), D(Y), T(XY)]) \\ &= (1 + s_1^2 + s_1^4 + \dots)(1 + s_2^2 + s_2^4 + \dots)(1 + s_1 s_2 + s_1^2 s_2^2 + \dots) \\ &= \frac{1}{(1 - s_1^2)(1 - s_2^2)(1 - s_1 s_2)}. \end{aligned}$$

Therefore

$$\begin{aligned} P(R) &= P(BI \oplus BX \oplus BY \oplus BXY) \\ &= P(B) + P(BX) + P(BY) + P(BXY) = (1 + s_1)(1 + s_2)P(B) \\ &= \frac{1}{(1 - s_1)(1 - s_2)(1 - s_1s_2)}. \end{aligned}$$

For (2) we note that by the exact sequence of Lemma 1

$$\begin{aligned} P(T_2^W(x_1, x_2)) &= P(k\langle x_1, x_2 \rangle) - P(R) \\ &= \frac{1}{1 - s_1 - s_2} - \frac{1}{(1 - s_1)(1 - s_2)(1 - s_1s_2)} \\ &= \frac{s_1s_2(s_1 + s_2 - s_1s_2)}{(1 - s_1)(1 - s_2)(1 - s_1s_2)(1 - s_1 - s_2)}. \end{aligned}$$

**2. Upper triangular matrices.** The object of study in this section is the ideal of identities of upper triangular  $2 \times 2$  matrices.

We first establish the notation that will be used in this section. Let  $A = k[x_{ij}^{(k)}; 1 \leq i \leq j \leq 2, 1 \leq k \leq n]$  be the commutative polynomial ring generated over  $k$  by the  $3n$  variables  $x_{ij}^{(k)}$ . By  $T_2^U(x_1, \dots, x_n)$  we mean the ideal of identities of upper triangular  $2 \times 2$  matrices in  $x_1, \dots, x_n$  with coefficients in  $k$ . Now let  $X_1, \dots, X_n$  be upper triangular  $2 \times 2$  generic matrices where

$$X_i = \begin{pmatrix} x_{11}^{(i)} & x_{12}^{(i)} \\ 0 & x_{22}^{(i)} \end{pmatrix}.$$

$R = k[X_1, \dots, X_n]$  denotes the algebra generated over  $k$  by  $X_1, \dots, X_n$ .

We begin with a version of the well known diagonalization technique.

**LEMMA 7.**  $R = k[X_1, X_2, \dots, X_n]$  is isomorphic (as  $k$ -algebras) to  $k[X, X_2, \dots, X_n]$  where

$$X = \begin{pmatrix} x_{11}^{(1)} & 0 \\ 0 & x_{22}^{(1)} \end{pmatrix}.$$

*Proof.* The matrix  $X_1$  is diagonalizable by some matrix  $T$  which may be taken upper triangular. Then

$$R \cong T^{-1}RT = k[X, T^{-1}X_2T, \dots, T^{-1}X_nT] \cong k[X, X_2, \dots, X_n].$$

In view of Lemma 7 from now on we will take  $R = k[X_1, \dots, X_n]$  where  $X_1 = X$ .

We grade  $k\langle x_1, \dots, x_n \rangle$  as in the previous section. Similarly  $A = k[x_{ij}^{(k)}; 1 \leq i \leq j \leq 2, 1 \leq k \leq n]$  and  $B = k[x_{ii}^{(k)}; i = 1, 2, 1 \leq k \leq n]$  are graded by giving each  $x_{ij}^{(1)}$  degree  $(1, 0, \dots, 0)$ , each  $x_{ij}^{(2)}$  degree  $(0, 1, \dots, 0)$ , etc. Also  $R$  is graded by assigning  $X_1$  degree  $(1, 0, \dots, 0)$ ,  $X_2$  degree  $(0, 1, \dots, 0)$ , etc.

With these gradings we state an obvious lemma which is analogous to Lemma 1.

LEMMA 8. *The sequence below, with the obvious maps, is an exact sequence of graded  $k$ -modules:*

$$0 \rightarrow T_2^U(x_1, \dots, x_n) \rightarrow k\langle x_1, \dots, x_n \rangle \rightarrow R \rightarrow 0.$$

The main theorem of this section is the evaluation of  $P(T_2^U(x_1, \dots, x_n))$  which will be proved by induction on  $n$ . In order to start the induction at  $n = 2$  we first calculate  $P(R_0)$  where  $R_0 = k[X_1, X_2]$ .

LEMMA 9. *The commutator ideal  $[R_0, R_0]$  equals*

$$k[x_{11}^{(1)}, x_{22}^{(1)}, x_{11}^{(2)}, x_{22}^{(2)}] \cdot [X_1, X_2].$$

*Proof.*  $[R_0, R_0]$  is the ideal of  $R_0$  generated by

$$[X_1, X_2] = \begin{pmatrix} 0 & (x_{11}^{(1)} - x_{22}^{(1)})x_{12}^{(2)} \\ 0 & 0 \end{pmatrix}.$$

Now notice that

$$X_i[X_1, X_2] = x_{11}^{(i)}[X_1, X_2]$$

and

$$[X_1, X_2]X_i = x_{22}^{(i)}[X_1, X_2].$$

Therefore

$$[R_0, R_0] \subseteq k[x_{11}^{(1)}, x_{22}^{(1)}, x_{11}^{(2)}, x_{22}^{(2)}] \cdot [X_1, X_2].$$

For the reverse inclusion if  $(x_{11}^{(1)})^a (x_{22}^{(1)})^b (x_{11}^{(2)})^c (x_{22}^{(2)})^d$  is any monomial in  $k[x_{11}^{(1)}, x_{22}^{(1)}, x_{11}^{(2)}, x_{22}^{(2)}]$  then one sees easily that

$$\begin{aligned} & (x_{11}^{(1)})^a (x_{22}^{(1)})^b (x_{11}^{(2)})^c (x_{22}^{(2)})^d [X_1, X_2] \\ &= X_1^a X_2^c [X_1, X_2] X_1^b X_2^d \in [R_0, R_0]. \end{aligned}$$

LEMMA 10.

$$P([R_0, R_0]) = \frac{s_1 s_2}{(1 - s_1)^2 (1 - s_2)^2}.$$

*Proof.* Since  $x_{11}^{(1)}, x_{22}^{(1)}, x_{11}^{(2)}, x_{22}^{(2)}$  have degrees  $(1, 0), (1, 0), (0, 1), (0, 1)$  respectively, we have

$$\begin{aligned} P([R_0, R_0]) &= P(k[x_{11}^{(1)}, x_{22}^{(1)}, x_{11}^{(2)}, x_{22}^{(2)}] \cdot [X_1, X_2]) \\ &= s_1 s_2 P(k[x_{11}^{(1)}, x_{22}^{(1)}, x_{11}^{(2)}, x_{22}^{(2)}]) \\ &= s_1 s_2 (1 + s_1 + s_1^2 + \cdots)^2 (1 + s_2 + s_2^2 + \cdots)^2 \\ &= \frac{s_1 s_2}{(1 - s_1)^2 (1 - s_2)^2}. \end{aligned}$$

LEMMA 11.

$$P(R_0) = \frac{1 - s_1 - s_2 + 2s_1 s_2}{(1 - s_1)^2 (1 - s_2)^2}.$$

*Proof.* Since  $R_0/[R_0, R_0] \cong k[x_1, x_2]$ , a commutative polynomial ring, it follows that as  $k$ -spaces

$$R_0 \cong_k [R_0, R_0] \oplus_k k[x_1, x_2].$$

Therefore

$$\begin{aligned} P(R_0) &= P([R_0, R_0]) + P(k[x_1, x_2]) \\ &= \frac{s_1 s_2}{(1 - s_1)^2 (1 - s_2)^2} + \frac{1}{(1 - s_1)(1 - s_2)} = \frac{1 - s_1 - s_2 + 2s_1 s_2}{(1 - s_1)^2 (1 - s_2)^2}. \end{aligned}$$

In order to calculate  $P(T_2^U(x_1, \dots, x_n))$  it suffices to calculate  $P(k[X_1, \dots, X_n])$ . We proceed by induction on  $n$ , having established the case  $n = 2$ . The following lemma will be used to execute the inductive step.

LEMMA 12. *The ideal  $[X_1, R]$  of  $R$  equals  $[X_1, X_2]B \oplus_k [X_1, X_3]B \oplus_k \cdots \oplus_k [X_1, X_n]B$ .*

*Proof.* The ideal  $[X_1, R]$  is the ideal of  $R$  generated by

$$\begin{aligned} [X_1, X_2] &= \begin{pmatrix} 0 & (x_{11}^{(1)} - x_{22}^{(1)})x_{12}^{(2)} \\ 0 & 0 \end{pmatrix} \\ &\vdots \\ [X_1, X_n] &= \begin{pmatrix} 0 & (x_{11}^{(1)} - x_{22}^{(1)})x_{12}^{(n)} \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Notice that

$$X_i[X_1, X_j] = x_{11}^{(i)}[X_1, X_j]$$

and

$$[X_1, X_j]X_i = x_{22}^{(i)}[X_1, X_j].$$

The lemma now follows easily as in Lemma 9. Of course the sum above is direct since the  $x_{12}^{(k)}, 1 \leq k \leq n$ , are distinct indeterminates.

As an immediate consequence of Lemma 12 we may compute  $P([X_1, R])$ .

LEMMA 13.

$$P([X_1, R]) = \frac{s_1s_2 + s_1s_3 + \cdots + s_1s_n}{(1 - s_1)^2(1 - s_2)^2 \cdots (1 - s_n)^2}.$$

THEOREM 14.

$$P(R) = \frac{(2(1 - s_1) \cdots (1 - s_n)) + (s_1 + \cdots + s_n) - 1}{(1 - s_1)^2(1 - s_2)^2 \cdots (1 - s_n)^2}.$$

*Proof.* We induct on  $n$ . The case  $n = 2$  is Lemma 11 so we assume  $n \geq 3$  and that the theorem is true for  $n - 1$  variables.

$R$  has the following decomposition as a  $k$ -space:

$$R \cong_k R/[X_1, R] \oplus_k [X_1, R] \cong_k \bigoplus_{i=0}^{\infty} X_1^i k[X_2, \dots, X_n] \oplus_k [X_1, R].$$

Therefore,

$$\begin{aligned} P(R) &= P\left(\bigoplus_{i=0}^{\infty} X_1^i k[X_2, \dots, X_n]\right) + P([X_1, R]) \\ &= (1 + s_1 + s_1^2 + \cdots)P(k[X_2, \dots, X_n]) + P([X_1, R]). \end{aligned}$$

By the inductive hypothesis  $P(k[X_2, \dots, X_n])$  equals

$$\frac{(2(1 - s_2) \cdots (1 - s_n)) + (s_2 + \cdots + s_n) - 1}{(1 - s_2)^2(1 - s_3)^2 \cdots (1 - s_n)^2},$$

and by Lemma 13  $P([X_1, R])$  equals

$$\frac{s_1s_2 + \cdots + s_1s_n}{(1 - s_1)^2 \cdots (1 - s_n)^2}.$$



Thus

$$\begin{aligned} P(R) &= \frac{(2(1-s_2) \cdots (1-s_n)) + (s_2 + \cdots + s_n) - 1}{(1-s_1)(1-s_2)^2(1-s_3)^2 \cdots (1-s_n)^2} \\ &\quad + \frac{s_1 s_2 + \cdots + s_1 s_n}{(1-s_1)^2 \cdots (1-s_n)^2} \\ &= \frac{(2(1-s_1) \cdots (1-s_n)) + (s_1 + \cdots + s_n) - 1}{(1-s_1)^2(1-s_2)^2 \cdots (1-s_n)^2}. \end{aligned}$$

We now prove the main result of this section.

**THEOREM 15.**

$$P(T_2^U(x_1, \dots, x_n)) = \frac{((1-s_1) \cdots (1-s_n) - (1-s_1 - \cdots - s_n))^2}{(1-s_1 - \cdots - s_n)(1-s_1)^2 \cdots (1-s_n)^2}.$$

*Proof.* By the exact sequence of Lemma 8 we have

$$\begin{aligned} P(T_2^U(x_1, \dots, x_n)) &= P(k\langle x_1, \dots, x_n \rangle) - P(k[X_1, \dots, X_n]) \\ &= \frac{1}{1-s_1 - \cdots - s_n} - \frac{2((1-s_1) \cdots (1-s_n)) + (s_1 + \cdots + s_n) - 1}{(1-s_1)^2 \cdots (1-s_n)^2} \\ &= \frac{((1-s_1) \cdots (1-s_n) - (1-s_1 - \cdots - s_n))^2}{(1-s_1 - \cdots - s_n)(1-s_1)^2 \cdots (1-s_n)^2}. \end{aligned}$$

As an application of Theorem 15 we now give a precise description of  $T_2^U(x_1, \dots, x_n)$ . Let  $T_1(x_1, \dots, x_n)$  denote the commutator ideal of  $k\langle x_1, \dots, x_n \rangle$ . In other words,  $T_1(x_1, \dots, x_n)$  is the ideal of  $k\langle x_1, \dots, x_n \rangle$  such that the sequence

$$0 \rightarrow T_1(x_1, \dots, x_n) \rightarrow k\langle x_1, \dots, x_n \rangle \rightarrow k[x_1, \dots, x_n] \rightarrow 0$$

is exact. It follows that

$$\begin{aligned} P(T_1(x_1, \dots, x_n)) &= \frac{1}{1-s_1 - \cdots - s_n} - \frac{1}{(1-s_1) \cdots (1-s_n)} \\ &= \frac{(1-s_1) \cdots (1-s_n) - (1-s_1 - \cdots - s_n)}{(1-s_1 - \cdots - s_n)(1-s_1) \cdots (1-s_n)}. \end{aligned}$$

We will show that  $T_2^U(x_1, \dots, x_n) = (T_1(x_1, \dots, x_n))^2$ . To show one inclusion is very easy. It then suffices to show that both members have the same Poincaré series. To calculate the Poincaré series of  $(T_1(x_1, \dots, x_n))^2$  we need to make use of a combinatorial lemma, due to Formanek. We sketch a proof of the lemma.

LEMMA 16. (*Formanek*) Let  $I$  and  $J$  be homogeneously generated ideals of  $k\langle x_1, \dots, x_n \rangle$ . Then  $P(IJ) = P(I)P(J)(1 - s_1 - \dots - s_n)$ .

*Proof.* One first shows, using only elementary arguments, that  $I$  and  $J$  are free as left ideals on homogeneous generators. Let  $\alpha(i_1, \dots, i_n)$  equal the number of free generators of  $I$  considered as a left ideal of degree  $(i_1, \dots, i_n)$ . Define

$$G(I) = \sum_{i_1, \dots, i_n \geq 0} \alpha(i_1, \dots, i_n) s_1^{i_1} s_2^{i_2} \cdots s_n^{i_n}.$$

Similarly define  $G(J)$  and  $G(IJ)$ . Then  $G(IJ) = G(I)G(J)$  and  $P(I) = G(I)/(1 - s_1 - \dots - s_n)$ . The lemma follows.

THEOREM 17.  $T_2^U(x_1, \dots, x_n) = (T_1(x_1, \dots, x_n))^2$ .

*Proof.* We first show that  $(T_1(x_1, \dots, x_n))^2 \subseteq T_2^U(x_1, \dots, x_n)$ . Any element of  $(T_1(x_1, \dots, x_n))^2$  is a sum of terms of the form  $r_1[x_i, x_j]r_2[x_k, x_l]r_3$  where  $1 \leq i, j, k, l \leq n$  and  $r_1, r_2, r_3 \in k\langle x_1, \dots, x_n \rangle$ . The commutator of two upper triangular matrices is strictly upper triangular. Therefore each term of the form above is an identity for  $R$  since any finite product of upper triangular  $2 \times 2$  matrices where at least two of the factors are strictly upper triangular is zero. Therefore  $(T_1(x_1, \dots, x_n))^2 \subseteq T_2^U(x_1, \dots, x_n)$ .

As mentioned above it now suffices to show that  $(T_1(x_1, \dots, x_n))^2$  and  $T_2^U(x_1, \dots, x_n)$  have the same Poincaré series. By Lemma 16

$$\begin{aligned} P\left((T_1(x_1, \dots, x_n))^2\right) &= (1 - s_1 - \dots - s_n)(P(T_1(x_1, \dots, x_n)))^2 \\ &= (1 - s_1 - \dots - s_n) \left( \frac{(1 - s_1) \cdots (1 - s_n) - (1 - s_1 - \dots - s_n)}{(1 - s_1 - \dots - s_n)(1 - s_1) \cdots (1 - s_n)} \right)^2 \\ &= P(T_2^U(x_1, \dots, x_n)). \end{aligned}$$

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