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## **SUFFICIENCY AND RELATIVE ENTROPY IN $*$ -ALGEBRAS WITH APPLICATIONS IN QUANTUM SYSTEMS**

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# SUFFICIENCY AND RELATIVE ENTROPY IN $\ast$ -ALGEBRAS WITH APPLICATIONS IN QUANTUM SYSTEMS

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**The sufficiency and weak sufficiency in  $\ast$ -algebras are discussed. Some properties are studied concerning the relative entropy and the sufficiency for invariant states and KMS states in  $W^\ast$ - and  $C^\ast$ -dynamical systems.**

**Introduction.** The concept of sufficiency is very important in mathematical statistics. The abstract measure theoretic investigation of sufficient statistics was initiated by Halmos and Savage [13]. Kullback and Leibler [19] gave the characterization of sufficiency in terms of the information (i.e., the classical relative entropy). Umegaki [33, 34] studied the sufficiency and the relative entropy in the noncommutative case of semi-finite von Neumann algebras.

Araki [4, 5] extended the relative entropy to the case for normal positive linear functionals of general von Neumann algebras and showed its several properties. Furthermore Uhlmann [32] showed the general WYDL concavity using a quadratic interpolation theory and defined the relative entropy of positive linear functionals of arbitrary  $\ast$ -algebras.

In the previous paper [14], we discussed the sufficiency and the relative entropy in von Neumann algebras and gave the characterizations of invariant states and KMS states with respect to the modular automorphism group of a faithful normal state.

In this paper, we further develop the sufficiency and the relative entropy in  $\ast$ -algebras. In §1, we introduce besides the sufficiency another notion of weak sufficiency and establish the relation between them. In §2, we deal with the weak sufficiency of positive linear maps between  $\ast$ -algebras. In §3, we mention the Araki's and Uhlmann's relative entropies which are equal in the von Neumann algebra case. We further give a formula of relative entropy for states of  $C^\ast$ -algebras. In §4, we establish some properties of invariant states and KMS states in  $W^\ast$ -dynamical systems and  $C^\ast$ -dynamical systems through the relative entropy and the sufficiency. The theorems there improve or extend the results obtained in [14]. Finally we give an application to the Gibbs states of quantum lattice systems.

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**1. Sufficiency and weak sufficiency of \*-subalgebras.** In this paper, we shall assume that all \*-algebras,  $C^*$ -algebras and von Neumann algebras have the unity  $I$  and their \*-subalgebras always contain  $I$ . Let  $\mathcal{A}$  be a \*-algebra and  $\mathfrak{S}$  be the set of all states of  $\mathcal{A}$ .

**DEFINITION 1.1.** A \*-subalgebra  $\mathfrak{B}$  of  $\mathcal{A}$  is said to be *sufficient* for  $S \subset \mathfrak{S}$  if there exists a projection  $\varepsilon$  of  $\mathcal{A}$  onto  $\mathfrak{B}$  such that

- (i)  $\varepsilon(A^*) = \varepsilon(A)^*$  for all  $A \in \mathcal{A}$ ,
- (ii)  $\varepsilon(A)^*\varepsilon(A) \leq \varepsilon(A^*A)$  for all  $A \in \mathcal{A}$ ,
- (iii)  $\varepsilon(B_1AB_2) = B_1\varepsilon(A)B_2$  for all  $A \in \mathcal{A}$  and  $B_1, B_2 \in \mathfrak{B}$ ,
- (iv)  $\varphi = \varphi \circ \varepsilon$  for all  $\varphi \in S$ .

We here call a projection  $\varepsilon$  of  $\mathcal{A}$  onto  $\mathfrak{B}$  satisfying (i)–(iii) a *conditional expectation* of  $\mathcal{A}$  onto  $\mathfrak{B}$ . If  $\mathcal{A}$  is a  $C^*$ -algebra and  $\mathfrak{B}$  is a  $C^*$ -subalgebra, then a conditional expectation of  $\mathcal{A}$  onto  $\mathfrak{B}$  is nothing but a norm one projection of  $\mathcal{A}$  onto  $\mathfrak{B}$  (cf. [31]).

We first give some examples of sufficiency in von Neumann algebras. Let  $\mathfrak{N}$  be a von Neumann algebra and  $\mathfrak{S}$  be the set of all normal states of  $\mathfrak{N}$ . The definition in [14] of sufficiency of a von Neumann subalgebra for  $S \subset \mathfrak{S}$  is somewhat different from Definition 1.1. However these are equivalent if  $S$  contains a faithful normal state (this is the case dealt in [14]).

**EXAMPLE 1.2.** Let  $\varphi \in \mathfrak{S}$  be faithful and  $\sigma_t^\varphi$  be its modular automorphism group (cf. [28]). We showed in [14] that the centralizer of  $Z_\varphi$  of  $\varphi$  is sufficient for the set of all  $\sigma_t^\varphi$ -invariant states in  $\mathfrak{S}$  and the center  $\mathfrak{Z} = \mathfrak{N} \cap \mathfrak{N}'$  is sufficient for the set of all states in  $\mathfrak{S}$  satisfying the KMS condition with respect to  $\sigma_t^\varphi$  (at  $\beta = 1$ ).

**EXAMPLE 1.3.** Assume that  $\mathfrak{N}$  is semi-finite with a faithful normal semi-finite trace  $\tau$  of  $\mathfrak{N}$ . For each  $\varphi \in \mathfrak{S}$ , there exists a unique positive self-adjoint operator  $\rho_\varphi = d\varphi/d\tau$  such that  $\varphi(A) = \tau(\rho_\varphi A)$  for all  $A \in \mathfrak{N}$ . For any set  $S \subset \mathfrak{S}$ , the von Neumann subalgebra  $\mathfrak{M}$  generated by  $\{d\varphi/d\tau : \varphi \in S\}$  is proved to be sufficient for  $S$  (see [16, p. 72]).

**EXAMPLE 1.4.** Let  $\{\mathfrak{N}, G, \alpha\}$  be a  $W^*$ -dynamical system where  $g \mapsto \alpha_g$  is a representation of a group  $G$  in  $\text{Aut}(\mathfrak{N})$ . Let  $\mathfrak{N}^\alpha$  be the fixed point subalgebra of  $\alpha$  and  $\mathfrak{S}_\alpha$  be the set of all  $\alpha$ -invariant states in  $\mathfrak{S}$ . Then the

result of Kovács and Szűcs [18] asserts that if  $\mathfrak{N}$  is  $G$ -finite, i.e.,  $\varphi(A^*A) = 0$  for all  $\varphi \in \mathfrak{S}_\alpha$  implies  $A = 0$ , then  $\mathfrak{N}^\alpha$  is sufficient for  $\mathfrak{S}_\alpha$ .

For  $*$ -subalgebras  $\mathfrak{B}$  of  $\mathcal{A}$ , the existence of a conditional expectation of  $\mathcal{A}$  onto  $\mathfrak{B}$  is usually a rather strict condition. In the sequel, we introduce another weak notion of sufficiency by using cyclic representations of  $\mathcal{A}$ . Unbounded  $*$ -representations of  $*$ -algebras were studied in [23]. A *\*-representation*  $\pi$  of  $\mathcal{A}$  on a Hilbert space  $\mathcal{H}$  is a map of  $\mathcal{A}$  into linear operators all defined on a common dense domain  $D(\pi) \subset \mathcal{H}$  which satisfies  $\pi(I) = I$  and

(i)  $\pi(\alpha A + \beta B)\Phi = \alpha\pi(A)\Phi + \beta\pi(B)\Phi$  for all  $A, B \in \mathcal{A}$ ,  $\alpha, \beta \in \mathbb{C}$  and  $\Phi \in D(\pi)$ ,

(ii)  $\pi(A)D(\pi) \subset D(\pi)$  for all  $A \in \mathcal{A}$  and  $\pi(A)\pi(B)\Phi = \pi(AB)\Phi$  for all  $A, B \in \mathcal{A}$  and  $\Phi \in D(\pi)$ ,

(iii)  $\langle \Phi, \pi(A)\Psi \rangle = \langle \pi(A^*)\Phi, \Psi \rangle$  for all  $\Phi, \Psi \in D(\pi)$ , i.e.,  $\pi(A^*) \subset \pi(A)^*$  for all  $A \in \mathcal{A}$ .

The *unbounded commutant*  $\pi(\mathcal{A})^c$  of  $\pi(\mathcal{A})$  consists of all linear operators  $T: D(\pi) \rightarrow \mathcal{H}$  such that

$$\langle \Phi, T\pi(A)\Psi \rangle = \langle \pi(A^*)\Phi, T\Psi \rangle, \quad A \in \mathcal{A}, \Phi, \Psi \in D(\pi).$$

The *commutant*  $\pi(\mathcal{A})'$  of  $\pi(\mathcal{A})$  is the set of all bounded operators  $T$  on  $\mathcal{H}$  such that  $T \upharpoonright D(\pi) \in \pi(\mathcal{A})^c$ . For each positive linear functional  $\varphi$  of  $\mathcal{A}$ , the GNS construction gives rise to a cyclic representation  $\{\mathcal{H}_\varphi, \pi_\varphi, \Omega_\varphi\}$  of  $\mathcal{A}$  induced by  $\varphi$  which is unique up to unitary equivalence, that is,  $\pi_\varphi$  is a  $*$ -representation of  $\mathcal{A}$  on a Hilbert space  $\mathcal{H}_\varphi$  with  $\Omega_\varphi \in D(\pi_\varphi)$  such that

$$D(\pi_\varphi) = \pi_\varphi(\mathcal{A})\Omega_\varphi, \quad \mathcal{H}_\varphi = \overline{\pi_\varphi(\mathcal{A})\Omega_\varphi},$$

$$\varphi(A) = \langle \Omega_\varphi, \pi_\varphi(A)\Omega_\varphi \rangle, \quad A \in \mathcal{A}.$$

If for every  $A \in \mathcal{A}$  there exists a  $c > 0$  with  $A^*A \leq cI$  (particularly if  $\mathcal{A}$  is a  $C^*$ -algebra), then  $\pi_\varphi$  becomes a bounded  $*$ -representation of  $\mathcal{A}$  on  $\mathcal{H}_\varphi$ . We shall use in this paper the following three conditions of absolute continuity.

(1) A positive linear functional  $\psi$  is *absolutely continuous* with respect to  $\varphi$  (we write  $\psi \ll \varphi$ ) if  $\varphi(A^*A) = 0$  implies  $\psi(A^*A) = 0$ .

(2) A linear functional  $\psi$  is *strongly absolutely continuous* with respect to  $\varphi$  (we write  $\psi < \varphi$ ) if for each sequence  $\{A_n\}$  in  $\mathcal{A}$ ,  $\varphi(A_n^*A_n) \rightarrow 0$  implies  $\psi(BA_n) \rightarrow 0$  for all  $B \in \mathcal{A}$ .

(3) A positive linear functional  $\psi$  is *dominated* by  $\varphi$  if  $\psi \leq c\varphi$  for some  $c > 0$ .

Note that for any positive  $\psi$ , (3) implies (2) and (2) implies (1). If  $\psi$  is a linear functional of  $\mathcal{A}$  with  $\psi < \varphi$ , then by [11, Theorem 1] there exists a

unique  $T \in \pi_\varphi(\mathcal{A})^c$  (we denote by  $T = d\psi/d\varphi$ ) such that

$$\psi(A) = \langle T\Omega_\varphi, \pi_\varphi(A)\Omega_\varphi \rangle, \quad A \in \mathcal{A}.$$

Then  $\psi$  is positive if and only if  $T$  is positive, and moreover  $\psi$  is dominated by  $\varphi$  if and only if  $T$  is bounded so that  $T \in \pi_\varphi(\mathcal{A})'$ . For each  $*$ -subalgebra  $\mathfrak{B}$  of  $\mathcal{A}$ , let  $\mathfrak{B}_\varphi = \pi_\varphi(\mathfrak{B})\Omega_\varphi$  and  $\overline{\mathfrak{B}_\varphi} = \overline{\pi_\varphi(\mathfrak{B})\Omega_\varphi}$ . For every  $A \in \mathcal{A}$ , we define a vector  $P_\varphi(A | \mathfrak{B})$  in  $\overline{\mathfrak{B}_\varphi}$  by

$$P_\varphi(A | \mathfrak{B}) = P_{\overline{\mathfrak{B}_\varphi}}(\pi_\varphi(A)\Omega_\varphi)$$

where  $P_{\overline{\mathfrak{B}_\varphi}}$  is the orthogonal projection onto  $\overline{\mathfrak{B}_\varphi}$ .

**DEFINITION 1.5.** A  $*$ -subalgebra  $\mathfrak{B}$  of  $\mathcal{A}$  is said to be *weakly sufficient* for  $S \subset \mathfrak{S}$  if for each  $A \in \mathcal{A}$  there exists a sequence  $\{B_n\}$  in  $\mathfrak{B}$  such that

$$P_\varphi(A | \mathfrak{B}) = s\text{-}\lim \pi_\varphi(B_n)\Omega_\varphi, \quad \varphi \in S.$$

**THEOREM 1.6.** Assume that there is a finite subset  $\{\varphi_1, \dots, \varphi_k\}$  of  $S$  such that every  $\varphi \in S$  is dominated by  $\rho = \sum_{i=1}^k \varphi_i$ . Then a  $*$ -subalgebra  $\mathfrak{B}$  of  $\mathcal{A}$  is weakly sufficient for  $S$  if and only if  $(d\varphi/d\rho)\mathfrak{B}_\rho \subset \overline{\mathfrak{B}_\rho}$  for every  $\varphi \in S$ .

*Proof.* Suppose that  $\mathfrak{B}$  is weakly sufficient for  $S$ . For each  $A \in \mathcal{A}$ , there exists a sequence  $\{B_n\}$  in  $\mathfrak{B}$  such that

$$P_\varphi(A | \mathfrak{B}) = s\text{-}\lim \pi_\varphi(B_n)\Omega_\varphi, \quad \varphi \in S.$$

Since  $\{\pi_\rho(B_n)\Omega_\rho\}$  is Cauchy, it follows that  $\Psi = s\text{-}\lim \pi_\rho(B_n)\Omega_\rho$  exists in  $\overline{\mathfrak{B}_\rho}$ . If  $B \in \mathfrak{B}$ , then we have

$$\begin{aligned} \|\pi_\rho(A)\Omega_\rho - \pi_\rho(B)\Omega_\rho\|^2 &= \sum_{i=1}^k \|\pi_{\varphi_i}(A)\Omega_{\varphi_i} - \pi_{\varphi_i}(B)\Omega_{\varphi_i}\|^2 \\ &\geq \sum_{i=1}^k \|\pi_{\varphi_i}(A)\Omega_{\varphi_i} - P_{\varphi_i}(A | \mathfrak{B})\|^2 \\ &= \lim \sum_{i=1}^k \|\pi_{\varphi_i}(A)\Omega_{\varphi_i} - \pi_{\varphi_i}(B_n)\Omega_{\varphi_i}\|^2 \\ &= \lim \|\pi_\rho(A)\Omega_\rho - \pi_\rho(B_n)\Omega_\rho\|^2 \\ &= \|\pi_\rho(A)\Omega_\rho - \Psi\|^2, \end{aligned}$$

so that  $P_\rho(A | \mathfrak{B}) = \Psi = s\text{-}\lim \pi_\rho(B_n)\Omega_\rho$ . For each  $\varphi \in S$ , let  $T = d\varphi/d\rho$  and  $\hat{T} = d(\varphi \upharpoonright \mathfrak{B})/d(\rho \upharpoonright \mathfrak{B})$  where the cyclic representation of  $\mathfrak{B}$  induced

by  $\rho \upharpoonright \mathfrak{B}$  is given by  $\{\overline{\mathfrak{B}_\rho}, \pi_\rho \upharpoonright \mathfrak{B}, \Omega_\rho\}$ . Then for every  $B \in \mathfrak{B}$  we have

$$\begin{aligned} \langle T\pi_\rho(B)\Omega_\rho, \pi_\rho(A)\Omega_\rho \rangle &= \varphi(B^*A) = \langle \pi_\varphi(B)\Omega_\varphi, P_\varphi(A | \mathfrak{B}) \rangle \\ &= \lim \varphi(B^*B_n) = \langle \hat{T}\pi_\rho(B)\Omega_\rho, P_\rho(A | \mathfrak{B}) \rangle \\ &= \langle \hat{T}\pi_\rho(B)\Omega_\rho, \pi_\rho(A)\Omega_\rho \rangle. \end{aligned}$$

Since this holds for each  $A \in \mathcal{A}$ , we obtain

$$T\pi_\rho(B)\Omega_\rho = \hat{T}\pi_\rho(B)\Omega_\rho \in \overline{\mathfrak{B}_\rho}, \quad B \in \mathfrak{B},$$

and hence  $T\mathfrak{B}_\rho \subset \overline{\mathfrak{B}_\rho}$ .

Conversely suppose that  $(d\varphi/d\rho)\mathfrak{B}_\rho \subset \overline{\mathfrak{B}_\rho}$  for all  $\varphi \in S$ . Let  $A \in \mathcal{A}$  and take a sequence  $\{B_n\}$  in  $\mathfrak{B}$  such that  $P_\rho(A | \mathfrak{B}) = s\text{-}\lim \pi_\rho(B_n)\Omega_\rho$ . For each  $\varphi \in S$ , since  $\varphi$  is dominated by  $\rho$ , it follows that  $\{\pi_\varphi(B_n)\Omega_\varphi\}$  is Cauchy, so that  $\Phi = s\text{-}\lim \pi_\varphi(B_n)\Omega_\varphi$  exists in  $\overline{\mathfrak{B}_\varphi}$ . If  $B \in \mathfrak{B}$ , then we have

$$\begin{aligned} \langle \pi_\varphi(B)\Omega_\varphi, P_\varphi(A | \mathfrak{B}) \rangle &= \varphi(B^*A) = \langle (d\varphi/d\rho)\pi_\rho(B)\Omega_\rho, P_\rho(A | \mathfrak{B}) \rangle \\ &= \lim \varphi(B^*B_n) = \langle \pi_\varphi(B)\Omega_\varphi, \Phi \rangle, \end{aligned}$$

and hence  $P_\varphi(A | \mathfrak{B}) = \Phi = s\text{-}\lim \pi_\varphi(B_n)\Omega_\varphi$ . Thus  $\mathfrak{B}$  is weakly sufficient for  $S$ .  $\square$

**REMARK.** Theorem 1.6 is considered as the noncommutative extension of Halmos-Savage's theorem [13]. For the proof of "only if" part of Theorem 1.6, we need only  $\varphi < \rho$  for every  $\varphi \in S$ . If  $\pi_\rho$  is a bounded  $*$ -representation (particularly if  $\mathcal{A}$  is a  $C^*$ -algebra), we see that  $(d\varphi/d\rho)\mathfrak{B}_\rho \subset \overline{\mathfrak{B}_\rho}$  is equivalent to  $(d\varphi/d\rho)\Omega_\rho \in \overline{\mathfrak{B}_\rho}$  since  $T\pi_\rho(A)\Omega_\rho = \pi_\rho(A)T\Omega_\rho$  for all  $A \in \mathcal{A}$  and  $T \in \pi_\rho(\mathcal{A})^\circ$ .

In the following theorem, we state the elementary facts of weak sufficiency which are immediately seen from the definition and Theorem 1.6.

**THEOREM 1.7.** (1) *If a  $*$ -subalgebra  $\mathfrak{B}$  of  $\mathcal{A}$  is weakly sufficient for  $\{\varphi, \psi\}$  and  $\varphi = \psi$  on  $\mathfrak{B}$ , then  $\varphi = \psi$  on  $\mathcal{A}$ .*

*When the assumption in Theorem 1.6 is satisfied, then:*

(2) *If a  $*$ -subalgebra  $\mathfrak{B}$  of  $\mathcal{A}$  is weakly sufficient for  $S$ , then  $\mathfrak{B}$  is weakly sufficient for the convex hull of  $S$ .*

(3) *If a  $*$ -subalgebra  $\mathfrak{B}$  of  $\mathcal{A}$  is weakly sufficient for  $S$  and a  $*$ -subalgebra  $\mathcal{C}$  of  $\mathfrak{B}$  is weakly sufficient for  $\{\varphi \upharpoonright \mathfrak{B} : \varphi \in S\}$ , then  $\mathcal{C}$  is weakly sufficient for  $S$ .*

(4) *If a  $*$ -subalgebra  $\mathfrak{B}$  of a  $C^*$ -algebra  $\mathcal{A}$  is weakly sufficient for  $S$ , then any  $*$ -subalgebra  $\mathcal{C}$  with  $\mathfrak{B} \subset \mathcal{C} \subset \mathcal{A}$  is weakly sufficient for  $S$ .*

**THEOREM 1.8.** (1) *If a  $*$ -subalgebra  $\mathfrak{B}$  of  $\mathcal{A}$  is sufficient for  $S$ , then  $\mathfrak{B}$  is weakly sufficient for  $S$ .*

(2) *Assume that there is a  $\varphi \in S$  such that  $\psi < \varphi$  for all  $\psi \in S$ . Then a  $*$ -subalgebra  $\mathfrak{B}$  of  $\mathcal{A}$  is sufficient for  $S$  if and only if  $\mathfrak{B}$  is weakly sufficient for  $S$  and there exists a conditional expectation  $\varepsilon_\varphi$  of  $\mathcal{A}$  onto  $\mathfrak{B}$  with  $\varphi = \varphi \circ \varepsilon_\varphi$ .*

*Proof.* (1) Let  $\varepsilon$  be a conditional expectation of  $\mathcal{A}$  onto  $\mathfrak{B}$  with  $\varphi = \varphi \circ \varepsilon$  for all  $\varphi \in S$ . If  $A \in \mathcal{A}$ ,  $B \in \mathfrak{B}$  and  $\varphi \in S$ , then we have

$$\begin{aligned} \langle P_\varphi(A | \mathfrak{B}), \pi_\varphi(B) \Omega_\varphi \rangle &= \varphi(A^*B) = \varphi(\varepsilon(A)^*B) \\ &= \langle \pi_\varphi(\varepsilon(A)) \Omega_\varphi, \pi_\varphi(B) \Omega_\varphi \rangle, \end{aligned}$$

and hence  $P_\varphi(A | \mathfrak{B}) = \pi_\varphi(\varepsilon(A)) \Omega_\varphi$ . Thus  $\mathfrak{B}$  is weakly sufficient for  $S$ .

(2) Suppose that  $\mathfrak{B}$  is weakly sufficient for  $S$  and there exists a conditional expectation  $\varepsilon_\varphi$  of  $\mathcal{A}$  onto  $\mathfrak{B}$  with  $\varphi = \varphi \circ \varepsilon_\varphi$ . We show that  $\psi = \psi \circ \varepsilon_\varphi$  for all  $\psi \in S$ . For each  $\psi \in S$ , since  $(d\psi/d\varphi) \Omega_\varphi \in \overline{\mathfrak{B}_\varphi}$  by Theorem 1.6 (Remark), we can choose  $\{B_n\}$  in  $\mathfrak{B}$  such that

$$(d\psi/d\varphi) \Omega_\varphi = s\text{-}\lim \pi_\varphi(B_n) \Omega_\varphi.$$

Then  $\psi = \psi \circ \varepsilon_\varphi$  follows from

$$\begin{aligned} \psi(\varepsilon_\varphi(A)) &= \langle (d\psi/d\varphi) \Omega_\varphi, \pi_\varphi(\varepsilon_\varphi(A)) \Omega_\varphi \rangle \\ &= \lim \langle \pi_\varphi(B_n) \Omega_\varphi, \pi_\varphi(\varepsilon_\varphi(A)) \Omega_\varphi \rangle = \lim \varphi(B_n^* \varepsilon_\varphi(A)) \\ &= \lim \varphi(B_n^* A) = \langle (d\psi/d\varphi) \Omega_\varphi, \pi_\varphi(A) \Omega_\varphi \rangle = \psi(A), \quad A \in \mathcal{A}. \quad \square \end{aligned}$$

**EXAMPLE 1.9.** We recall the usual concept of sufficiency in the classical probability theory (cf. [7, 13]). Let  $(X, \mathfrak{F})$  be a measurable space and  $S$  be a set of probability measures on  $\mathfrak{F}$ . A  $\sigma$ -subalgebra  $\mathcal{G}$  of  $\mathfrak{F}$  is sufficient for  $S$  if and only if for each  $A \in \mathfrak{F}$  there exists a  $\mathcal{G}$ -measurable function  $g$  such that  $g = E_\mu(1_A | \mathcal{G})$  a.e.  $[\mu]$  for every  $\mu \in S$ , where  $E_\mu(1_A | \mathcal{G})$  denotes the conditional expectation of the characteristic function  $1_A$  of  $A$  with respect to  $\mu$  and  $\mathcal{G}$ . Let  $\mathcal{A}$  (resp.  $\mathfrak{B}$ ) be the set of all complex-valued  $\mathfrak{F}$  (resp.  $\mathcal{G}$ )-measurable simple functions. Under the pointwise operations,  $\mathcal{A}$  becomes a  $*$ -algebra and  $\mathfrak{B}$  is a  $*$ -subalgebra of  $\mathcal{A}$ . Each  $\mu \in S$  is naturally regarded as a state of  $\mathcal{A}$ . The cyclic representation  $\{\mathcal{H}_\mu, \pi_\mu, \Omega_\mu\}$  is given as follows:  $\mathcal{H}_\mu = L^2(X, \mathfrak{F}, \mu)$ ,  $\pi_\mu(f)$  is the multiplication operator by  $f \in \mathcal{A}$ , and  $\Omega_\mu = 1$ . Moreover  $\overline{\mathfrak{B}_\mu} = L^2(X, \mathcal{G}, \mu)$  and  $P_\mu(f | \mathfrak{B}) = E_\mu(f | \mathcal{G})$ . Then it is easy to see that if  $S$  is dominated, i.e., there is a measure  $m$  on  $\mathfrak{F}$  with  $\mu \ll m$  for all  $\mu \in S$ , then  $\mathcal{G}$  is sufficient for  $S$  if and only if  $\mathfrak{B}$  is weakly sufficient for  $S$ .

EXAMPLE 1.10. Let  $\mathfrak{N}$  be a von Neumann algebra acting on a Hilbert space  $\mathcal{H}$  with a cyclic and separating vector  $\Omega$  with  $\|\Omega\| = 1$ , and  $\varphi$  be a faithful normal state given by  $\varphi(A) = \langle \Omega, A\Omega \rangle$ . For each von Neumann subalgebra  $\mathfrak{M}$  of  $\mathfrak{N}$ , let  $S$  be the set of all states  $\psi$  defined by  $\psi(A) = \langle T\Omega, A\Omega \rangle$  with  $T \in \mathfrak{N}'_+$ ,  $T\Omega \in \overline{\mathfrak{M}\Omega}$  and  $\|T^{1/2}\Omega\| = 1$ . Then it follows from Theorem 1.6 that  $\mathfrak{M}$  is weakly sufficient for  $S$ . Furthermore Theorem 1.8 shows that  $\mathfrak{M}$  is sufficient for  $S$  if and only if there exists a conditional expectation  $\varepsilon_\varphi$  of  $\mathfrak{N}$  onto  $\mathfrak{M}$  with  $\varphi = \varphi \circ \varepsilon_\varphi$ , which is if and only if  $\mathfrak{M}$  is invariant under the modular automorphism group  $\sigma_t^\varphi$  (cf. [29]).

**2. Weak sufficiency of positive linear maps.** In this section, let  $\mathcal{Q}$  and  $\mathfrak{B}$  be two  $*$ -algebras and  $\gamma: \mathfrak{B} \rightarrow \mathcal{Q}$  be a linear map such that  $\gamma(I) = I$ ,  $\gamma(B^*) = \gamma(B)^*$  and  $\gamma(B)^*\gamma(B) \leq \gamma(B^*B)$  for all  $B \in \mathfrak{B}$ . We also assume that for every  $B \in \mathfrak{B}$  there is a  $c > 0$  with  $B^*B \leq cI$ , which is satisfied if  $\mathfrak{B}$  is a  $C^*$ -algebra. Let  $\mathcal{S}_\mathcal{Q}$  and  $\mathcal{S}_\mathfrak{B}$  be the sets of all states of  $\mathcal{Q}$  and  $\mathfrak{B}$ . Then it is immediate that  $\varphi \in \mathcal{S}_\mathcal{Q}$  implies  $\varphi \circ \gamma \in \mathcal{S}_\mathfrak{B}$ . For each  $\varphi \in \mathcal{S}_\mathcal{Q}$  and  $A \in \mathcal{Q}$ , define a linear functional  $\varphi_A$  of  $\mathcal{Q}$  by  $\varphi_A(A_1) = \varphi(A^*A_1)$ . Then we have  $\varphi_A \circ \gamma < \varphi \circ \gamma$  since

$$\begin{aligned} |(\varphi_A \circ \gamma)(BB_1)| &= |\varphi(A^*\gamma(BB_1))| \\ &\leq \varphi(A^*A)^{1/2} \varphi(\gamma(BB_1)^*\gamma(BB_1))^{1/2} \\ &\leq \varphi(A^*A)^{1/2} \varphi(\gamma(B_1^*B^*BB_1))^{1/2} \\ &\leq \varphi(A^*A)^{1/2} c^{1/2} (\varphi \circ \gamma)(B_1^*B_1)^{1/2} \end{aligned}$$

for every  $B, B_1 \in \mathfrak{B}$  where  $B^*B \leq cI$ . Therefore  $d(\varphi_A \circ \gamma)/d(\varphi \circ \gamma) \in \pi_{\varphi \circ \gamma}(\mathfrak{B})^c$  is defined.

DEFINITION 2.1. We call  $\gamma$  to be *weakly sufficient* for  $S$  if for each  $A \in \mathcal{Q}$  there exists a sequence  $\{B_n\}$  in  $\mathfrak{B}$  such that

$$[d(\varphi_A \circ \gamma)/d(\varphi \circ \gamma)]\Omega_{\varphi \circ \gamma} = s\text{-}\lim \pi_{\varphi \circ \gamma}(B_n)\Omega_{\varphi \circ \gamma}, \quad \varphi \in S.$$

Definition 2.1 is compatible with Definition 1.5. Indeed we have

THEOREM 2.2. *Let  $\gamma: \mathfrak{B} \rightarrow \mathcal{Q}$  be a  $*$ -homomorphism. Then  $\gamma$  is weakly sufficient for  $S \subset \mathcal{S}_\mathcal{Q}$  if and only if the  $*$ -subalgebra  $\gamma\mathfrak{B}$  of  $\mathcal{Q}$  is weakly sufficient for  $S$ .*

*Proof.* If  $\{\mathcal{H}_\varphi, \pi_\varphi, \Omega_\varphi\}$  is the cyclic representation of  $\mathcal{Q}$  induced by  $\varphi \in \mathcal{S}_\mathcal{Q}$ , then the cyclic representation of  $\mathfrak{B}$  induced by  $\varphi \circ \gamma$  is obtained

by  $\{(\overline{\gamma\mathfrak{B}})_\varphi, \pi_\varphi \circ \gamma, \Omega_\varphi\}$ . Now it suffices to show that

$$[d(\varphi_A \circ \gamma)/d(\varphi \circ \gamma)]\Omega_\varphi = P_\varphi(A \mid \gamma\mathfrak{B}), \quad \varphi \in \mathfrak{S}_\mathcal{Q}, A \in \mathcal{Q}.$$

This follows from

$$\begin{aligned} \langle [d(\varphi_A \circ \gamma)/d(\varphi \circ \gamma)]\Omega_\varphi, \pi_\varphi(\gamma B)\Omega_\varphi \rangle &= (\varphi_A \circ \gamma)(B) \\ &= \varphi(A^*(\gamma B)) = \langle \pi_\varphi(A)\Omega_\varphi, \pi_\varphi(\gamma B)\Omega_\varphi \rangle \\ &= \langle P_\varphi(A \mid \gamma\mathfrak{B}), \pi_\varphi(\gamma B)\Omega_\varphi \rangle, \quad B \in \mathfrak{B}. \quad \square \end{aligned}$$

We assume further that  $\mathcal{Q}$  is abelian and  $\gamma: \mathfrak{B} \rightarrow \mathcal{Q}$  is completely positive, i.e.,

$$\sum_{i,j=1}^n A_i^* \gamma(B_i^* B_j) A_j \geq 0$$

for every  $A_1, \dots, A_n \in \mathcal{Q}$  and  $B_1, \dots, B_n \in \mathfrak{B}$ . Note (see [30, IV. 3]) that when  $\mathcal{Q}$  and  $\mathfrak{B}$  are  $C^*$ -algebras, any completely positive map  $\gamma: \mathfrak{B} \rightarrow \mathcal{Q}$  with  $\gamma(I) = I$  satisfies automatically  $\gamma(B)^* \gamma(B) \leq \gamma(B^* B)$  for all  $B \in \mathfrak{B}$ , and any positive linear map  $\gamma: \mathfrak{B} \rightarrow \mathcal{Q}$  is completely positive if either  $\mathcal{Q}$  or  $\mathfrak{B}$  is abelian. Let  $\mathcal{Q} \otimes \mathfrak{B}$  be the  $*$ -algebraic tensor product of  $\mathcal{Q}$  and  $\mathfrak{B}$ . For each  $\varphi \in \mathfrak{S}_\mathcal{Q}$ , we can define the compound state  $\varphi \otimes \gamma$  of  $\mathcal{Q} \otimes \mathfrak{B}$  by

$$(\varphi \otimes \gamma)(A \otimes B) = (\varphi_{A^*} \circ \gamma)(B) = \varphi(A(\gamma B)), \quad A \in \mathcal{Q}, B \in \mathfrak{B},$$

since

$$(\varphi \otimes \gamma) \left( \left( \sum_{i=1}^n A_i \otimes B_i \right)^* \left( \sum_{i=1}^n A_i \otimes B_i \right) \right) = \varphi \left( \sum_{i,j=1}^n A_i^* \gamma(B_i^* B_j) A_j \right) \geq 0.$$

Identifying  $\mathcal{Q}$  and  $\mathfrak{B}$  with  $*$ -subalgebras  $\mathcal{Q} \otimes I$  and  $I \otimes \mathfrak{B}$  of  $\mathcal{Q} \otimes \mathfrak{B}$ , we then have

**THEOREM 2.3.** (1)  $\mathcal{Q}$  is sufficient for  $\{\varphi \otimes \gamma: \varphi \in \mathfrak{S}_\mathcal{Q}\}$ .

(2)  $\gamma$  is weakly sufficient for  $S \subset \mathfrak{S}_\mathcal{Q}$  if and only if  $\mathfrak{B}$  is weakly sufficient for  $\{\varphi \otimes \gamma: \varphi \in S\}$ .

*Proof.* (1) Define  $\varepsilon: \mathcal{Q} \otimes \mathfrak{B} \rightarrow \mathcal{Q}$  by  $\varepsilon(A \otimes B) = A\gamma(B)$ ,  $A \in \mathcal{Q}$ ,  $B \in \mathfrak{B}$ . Since  $\gamma$  is completely positive and  $\mathcal{Q}$  is abelian, it follows that  $\varepsilon$  is a conditional expectation of  $\mathcal{Q} \otimes \mathfrak{B}$  onto  $\mathcal{Q}$ . Hence (1) is seen from  $(\varphi \otimes \gamma) \circ \varepsilon = \varphi \otimes \gamma$  for all  $\varphi \in \mathfrak{S}_\mathcal{Q}$ .

(2) For  $\varphi \in \mathfrak{S}_\mathcal{Q}$ , let  $\{\mathcal{H}_{\tilde{\varphi}}, \pi_{\tilde{\varphi}}, \Omega_{\tilde{\varphi}}\}$  be the cyclic representation of  $\mathcal{Q} \otimes \mathfrak{B}$  induced by  $\tilde{\varphi} = \varphi \otimes \gamma$ . Since  $\tilde{\varphi} \upharpoonright \mathfrak{B} = \varphi \circ \gamma$ , the cyclic representation of  $\mathfrak{B}$  induced by  $\varphi \circ \gamma$  is given by  $\{\overline{\mathfrak{B}_{\tilde{\varphi}}}, \pi_{\tilde{\varphi}} \upharpoonright \mathfrak{B}, \Omega_{\tilde{\varphi}}\}$ . Let  $A \in \mathcal{Q}$ ,

$B \in \mathfrak{B}$  and  $T = d(\varphi_A \circ \gamma)/d(\varphi \circ \gamma) \in \pi_{\tilde{\varphi}}(\mathfrak{B})^c$ . It follows that

$$\begin{aligned} P_{\tilde{\varphi}}(A \otimes B | \mathfrak{B}) &= [d(\tilde{\varphi}_{A \otimes B} \upharpoonright \mathfrak{B})/d(\tilde{\varphi} \upharpoonright \mathfrak{B})]\Omega_{\tilde{\varphi}} \\ &= [d(\varphi_A \circ \gamma)_B/d(\varphi \circ \gamma)]\Omega_{\tilde{\varphi}} = T\pi_{\tilde{\varphi}}(B)\Omega_{\tilde{\varphi}}, \end{aligned}$$

where the first equality is a special case of the equation in the proof of Theorem 2.2 and the last equality is seen from

$$\begin{aligned} &\langle [d(\varphi_A \circ \gamma)_B/d(\varphi \circ \gamma)]\Omega_{\tilde{\varphi}}, \pi_{\tilde{\varphi}}(B_1)\Omega_{\tilde{\varphi}} \rangle \\ &= (\varphi_A \circ \gamma)(B^*B_1) = \langle T\pi_{\tilde{\varphi}}(B)\Omega_{\tilde{\varphi}}, \pi_{\tilde{\varphi}}(B_1)\Omega_{\tilde{\varphi}} \rangle, \quad B_1 \in \mathfrak{B}. \end{aligned}$$

The “if” part of (2) is now immediate by taking  $B = I$ . Conversely if  $\gamma$  is weakly sufficient for  $S$ , then there exists a sequence  $\{B_n\}$  in  $\mathfrak{B}$  such that

$$T\Omega_{\tilde{\varphi}} = s\text{-}\lim \pi_{\tilde{\varphi}}(B_n)\Omega_{\tilde{\varphi}}, \quad \varphi \in S.$$

Since  $\pi_{\tilde{\varphi}}(B)$  is bounded from  $B^*B \leq cI$ , we have

$$P_{\tilde{\varphi}}(A \otimes B | \mathfrak{B}) = \pi_{\tilde{\varphi}}(B)T\Omega_{\tilde{\varphi}} = s\text{-}\lim \pi_{\tilde{\varphi}}(BB_n)\Omega_{\tilde{\varphi}}, \quad \varphi \in S.$$

Hence  $\mathfrak{B}$  is weakly sufficient for  $\{\varphi \otimes \gamma: \varphi \in S\}$ .  $\square$

**EXAMPLE 2.4.** Let  $(X, \mathcal{F})$  and  $(Y, \mathcal{G})$  be two measurable spaces and  $\nu$  be a channel distribution from  $(X, \mathcal{F})$  to  $(Y, \mathcal{G})$ , i.e.,  $\nu$  is a real-valued function on  $X \times \mathcal{G}$  such that for every  $x \in X$ ,  $\nu(x, \cdot)$  is a probability measure on  $\mathcal{G}$  and for every  $B \in \mathcal{G}$ ,  $\nu(\cdot, B)$  is  $\mathcal{F}$ -measurable on  $X$ . Let  $\mathfrak{B}(X)$  and  $\mathfrak{B}(Y)$  be the abelian  $C^*$ -algebras of bounded complex-valued measurable functions on  $X$  and  $Y$ . Define a positive linear map  $\gamma: \mathfrak{B}(Y) \rightarrow \mathfrak{B}(X)$  by

$$(\gamma g)(x) = \int_Y g(y) \nu(x, dy), \quad x \in X, g \in \mathfrak{B}(Y).$$

Let  $S$  be a set of probability measures on  $\mathcal{F}$ . For each  $\mu \in S$ ,  $\mu \otimes \gamma$  is given by

$$(\mu \otimes \gamma)(f \otimes g) = \int_{X \times Y} f \otimes g d(\mu \otimes \nu), \quad f \in \mathfrak{B}(X), g \in \mathfrak{B}(Y),$$

where  $\mu \otimes \nu$  is the probability measure on  $\mathcal{F} \otimes \mathcal{G}$  defined by  $(\mu \otimes \nu) \times (A \times B) = \int_A \nu(x, B) d\mu$ . Then we see in connection with Theorem 2.3(2) that  $\gamma$  is weakly sufficient for  $S$  if and only if the  $\sigma$ -subalgebra  $X \times \mathcal{G} = \{X \times B: B \in \mathcal{G}\}$  of  $\mathcal{F} \otimes \mathcal{G}$  is sufficient in the classical sense for  $\{\mu \otimes \nu: \mu \in S\}$ .

EXAMPLE 2.5. Let  $\mathfrak{N}$  be a von Neumann algebra. An  $\mathfrak{N}$ -valued PO-measure  $M$  on a measurable space  $(X, \mathfrak{F})$  is a map  $M: \mathfrak{F} \rightarrow \mathfrak{N}$  such that  $M(F) \geq 0$  for all  $F \in \mathfrak{F}$  and  $\sum_{n=1}^{\infty} M(F_n) = I$  ( $\sigma$ -weakly) for every countable measurable partition  $\{F_n\}$  of  $X$ . Let  $\mathfrak{B}(X)$  be the abelian  $C^*$ -algebra of bounded measurable functions on  $X$ . We define a positive linear map  $\gamma: \mathfrak{B}(X) \rightarrow \mathfrak{N}$  with  $\gamma(1) = I$  by

$$\varphi(\gamma(f)) = \int_X f d(\varphi \circ M), \quad f \in \mathfrak{B}(X), \varphi \in \mathfrak{N}_*.$$

For each  $\varphi \in \mathfrak{S}$ , the cyclic representation  $\{\mathcal{H}_{\varphi \circ \gamma}, \pi_{\varphi \circ \gamma}, \Omega_{\varphi \circ \gamma}\}$  of  $\mathfrak{B}(X)$  induced by  $\varphi \circ \gamma$  is given as follows  $\mathcal{H}_{\varphi \circ \gamma} = L^2(X, \varphi \circ M)$ ,  $\pi_{\varphi \circ \gamma}(f)$  is the multiplication operator by  $f$ , and  $\Omega_{\varphi \circ \gamma} = 1$ . For  $A \in \mathfrak{N}$ ,  $d(\varphi_A \circ \gamma)/d(\varphi \circ \gamma)$  is identical to the Radon-Nikodym derivative  $d(\varphi_A \circ M)/d(\varphi \circ M)$  which is in  $L^2(X, \varphi \circ M)$ . Now assume that  $\mathfrak{N}$  is  $\sigma$ -finite, so that  $\mathfrak{N}$  has a faithful normal state. Then it is proved that  $\gamma$  is weakly sufficient for  $S \subset \mathfrak{S}$  if and only if for every  $A \in \mathfrak{N}$  there exists a measurable function  $f$  on  $X$  satisfying

$$d(\varphi_A \circ M)/d(\varphi \circ M) = f \quad \text{a.e. } [\varphi \circ M], \quad \varphi \in S.$$

Further assume that  $M$  is pure, i.e.,  $M$  is a spectral measure. Then  $\gamma$  is a  $*$ -homomorphism and  $\gamma(\mathfrak{B}(X))$  is equal to the subalgebra  $\mathfrak{M} = \{M(F): F \in \mathfrak{F}\}''$ . Hence Theorem 2.2 shows that  $\gamma$  is weakly sufficient for  $S$  if and only if  $\mathfrak{M}$  is weakly sufficient for  $S$ .

EXAMPLE 2.6. Let  $\mathcal{A}$  be a  $C^*$ -algebra and  $C(\mathfrak{S})$  be the abelian  $C^*$ -algebra of continuous functions on  $\mathfrak{S}$ . Define a positive linear map  $\gamma: \mathcal{A} \rightarrow C(\mathfrak{S})$  with  $\gamma(I) = 1$  by  $(\gamma A)(\omega) = \omega(A)$ ,  $A \in \mathcal{A}$ ,  $\omega \in \mathfrak{S}$ . For each  $\rho \in \mathfrak{S}$  and each abelian von Neumann subalgebra  $\mathfrak{B}$  of  $\pi_\rho(\mathcal{A})'$ , we take the  $\mathfrak{B}$ -orthogonal measure  $\lambda$  of  $\rho$  (cf. [30, p. 241]). Now assume that  $\mathcal{A}$  is separable and  $\mathfrak{B} \subset \mathfrak{B}_\rho = \pi_\rho(\mathcal{A})'' \cap \pi_\rho(\mathcal{A})'$ , i.e.,  $\lambda$  is a subcentral measure of  $\rho$ , and let  $S$  be the set of all Borel probability measures  $\mu$  on  $\mathfrak{S}$  with  $\mu \ll \lambda$ . Then  $\gamma$  is weakly sufficient for  $S$ . This is proved as follows. There is a  $*$ -isomorphism  $\theta$  of  $L^\infty(\mathfrak{S}, \lambda)$  onto  $\mathfrak{B}$  such that

$$\langle \Omega_\rho, \theta(f) \pi_\rho(A) \Omega_\rho \rangle = \int_{\mathfrak{S}} f(\omega) \omega(A) d\lambda(\omega), \quad A \in \mathcal{A}, f \in L^\infty(\mathfrak{S}, \lambda).$$

For each  $\mu \in S$  and  $f \in C(\mathfrak{S})$ , taking  $g_{\mu n} = \min((d\mu/d\lambda)^{1/2}, n)$  we obtain

$$(\mu \circ \gamma)(A) = \int_{\mathfrak{S}} \omega(A) d\mu(\omega) = \lim \langle \theta(g_{\mu_n})\Omega_\rho, \pi_\rho(A)\theta(g_{\mu_n})\Omega_\rho \rangle,$$

$$\begin{aligned} (\mu_f \circ \gamma)(A) &= \int_{\mathfrak{S}} \bar{f}(\omega) \omega(A) d\mu(\omega) \\ &= \lim \langle \theta(f)\theta(g_{\mu_n})\Omega_\rho, \pi_\rho(A)\theta(g_{\mu_n})\Omega_\rho \rangle. \end{aligned}$$

Since  $\{g_{\mu_n}\}$  is Cauchy in  $L^2(\mathfrak{S}, \lambda)$ , it follows that  $\Phi_\mu = s\text{-}\lim \theta(g_{\mu_n})\Omega_\rho$  exists and

$$\begin{aligned} (\mu \circ \gamma)(A) &= \langle \Phi_\mu, \pi_\rho(A)\Phi_\mu \rangle, \\ (\mu_f \circ \gamma)(A) &= \langle \theta(f)\Phi_\mu, \pi_\rho(A)\Phi_\mu \rangle, \quad A \in \mathcal{Q}. \end{aligned}$$

Hence the cyclic representation of  $\mathcal{Q}$  induced by  $\mu \circ \gamma$  is given by

$$\{\overline{\pi_\rho(\mathcal{Q})\Phi_\mu}, \pi_\rho(\cdot) \upharpoonright \overline{\pi_\rho(\mathcal{Q})\Phi_\mu}, \Phi_\mu\}$$

and we have

$$[d(\mu_f \circ \gamma)/d(\mu \circ \gamma)]\Phi_\mu = \theta(f)\Phi_\mu.$$

Since  $\mathcal{Q}$  is separable, there exists a sequence  $\{A_n\}$  in  $\mathcal{Q}$  such that  $\pi_\rho(A_n) \rightarrow \theta(f)$  (strongly), and hence

$$[d(\mu_f \circ \gamma)/d(\mu \circ \gamma)]\Phi_\mu = s\text{-}\lim \pi_\rho(A_n)\Phi_\mu, \quad \mu \in S.$$

This shows that  $\gamma$  is weakly sufficient for  $S$ .

A linear map  $\gamma: \mathfrak{B} \rightarrow \mathcal{Q}$  considered here describes more or less a quantum communication channel with the input space  $\mathcal{Q}$  and the output space  $\mathfrak{B}$  (cf. [15, 21]). Examples 2.4–2.6 provide classical-classical, quantum-classical and classical-quantum channels. Roughly speaking, the physical meaning of weak sufficiency of  $\gamma$  is that the indirect measurement through  $\gamma$  gives as much information (measured by the relative entropy) as the direct measurement of observables in  $\mathcal{Q}$  given a set  $S$  of input states (see §§3, 4).

**3. Relative entropy of states of  $*$ -algebras.** We begin with the definitions of Araki's relative entropy and Uhlmann's relative entropy.

(I) *Araki's relative entropy.* Let  $(\mathfrak{N}, \mathfrak{H}, J, \mathfrak{P})$  be a standard form of a von Neumann algebra  $\mathfrak{N}$  (cf. [2, 12]). Araki [4, 5] defined the relative entropy of normal positive linear functionals  $\varphi$  and  $\psi$  of  $\mathfrak{N}$  as follows.

There exist unique vector representatives  $\Phi$  and  $\Psi$  in  $\mathcal{P}$  such that  $\varphi(A) = \langle \Phi, A\Phi \rangle$  and  $\psi(A) = \langle \Psi, A\Psi \rangle$  for all  $A \in \mathfrak{N}$ . The operator  $S_{\Psi, \Phi}$  with the domain

$$D(S_{\Psi, \Phi}) = \mathfrak{N}\Phi + (I - s^{\mathfrak{N}'}(\Phi))$$

is defined by

$$S_{\Psi, \Phi}(A\Phi + \Omega) = s^{\mathfrak{N}}(\Phi)A^*\Psi, \quad A \in \mathfrak{N}, s^{\mathfrak{N}'}(\Phi)\Omega = 0,$$

where  $s^{\mathfrak{N}}(\Phi)$  denotes the  $\mathfrak{N}$ -support of  $\Phi$ . Then  $S_{\Psi, \Phi}$  is a closable conjugate-linear operator and the relative modular operator  $\Delta_{\Psi, \Phi}$  is defined by  $\Delta_{\Psi, \Phi} = (S_{\Psi, \Phi})^* \overline{S_{\Psi, \Phi}}$ . Let  $\Delta_{\Psi, \Phi} = \int_0^\infty \lambda d e_{\Psi, \Phi}(\lambda)$  be the spectral decomposition of  $\Delta_{\Psi, \Phi}$ . The Araki's relative entropy  $S(\psi | \varphi)$  is now given by

$$S(\psi | \varphi) = \begin{cases} \int_{+0}^\infty \log \lambda d \langle \Psi, e_{\Psi, \Phi}(\lambda) \Psi \rangle & \text{if } \psi \ll \varphi, \\ +\infty & \text{otherwise.} \end{cases}$$

Note that the relative entropy  $S(\psi | \varphi)$  is independent of the choice of a standard form of  $\mathfrak{N}$  which is unique up to unitary equivalence. We used in [14] the notation  $S(\varphi | \psi)$  instead of  $S(\psi | \varphi)$ .

(II) *Uhlmann's relative entropy*. Let  $\mathcal{L}$  be a complex linear space. Given two seminorms  $p$  and  $q$  on  $\mathcal{L}$ , the quadratical mean  $QM(p, q)$  is defined by

$$QM(p, q)(x) = \sup_{\alpha \in H} \alpha(x, x)^{1/2}, \quad x \in \mathcal{L},$$

where  $H$  is the set of all positive hermitian forms  $\alpha$  on  $\mathcal{L}$  satisfying  $|\alpha(x, y)| \leq p(x)q(y)$  for all  $x, y \in \mathcal{L}$ . A function  $t \mapsto p_t$  on  $[0, 1]$  whose values are seminorms on  $\mathcal{L}$  is called a quadratical interpolation from  $p$  to  $q$  if for every  $x \in \mathcal{L}$  the function  $t \mapsto p_t(x)$  is continuous and if the following properties hold:

$$\begin{aligned} p_t &= QM(p_{t_1}, p_{t_2}), & t &= (t_1 + t_2)/2, t_1, t_2 \in [0, 1], \\ p_{1/2} &= QM(p, q), \\ p_{t/2} &= QM(p, p_t), & t &\in [0, 1], \\ p_{(1+t)/2} &= QM(q, p_t), & t &\in [0, 1]. \end{aligned}$$

Uhlmann [32] showed that for each positive hermitian forms  $\alpha$  and  $\beta$  there exists a unique function  $t \mapsto QF_t(\alpha, \beta)$  on  $[0, 1]$  with values in the set of

positive hermitian forms on  $\mathcal{L}$  such that the function  $p_t$  given by  $p_t(x) = QF_t(\alpha, \beta)(x, x)^{1/2}$  is the quadratical interpolation from  $\alpha(x, x)^{1/2}$  to  $\beta(x, x)^{1/2}$ , and defined the relative entropy functional  $S(\alpha; \beta)(x)$  of  $\alpha$  and  $\beta$  by

$$S(\alpha; \beta)(x) = -\liminf_{t \rightarrow +0} \frac{1}{t} \{QF_t(\alpha, \beta)(x, x) - \alpha(x, x)\}, \quad x \in \mathcal{L}.$$

Now let  $\mathcal{A}$  be a  $*$ -algebra, and  $\varphi$  and  $\psi$  be positive linear functionals of  $\mathcal{A}$ . The Uhlmann's relative entropy  $S(\psi | \varphi)$  is defined by

$$S(\psi | \varphi) = S(\psi^R; \varphi^L)(I),$$

where  $\varphi^L$  and  $\psi^R$  are the positive hermitian forms given by  $\varphi^L(A, B) = \varphi(A^*B)$  and  $\psi^R(A, B) = \psi(BA^*)$ .

For each normal positive linear functionals  $\varphi$  and  $\psi$  of a von Neumann algebra  $\mathfrak{N}$ , the Uhlmann's relative entropy is equal to the Araki's relative entropy. We here contain the proof for completeness.

Let  $\mathcal{K}$  be the domain of  $(I + \Delta_{\Psi, \Phi})^{1/2}$ , which becomes a Hilbert space with an inner product:

$$(\Omega_1, \Omega_2) = \langle (I + \Delta_{\Psi, \Phi})^{1/2}\Omega_1, (I + \Delta_{\Psi, \Phi})^{1/2}\Omega_2 \rangle, \quad \Omega_1, \Omega_2 \in \mathcal{K}.$$

The operators  $(I + \Delta_{\Psi, \Phi})^{-1}$  and  $\Delta_{\Psi, \Phi}(I + \Delta_{\Psi, \Phi})^{-1}$  are positive bounded linear operators on  $\mathcal{K}$ . Define positive hermitian forms  $\alpha$  and  $\beta$  on  $\mathcal{K}$  by

$$\alpha(\Omega_1, \Omega_2) = (\Omega_1, \Delta_{\Psi, \Phi}(I + \Delta_{\Psi, \Phi})^{-1}\Omega_2),$$

$$\beta(\Omega_1, \Omega_2) = (\Omega_1, (I + \Delta_{\Psi, \Phi})^{-1}\Omega_2).$$

We then have (cf. [24], [32, Example 4])

$$\begin{aligned} QF_t(\alpha, \beta)(\Omega, \Omega) &= \left( \Omega, [\Delta_{\Psi, \Phi}(I + \Delta_{\Psi, \Phi})^{-1}]^{1-t} [(I + \Delta_{\Psi, \Phi})^{-1}]^t \Omega \right) \\ &= \langle \Omega, (\Delta_{\Psi, \Phi})^{1-t} \Omega \rangle, \quad t \in (0, 1), \Omega \in \mathcal{K}. \end{aligned}$$

Since  $\mathfrak{N}\Phi \subset \mathcal{K}$  and

$$\psi^R(A, B) = (A\Phi, \Delta_{\Psi, \Phi}(I + \Delta_{\Psi, \Phi})^{-1}B\Phi),$$

$$\varphi^L(A, B) = (A\Phi, (I + \Delta_{\Psi, \Phi})^{-1}B\Phi),$$

it is easy to check that

$$QF_t(\psi^R, \varphi^L)(A, A) = QF_t(\alpha, \beta)(A\Phi, A\Phi), \quad A \in \mathfrak{N}.$$

Take the spectral decomposition  $\Delta_{\Psi, \Phi} = \int_0^\infty \lambda d\epsilon_{\Psi, \Phi}(\lambda)$ . If  $\psi \ll \varphi$ , then  $(\Delta_{\Psi, \Phi})^{1/2}\Phi = JS_{\Psi, \Phi}\Phi = J\Psi = \Psi$  and hence  $\psi^R(I, I) = \|(\Delta_{\Psi, \Phi})^{1/2}\Phi\|^2$ .

We have

$$\begin{aligned}
S(\psi^R; \varphi^L)(I) &= -\liminf_{t \rightarrow +0} \frac{1}{t} \left\{ \langle \Phi, (\Delta_{\Psi, \Phi})^{1-t} \Phi \rangle - \langle \Phi, \Delta_{\Psi, \Phi} \Phi \rangle \right\} \\
&= -\liminf_{t \rightarrow +0} \int_{+0}^{\infty} \lambda \frac{\lambda^{-t} - 1}{t} d \langle \Phi, e_{\Psi, \Phi}(\lambda) \Phi \rangle \\
&= \int_{+0}^{\infty} \lambda \log \lambda d \langle \Phi, e_{\Psi, \Phi}(\lambda) \Phi \rangle \\
&= \int_{+0}^{\infty} \log \lambda d \langle \Psi, e_{\Psi, \Phi}(\lambda) \Psi \rangle,
\end{aligned}$$

because the function  $(\lambda^{-t} - 1)/t$  converges decreasingly to  $-\log \lambda$  as  $t \rightarrow +0$ . If  $\psi \ll \varphi$  does not hold, then  $\psi^R(I, I) < \|(\Delta_{\Psi, \Phi})^{1/2} \Phi\|^2$  and hence  $S(\psi^R; \varphi^L)(I) = +\infty$ . Thus the Uhlmann's relative entropy is equal to the Araki's one.

**LEMMA 3.1.** *Let  $\mathcal{Q}$  be a  $C^*$ -algebra and  $\pi$  be a nondegenerate representation of  $\mathcal{Q}$  on a Hilbert space. If  $\varphi$  and  $\psi$  are positive linear functionals of  $\mathcal{Q}$  having the normal extensions  $\tilde{\varphi}$  and  $\tilde{\psi}$  to  $\pi(\mathcal{Q})''$  such that  $\varphi(A) = \tilde{\varphi}(\pi(A))$  and  $\psi(A) = \tilde{\psi}(\pi(A))$ , then  $S(\psi | \varphi) = S(\tilde{\psi} | \tilde{\varphi})$ .*

*Proof.* According to the Uhlmann's definition of relative entropy, it suffices to show that

$$QF_t(\psi^R, \varphi^L)(A, A) = QF_t(\tilde{\psi}^R, \tilde{\varphi}^L)(\pi(A), \pi(A)), \quad t \in [0, 1], A \in \mathcal{Q}.$$

Let  $\Gamma$  be the set of  $t \in [0, 1]$  for which the above equation holds for every  $A \in \mathcal{Q}$ . Let  $H$  be the set of all positive hermitian forms  $\alpha$  on  $\mathcal{Q}$  satisfying

$$|\alpha(A_1, A_2)| \leq \psi^R(A_1, A_1)^{1/2} \varphi^L(A_2, A_2)^{1/2}, \quad A_1, A_2 \in \mathcal{Q},$$

and  $\tilde{H}$  be the set of all positive hermitian forms  $\tilde{\alpha}$  on  $\pi(\mathcal{Q})''$  satisfying

$$|\tilde{\alpha}(Q_1, Q_2)| \leq \tilde{\psi}^R(Q_1, Q_2)^{1/2} \tilde{\varphi}^L(Q_2, Q_2)^{1/2}, \quad Q_1, Q_2 \in \pi(\mathcal{Q})''.$$

If  $\tilde{\alpha} \in \tilde{H}$ , then the form  $\alpha$  on  $\mathcal{Q}$  defined by  $\alpha(A_1, A_2) = \tilde{\alpha}(\pi(A_1), \pi(A_2))$  is in  $H$ . Conversely if  $\alpha \in H$ , then there exists a positive hermitian form  $\tilde{\alpha}$  on  $\pi(\mathcal{Q})$  such that  $\alpha(A_1, A_2) = \tilde{\alpha}(\pi(A_1), \pi(A_2))$  and hence

$$\begin{aligned}
&|\tilde{\alpha}(\pi(A_1), \pi(A_2))| \\
&\leq \tilde{\psi}^R(\pi(A_1), \pi(A_2))^{1/2} \tilde{\varphi}^L(\pi(A_2), \pi(A_2))^{1/2}, \quad A_1, A_2 \in \mathcal{Q}.
\end{aligned}$$

By the Kaplansky density theorem,  $\hat{\alpha}$  can be uniquely extended to a positive hermitian form  $\tilde{\alpha}$  on  $\pi(\mathcal{Q})''$  which is in  $\tilde{H}$ . Therefore

$$\begin{aligned} QF_{1/2}(\psi^R, \varphi^L)(A, A) &= \sup_{\alpha \in H} \alpha(A, A) \\ &= \sup_{\tilde{\alpha} \in \tilde{H}} \tilde{\alpha}(\pi(A), \pi(A)) \\ &= QF_{1/2}(\tilde{\psi}^R, \tilde{\varphi}^L)(\pi(A), \pi(A)), \quad A \in \mathcal{Q}. \end{aligned}$$

This implies  $1/2 \in \Gamma$ . Noting that

$$QF_t(\psi^R, \varphi^L)(A, A) \leq \psi^R(A, A)^{1-t} \varphi^L(A, A)^t, \quad t \in [0, 1], A \in \mathcal{Q},$$

we can see by the similar arguments that  $t \in \Gamma$  implies  $t/2 \in \Gamma$  and  $(1+t)/2 \in \Gamma$ , and that  $t_1, t_2 \in \Gamma$  implies  $(t_1 + t_2)/2 \in \Gamma$ . Since  $\Gamma$  is closed, we deduce that  $\Gamma = [0, 1]$ .  $\square$

In the above lemma, we can take as  $\pi$  the cyclic representation induced by  $\varphi + \psi$  or the universal representation of  $\mathcal{Q}$ .

We here remark that the relative entropy defined in (I) and (II) contains the usual relative entropies in the classical and quantum systems. Let  $(X, \mathcal{F})$  be a measurable space, and  $\mu$  and  $\nu$  be probability measures on  $\mathcal{F}$ . Take a measure  $m$  on  $\mathcal{F}$  with  $\mu, \nu \ll m$ . Then  $\mu$  and  $\nu$  are naturally regarded as normal states of the abelian von Neumann algebra  $\mathfrak{N} = L^\infty(X, m)$  acting on  $\mathcal{H} = L^2(X, m)$ . Then the relative entropy  $S(\nu | \mu)$  is equal to the classical relative entropy  $I(\nu | \mu)$  (known as the Kullback-Leibler information):

$$I(\nu | \mu) = \begin{cases} \int \frac{d\nu}{d\mu} \log \frac{d\nu}{d\mu} d\mu & \text{if } \nu \ll \mu, \\ +\infty & \text{otherwise.} \end{cases}$$

Indeed,  $\Phi = (d\mu/dm)^{1/2}$  and  $\Psi = (d\nu/dm)^{1/2}$  are vector representatives for  $\mu$  and  $\nu$ , and  $\Delta_{\Psi, \Phi}$  is the multiplication operator by  $1_{\text{supp}\Phi}(\Psi/\Phi)^2$  where  $1_{\text{supp}\Phi}$  is the characteristic function of the support of  $\Phi$ . If  $\nu \ll \mu$ , then we have

$$\begin{aligned} S(\nu | \mu) &= \int \Psi^2 1_{\text{supp}\Phi} \log(\Psi/\Phi)^2 dm \\ &= \int \frac{d\nu}{dm} \left( \log \frac{d\nu}{dm} - \log \frac{d\mu}{dm} \right) dm = I(\nu | \mu). \end{aligned}$$

Next let  $\varphi$  and  $\psi$  be normal states of the full von Neumann algebra  $\mathfrak{N} = \mathbf{B}(\mathfrak{H})$  on a Hilbert space  $\mathfrak{H}$ . Then  $\varphi$  and  $\psi$  are given by  $\varphi(A) = \text{Tr}(\rho_\varphi A)$  and  $\psi(A) = \text{Tr}(\rho_\psi A)$  with positive trace class operators  $\rho_\varphi$  and  $\rho_\psi$  on  $\mathfrak{H}$ , and we obtain

$$S(\psi | \varphi) = \text{Tr}(\rho_\psi \log \rho_\psi - \rho_\psi \log \rho_\varphi).$$

The relative entropy  $S(\psi | \varphi)$  has several basic properties such as joint convexity, monotonicity, lower semicontinuity, etc. (cf. [4, 5, 32]). The monotonicity is stated as follows (cf. [32, Proposition 18]). Let  $\mathcal{A}$  and  $\mathfrak{B}$  be  $*$ -algebras and  $\gamma: \mathfrak{B} \rightarrow \mathcal{A}$  be a linear map such that  $\gamma(I) = I$ ,  $\gamma(B^*) = \gamma(B)^*$  and  $\gamma(B)^* \gamma(B) \leq \gamma(B^* B)$  for all  $B \in \mathfrak{B}$ . If  $\varphi$  and  $\psi$  are positive linear functionals on  $\mathcal{A}$ , then

$$S(\psi \circ \gamma | \varphi \circ \gamma) \leq S(\psi | \varphi).$$

This monotonicity is applied to positive linear maps such as in Examples 2.4–2.6. Particularly if  $\mathfrak{B}$  is a  $*$ -subalgebra of  $\mathcal{A}$ , then we have  $S_{\mathfrak{B}}(\psi | \varphi) \leq S(\psi | \varphi)$  where  $S_{\mathfrak{B}}(\psi | \varphi)$  denotes the relative entropy of the restrictions  $\varphi \upharpoonright \mathfrak{B}$  and  $\psi \upharpoonright \mathfrak{B}$ .

In connection with Example 2.6, it is proved that the relative entropy of states of a  $C^*$ -algebra is equal to that of their decomposition measures in some cases.

**THEOREM 3.2.** *Let  $\mathcal{A}$  be a  $C^*$ -algebra and  $\mu, \nu$  be regular Borel probability measures on  $\mathfrak{S}$  with barycenters  $\varphi, \psi \in \mathfrak{S}$ . If there is a subcentral measure  $\lambda$  on  $\mathfrak{S}$  such that  $\mu, \nu \ll \lambda$ , then  $S(\psi | \varphi) = I(\nu | \mu)$ .*

*Proof.* Let  $\lambda$  be the  $\mathfrak{B}$ -orthogonal measure of  $\rho \in \mathfrak{S}$  with an abelian von Neumann subalgebra  $\mathfrak{B}$  of  $\mathfrak{B}_\rho = \pi_\rho(\mathcal{A})'' \cap \pi_\rho(\mathcal{A})'$ , and  $\theta$  be the  $*$ -isomorphism of  $L^\infty(\mathfrak{S}, \lambda)$  onto  $\mathfrak{B}$  such that

$$\langle \Omega_\rho, \theta(f) \pi_\rho(A) \Omega_\rho \rangle = \int_{\mathfrak{S}} f(\omega) \omega(A) d\lambda(\omega), \quad A \in \mathcal{A}, f \in L^\infty(\mathfrak{S}, \lambda).$$

As is seen in Example 2.6, there exists a  $\Phi_\mu \in \mathfrak{H}_\rho$  such that  $\varphi(A) = \langle \Phi_\mu, \pi_\rho(A) \Phi_\mu \rangle$  for all  $A \in \mathcal{A}$ . Hence  $\varphi$  has the normal extension  $\tilde{\varphi}$  to  $\pi_\rho(\mathcal{A})''$  and it is easily checked that

$$\tilde{\varphi}(\theta(f)) = \int_{\mathfrak{S}} f d\mu, \quad f \in L^\infty(\mathfrak{S}, \lambda).$$

Analogously  $\psi$  has the normal extension  $\tilde{\psi}$  to  $\pi_\rho(\mathcal{A})''$  satisfying

$$\tilde{\psi}(\theta(f)) = \int_{\mathfrak{S}} f d\nu, \quad f \in L^\infty(\mathfrak{S}, \lambda).$$

Using Lemma 3.1, we have

$$S(\psi | \varphi) = S(\tilde{\psi} | \tilde{\varphi}) \geq S_{\mathfrak{B}}(\tilde{\psi} | \tilde{\varphi}) = I(\nu | \mu).$$

The inverse inequality always holds by the monotonicity.  $\square$

**COROLLARY 3.3.** (1) *Let  $\varphi, \psi \in \mathfrak{S}$  which satisfy the KMS condition with respect to a strongly continuous one-parameter automorphism group  $\alpha_t$  of  $\mathcal{A}$ . If  $\mu$  and  $\nu$  are the central measures of  $\varphi$  and  $\psi$ , then  $S(\psi | \varphi) = I(\nu | \mu)$ .*

(2) *Let  $\{\mathcal{A}, G, \alpha\}$  be a  $C^*$ -dynamical system such that  $\alpha_G$  is a large group of automorphisms of  $\mathcal{A}$ , and  $\varphi, \psi \in \mathfrak{S}$  be  $\alpha$ -invariant. If  $\mu$  and  $\nu$  are the ergodic decomposition measures of  $\varphi$  and  $\psi$ , then  $S(\psi | \varphi) = I(\nu | \mu)$ .*

*Proof.* (1) Let  $K$  be the set of all states satisfying the KMS condition with respect to  $\alpha_t$ . Then  $K$  is a Choquet simplex and the central measure of  $\rho \in K$  is identical to the unique maximal measure on  $K$  representing  $\rho$  (cf. [8, p. 121]). Hence it follows that  $\lambda = (\mu + \nu)/2$  is the central measure of  $\rho = (\varphi + \psi)/2$ , so that Theorem 3.2 gives the desired equality.

(2) First note that the set  $\mathfrak{S}_\alpha$  of all  $\alpha$ -invariant states becomes a Choquet simplex, because the condition of large group implies the  $G$ -abelianness (cf. [10]). Hence  $\lambda = (\mu + \nu)/2$  is the ergodic decomposition measure of  $\rho = (\varphi + \psi)/2$ . It follows (cf. [26, Theorem 3.6], [27, Theorem 3.1]) that  $\lambda$  is the  $\mathfrak{B}$ -orthogonal measure of  $\rho$  with  $\mathfrak{B} = (\pi_\rho(\mathcal{A}) \cup U_\rho(G))' = \mathfrak{Z}_\rho \cap U_\rho(G)'$  where  $g \mapsto U_\rho(g)$  is the unitary representation of  $G$  on  $\mathcal{H}_\rho$  such that  $\pi_\rho(\alpha_g(A)) = U_\rho(g)\pi_\rho(A)U_\rho(g)^*$  and  $U_\rho(g)\Omega_\rho = \Omega_\rho$ . Thus we have the desired equality.  $\square$

**4. Relative entropy, sufficiency and KMS condition.** In this section, we establish some relations between the relative entropy, the sufficiency and the KMS condition in  $W^*$ -dynamical systems and  $C^*$ -dynamical systems. The following theorem is obvious from Definition 1.1 and the monotonicity of relative entropy.

**THEOREM 4.1.** *If a  $*$ -subalgebra  $\mathfrak{B}$  of  $\mathcal{A}$  is sufficient for  $\{\varphi, \psi\}$  in  $\mathfrak{S}$ , then  $S(\psi | \varphi) = S_{\mathfrak{B}}(\psi | \varphi)$ .*

**THEOREM 4.2.** *Let  $\mathfrak{N}$  be a von Neumann algebra and  $\mathfrak{S}$  be the set of all normal states of  $\mathfrak{N}$ .*

(1) *Let  $\{\mathfrak{N}, G, \alpha\}$  be a  $W^*$ -dynamical system. If  $\varphi, \psi \in \mathfrak{S}$  are  $\alpha$ -invariant, then  $S(\psi | \varphi) = S_{\mathfrak{N}^\alpha}(\psi | \varphi)$  where  $\mathfrak{N}^\alpha$  is the fixed point subalgebra of  $\alpha$ .*

(2) Let  $\alpha_t$  be a strongly continuous one-parameter automorphism group of  $\mathfrak{N}$ . If  $\varphi, \psi \in \mathfrak{S}$  satisfy the KMS condition with respect to  $\alpha_t$ , then  $S(\psi | \varphi) = S_{\mathfrak{Z}}(\psi | \varphi)$  where  $\mathfrak{Z} = \mathfrak{N} \cap \mathfrak{N}'$ .

*Proof.* (1) Let  $s(\varphi)$  and  $s(\psi)$  be the support projections of  $\varphi$  and  $\psi$ , which are in  $\mathfrak{N}^\alpha$  from the  $\alpha$ -invariance of  $\varphi$  and  $\psi$ . Since  $S(\psi | \varphi) = S_{\mathfrak{N}^\alpha}(\psi | \varphi) = +\infty$  if  $s(\psi) \leq s(\varphi)$  does not hold, we assume that  $s(\psi) \leq s(\varphi)$ . Letting  $e = s(\varphi)$ , we can define a  $W^*$ -dynamical system  $\{\hat{\mathfrak{N}}, G, \hat{\alpha}\}$  by  $\hat{\mathfrak{N}} = e\mathfrak{N}e$  and  $\hat{\alpha}_g = \alpha_g \upharpoonright \hat{\mathfrak{N}}$ . Then  $\hat{\varphi} = \varphi \upharpoonright \hat{\mathfrak{N}}$  and  $\hat{\psi} = \psi \upharpoonright \hat{\mathfrak{N}}$  are  $\hat{\alpha}$ -invariant. Since  $\hat{\varphi}$  is faithful, it follows (see Example 1.4) that  $\hat{\mathfrak{N}}^{\hat{\alpha}} = e\mathfrak{N}^\alpha e$  is sufficient for  $\{\hat{\varphi}, \hat{\psi}\}$ . Hence we have  $S(\hat{\psi} | \hat{\varphi}) = S_{e\mathfrak{N}^\alpha e}(\hat{\psi} | \hat{\varphi})$  by Theorem 4.1. It now suffices to show the equations:

$$S(\psi | \varphi) = S(\hat{\psi} | \hat{\varphi}) \quad \text{and} \quad S_{\mathfrak{N}^\alpha}(\psi | \varphi) = S_{e\mathfrak{N}^\alpha e}(\hat{\psi} | \hat{\varphi}).$$

Define a linear map  $\gamma: \mathfrak{N} \rightarrow \hat{\mathfrak{N}}$  by  $\gamma(A) = eAe$ . Then we have  $\gamma(I) = e$ ,  $\gamma(A^*) = \gamma(A)^*$  and  $\gamma(A)^*\gamma(A) \leq \gamma(A^*A)$  for all  $A \in \mathfrak{N}$ . Since  $\varphi = \hat{\varphi} \circ \gamma$  and  $\psi = \hat{\psi} \circ \gamma$ , the monotonicity gives  $S(\psi | \varphi) \leq S(\hat{\psi} | \hat{\varphi})$ . Next define a linear map  $\hat{\gamma}: \hat{\mathfrak{N}} \rightarrow \mathfrak{N}$  by  $\hat{\gamma}(B) = B + \hat{\varphi}(B)(I - e)$ . Then we have  $\hat{\gamma}(e) = I$ ,  $\hat{\gamma}(B^*) = \hat{\gamma}(B)^*$  and

$$\begin{aligned} \hat{\gamma}(B)^*\hat{\gamma}(B) &= B^*B + |\hat{\varphi}(B)|^2(I - e) \\ &\leq B^*B + \hat{\varphi}(B^*B)(I - e) = \hat{\gamma}(B^*B), \quad B \in \hat{\mathfrak{N}}. \end{aligned}$$

Since  $\hat{\varphi} = \varphi \circ \gamma$  and  $\hat{\psi} = \psi \circ \gamma$ , the monotonicity again gives  $S(\hat{\psi} | \hat{\varphi}) \leq S(\psi | \varphi)$ . We hence obtain the first equation and analogously the second equation.

(2) By the KMS condition, the support projections  $s(\varphi)$  and  $s(\psi)$  are in  $\mathfrak{Z}$  (cf. [22, Lemma 5.1]). Letting  $s(\psi) \leq s(\varphi) = e$ , we define  $\hat{\mathfrak{N}} = \mathfrak{N}e$  and  $\hat{\alpha}_t = \alpha_t \upharpoonright \hat{\mathfrak{N}}$ . Then  $\hat{\varphi} = \varphi \upharpoonright \hat{\mathfrak{N}}$  and  $\hat{\psi} = \psi \upharpoonright \hat{\mathfrak{N}}$  satisfy the KMS condition with respect to  $\hat{\alpha}_t$ . Since  $\hat{\varphi}$  is faithful and hence  $\hat{\alpha}_t = \sigma_t^{\hat{\varphi}}$  the modular automorphism group of  $\hat{\varphi}$ , it follows (see Example 1.2) that  $\hat{\mathfrak{Z}} = \mathfrak{Z}e$  is sufficient for  $\{\hat{\varphi}, \hat{\psi}\}$ . As in the proof of (1), we thus have

$$S(\psi | \varphi) = S(\hat{\psi} | \hat{\varphi}) = S_{\mathfrak{Z}e}(\hat{\psi} | \hat{\varphi}) = S_{\mathfrak{Z}}(\psi | \varphi). \quad \square$$

**THEOREM 4.3.** *Let  $\alpha_t$  be a strongly continuous one-parameter automorphism group of  $\mathfrak{N}$  and  $\varphi, \psi \in \mathfrak{S}$ . Assume that  $\varphi$  satisfies the KMS condition with respect to  $\alpha_t$ .*

(1) *If  $S(\psi | \varphi) = S_{\mathfrak{N}^\alpha}(\psi | \varphi) < +\infty$ , then  $\psi$  is  $\alpha_t$ -invariant.*

(2) *If  $S(\psi | \varphi) = S_{\mathfrak{Z}}(\psi | \varphi) < +\infty$ , then  $\psi$  satisfies the KMS condition with respect to  $\alpha_t$ .*

*Proof.* By the assumptions, we have  $s(\varphi) \in \mathfrak{J} \cap \mathfrak{N}^\alpha$  and  $s(\psi) \leq s(\varphi)$ . As is seen from the proof of Theorem 4.2, we may suppose that  $\varphi$  is faithful, so that  $\alpha_t = \sigma_t^\varphi$  the modular automorphism group and  $\mathfrak{N}^\alpha = Z_\varphi$  the centralizer of  $\varphi$  (cf. [28, Lemma 15.8]). In [14, Corollaries 4.2 and 4.3], we proved (1) and (2) for the case when also  $\psi$  is faithful. Now let  $\psi$  be not faithful and  $p = s(\psi)$ .

(1) We first show that  $p \in Z_\varphi$ . Let  $\mathfrak{H} = (Z_\varphi \cup \{p\})''$ ,  $\hat{\varphi} = \varphi \upharpoonright \mathfrak{H}$  and  $\hat{\psi} = \psi \upharpoonright \mathfrak{H}$ . Then  $\hat{\varphi}$  is a trace of  $\mathfrak{H}$  and we have  $S(\hat{\psi} | \hat{\varphi}) = S_{Z_\varphi}(\hat{\psi} | \hat{\varphi}) < +\infty$  by the assumption. Let  $\varepsilon$  be the conditional expectation of  $\mathfrak{H}$  onto  $Z_\varphi$  with  $\hat{\varphi} \circ \varepsilon = \hat{\varphi}$ . Define  $\hat{\psi}' = \hat{\psi} \circ \varepsilon$ ,  $\hat{\psi}_t = (1-t)\hat{\psi} + t\hat{\varphi}$  and  $\hat{\psi}'_t = \hat{\psi}_t \circ \varepsilon = (1-t)\hat{\psi}' + t\hat{\varphi}$  for  $0 < t < 1$ . Since  $\hat{\psi}_t$  is faithful, it follows by [14, Theorem 3.3] that

$$(*) \quad \|\hat{\psi}'_t - \hat{\psi}_t\| \leq \left\{ 2 \left( S(\hat{\psi}_t | \hat{\varphi}) - S_{Z_\varphi}(\hat{\psi}_t | \hat{\varphi}) \right) \right\}^{1/2}, \quad 0 < t < 1.$$

Since  $\hat{\varphi}$  is a trace, there exists a positive self-adjoint operator  $h$  affiliated with  $\mathfrak{H}$  such that  $\hat{\psi}(A) = \hat{\varphi}(hA)$  for all  $A \in \mathfrak{H}$ . Take the spectral decomposition  $h = \int_0^\infty \lambda \, de(\lambda)$ . Noting that  $\Delta_{\hat{\psi}, \hat{\varphi}} = h$  and  $\Delta_{\hat{\psi}_t, \hat{\varphi}} = (1-t)h + tI$  where  $\hat{\varphi}$ ,  $\hat{\psi}$ , and  $\hat{\psi}_t$  are vector representatives of  $\hat{\varphi}$ ,  $\hat{\psi}$  and  $\hat{\psi}_t$  in the standard form of  $\mathfrak{H}$ , we have

$$S(\hat{\psi} | \hat{\varphi}) = \int_0^\infty \lambda \log \lambda \, d\hat{\varphi}(e(\lambda)),$$

$$S(\hat{\psi}_t | \hat{\varphi}) = \int_0^\infty [(1-t)\lambda + t] \log[(1-t)\lambda + t] \, d\hat{\varphi}(e(\lambda)).$$

Since

$$-\frac{1}{e} \leq [(1-t)\lambda + t] \log[(1-t)\lambda + t] \leq (1-t)\lambda \log \lambda,$$

it follows from the Lebesgue's convergence theorem that

$$S(\hat{\psi} | \hat{\varphi}) = \lim_{t \rightarrow +0} S(\hat{\psi}_t | \hat{\varphi}),$$

and analogously

$$S_{Z_\varphi}(\hat{\psi} | \hat{\varphi}) = \lim_{t \rightarrow +0} S_{Z_\varphi}(\hat{\psi}_t | \hat{\varphi}).$$

By letting  $t \rightarrow +0$  in (\*), we obtain  $\hat{\psi}' = \hat{\psi}$ , which implies that  $Z_\varphi$  is sufficient for  $\{\hat{\varphi}, \hat{\psi}\}$ . Then it is easy to see that  $h$  is affiliated with  $Z_\varphi$ , so that  $p = s(h) \in Z_\varphi$ . Now define a faithful state  $\bar{\psi} = c\psi + (1-c)\bar{\varphi}$  where

$c = \varphi(p) < 1$  and  $\bar{\varphi} = (1 - c)^{-1}\varphi_{I-p}$ . Since  $s(\psi) \perp s(\bar{\varphi})$ , by [5, Theorem 3.6] we have

$$\begin{aligned} S(\bar{\psi}|\varphi) &= cS(\psi|\varphi) + (1 - c)S(\bar{\varphi}|\varphi) \\ &\quad + c \log c + (1 - c) \log(1 - c), \\ S_{Z_\varphi}(\bar{\psi}|\varphi) &= cS_{Z_\varphi}(\psi|\varphi) + (1 - c)S_{Z_\varphi}(\bar{\varphi}|\varphi) \\ &\quad + c \log c + (1 - c) \log(1 - c). \end{aligned}$$

Since  $\bar{\varphi}$  is  $\sigma_t^\varphi$ -invariant and  $\bar{\varphi} \leq (1 - c)^{-1}\varphi$ , it follows from Theorem 4.2 (1) that  $S(\bar{\varphi}|\varphi) = S_{Z_\varphi}(\bar{\varphi}|\varphi) < +\infty$ , and hence  $S(\bar{\psi}|\varphi) = S_{Z_\varphi}(\bar{\psi}|\varphi) < +\infty$ . This implies by [14, Corollary 4.2] that  $\bar{\psi}$  is  $\sigma_t^\varphi$ -invariant. Thus  $\psi$  is  $\sigma_t^\varphi$ -invariant.

(2) Substituting  $\mathfrak{Z}$  for  $Z_\varphi$  in the proof of (1), we can show that  $p \in \mathfrak{Z}$  and a faithful state  $\bar{\psi}$  defined as above satisfies the KMS condition with respect to  $\sigma_t^\varphi$ , and thus  $\psi$  satisfies the same condition.  $\square$

Let  $\mathcal{Q}$  be a  $C^*$ -algebra and  $\alpha_t$  be a strongly continuous one-parameter automorphism group of  $\mathcal{Q}$ . Let  $\varphi \in \mathfrak{S}$  and  $\{\mathcal{H}_\varphi, \pi_\varphi, \Omega_\varphi\}$  be the cyclic representation of  $\mathcal{Q}$  induced by  $\varphi$ . Suppose that  $\varphi$  satisfies the KMS condition with respect to  $\alpha_t$ . Since  $\varphi$  is  $\alpha_t$ -invariant, there is a strongly continuous one-parameter unitary group  $U_\varphi(t)$  on  $\mathcal{H}_\varphi$  such that  $U_\varphi(t)\Omega_\varphi = \Omega_\varphi$  and

$$\pi_\varphi(\alpha_t(A)) = U_\varphi(t)\pi_\varphi(A)U_\varphi(t)^*, \quad t \in \mathbf{R}, A \in \mathcal{Q}.$$

The normal extensions  $\tilde{\varphi}$  and  $\tilde{\alpha}_t$  of  $\varphi$  and  $\alpha_t$  to  $\pi_\varphi(\mathcal{Q})''$  are given by

$$\begin{aligned} \tilde{\varphi}(Q) &= \langle \Omega_\varphi, Q\Omega_\varphi \rangle, \quad Q \in \pi_\varphi(\mathcal{Q})'', \\ \tilde{\alpha}_t(Q) &= U_\varphi(t)QU_\varphi(t)^*, \quad t \in \mathbf{R}, Q \in \pi_\varphi(\mathcal{Q})'', \end{aligned}$$

and it is known (cf. [1, Lemma 2.4]) that  $\tilde{\varphi}$  satisfies the KMS condition with respect to  $\tilde{\alpha}_t$ , i.e.,  $\tilde{\alpha}_t = \sigma_t^{\tilde{\varphi}}$  the modular automorphism group of  $\tilde{\varphi}$ . Then we have

**THEOREM 4.4.** *Let  $\mathcal{Q}$ ,  $\alpha_t$  and  $\varphi$  be as above. For each  $\psi \in \mathfrak{S}$  with  $\psi \prec \varphi$ , let  $\tilde{\psi}$  be the normal extension of  $\psi$  to  $\pi_\varphi(\mathcal{Q})''$ . Then the following conditions are equivalent:*

- (i)  $\psi$  satisfies the KMS condition with respect to  $\alpha_t$ ;
- (ii)  $\mathfrak{Z}_\varphi = \pi_\varphi(\mathcal{Q})'' \cap \pi_\varphi(\mathcal{Q})'$  is sufficient for  $\{\tilde{\varphi}, \tilde{\psi}\}$ ;
- (iii)  $\mathfrak{Z}_\varphi$  is weakly sufficient for  $\{\tilde{\varphi}, \tilde{\psi}\}$ ;
- (iv)  $(d\psi/d\varphi)\Omega_\varphi \in \mathfrak{Z}_\varphi\Omega_\varphi$ ;

- (v)  $(D\tilde{\varphi}: D(\tilde{\varphi} + \tilde{\psi}))_t \in \mathfrak{Z}_\varphi$  for all  $t \in \mathbf{R}$  where  $(D\tilde{\varphi}: D(\tilde{\varphi} + \tilde{\psi}))_t$  is the Connes Radon-Nikodym derivative (cf. [9]);
- (vi)  $S(\psi | \varphi) = S_{\mathfrak{Z}_\varphi}(\tilde{\psi} | \tilde{\varphi})$ .

*Proof.* Note that  $\tilde{\psi}$  is given by

$$\tilde{\psi}(Q) = \langle (d\psi/d\varphi)\Omega_\varphi, Q\Omega_\varphi \rangle, \quad Q \in \pi_\varphi(\mathcal{A})'',$$

and hence  $\tilde{\psi} < \tilde{\varphi}$ . Since there exists a conditional expectation  $\varepsilon_{\tilde{\varphi}}$  of  $\pi_\varphi(\mathcal{A})''$  onto  $\mathfrak{Z}_\varphi$  with  $\tilde{\varphi} = \tilde{\varphi} \circ \varepsilon_{\tilde{\varphi}}$ , the equivalence of (ii), (iii) and (iv) follows from Theorem 1.6 (Remark) and the proof of Theorem 1.8. Because the KMS condition of  $\psi$  with respect to  $\alpha_t$  and the same of  $\tilde{\psi}$  with respect to  $\tilde{\alpha}_t$  are equivalent, it follows from [14, Theorem 2.3] that (i) and (ii) are equivalent. Since  $(D\tilde{\varphi}: D(\tilde{\varphi} + \tilde{\psi}))_t = (D(\tilde{\varphi} + \tilde{\psi}): D\tilde{\varphi})_t^*$ , we see by [14, Lemma 2.1] the equivalence of (ii) and (v). Finally the equivalence of (i) and (vi) follows from Theorems 4.2 (2) and 4.3 (2) and Lemma 3.1 if we prove  $S_{\mathfrak{Z}_\varphi}(\tilde{\psi} | \tilde{\varphi}) < +\infty$ . There exists a positive self-adjoint operator  $h$  affiliated with  $\mathfrak{Z}_\varphi$  such that  $\tilde{\psi}(Q) = \tilde{\varphi}(hQ)$  for all  $Q \in \mathfrak{Z}_\varphi$ . Take the spectral decomposition  $h = \int_0^\infty \lambda \, de(\lambda)$ . Then the condition  $\tilde{\psi} < \tilde{\varphi}$  gives rise to  $\tilde{\varphi}(h^2) = \int_0^\infty \lambda^2 \, d\tilde{\varphi}(e(\lambda)) < +\infty$ . Hence we have

$$\begin{aligned} S_{\mathfrak{Z}_\varphi}(\tilde{\psi} | \tilde{\varphi}) &= \int_0^\infty \lambda \log \lambda \, d\tilde{\varphi}(e(\lambda)) \\ &\leq \int_0^\infty \lambda^2 \, d\tilde{\varphi}(e(\lambda)) < +\infty. \end{aligned} \quad \square$$

**REMARK.** Assuming only that  $\psi$  has the normal extension  $\tilde{\psi}$  to  $\pi_\varphi(\mathcal{A})''$  (which is necessarily a vector state), we obtain the equivalence of the conditions (i), (ii) and (v) in Theorem 4.4, which imply (vi) and are implied by the equality (vi) with a finite value. For the case of  $\psi$  being dominated by  $\varphi$ , the condition (iv) can be replaced by  $d\psi/d\varphi \in \mathfrak{Z}_\varphi$  (see e.g. [17, p. 104]). Also for the  $\alpha_t$ -invariance of  $\psi \in \mathfrak{S}$  with  $\psi < \varphi$ , we can obtain the similar equivalent conditions by substituting  $Z_{\tilde{\varphi}} = \pi_\varphi(\mathcal{A})'' \cap U_\varphi(\mathbf{R})'$  for  $\mathfrak{Z}_\varphi$  in the above conditions (ii)–(vi).

Theorem 4.4 finds an application in quantum lattice systems. Let  $L$  be a countable set and  $\mathcal{H}_0$  a finite-dimensional Hilbert space. For each nonempty finite set  $\Lambda \subset L$ , let  $\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{H}_x$  with  $\mathcal{H}_x = \mathcal{H}_0$  and  $\mathcal{A}_\Lambda = \mathbf{B}(\mathcal{H}_\Lambda)$ . Then the quantum lattice system on  $L$  is described by the quasi-local  $C^*$ -algebra  $\mathcal{A} = \overline{\bigcup_{\Lambda \subset L} \mathcal{A}_\Lambda}$ . An interaction  $\Phi$  is defined as a function from finite subsets  $\Lambda \subset L$  into the self-adjoint elements of  $\mathcal{A}$  such that  $\Phi(\Lambda) \in \mathcal{A}_\Lambda$ . Let  $\varphi$  be a state of  $\mathcal{A}$  satisfying the Gibbs condition with

respect to  $\Phi$  (see [8] for the definition). Now assume that  $\Phi$  satisfies

$$\sum_{n=0}^{\infty} e^{rn} \left( \sup_{x \in L} \sum_{\substack{\Lambda \ni x \\ |\Lambda| = n+1}} \|\Phi(\Lambda)\| \right) < +\infty$$

for some  $r > 0$ . Then the strongly continuous one-parameter automorphism group  $\alpha_t^\Phi$  of  $\mathcal{Q}$  can be given by

$$\alpha_t^\Phi(A) = \lim_{\Lambda \rightarrow L} e^{itH_\Phi(\Lambda)} A e^{-itH_\Phi(\Lambda)}, \quad A \in \mathcal{Q}, t \in \mathbf{R},$$

where  $H_\Phi(\Lambda) = \sum_{X \subset \Lambda} \Phi(X)$ . It is known (cf. [8, p. 268]) that  $\psi \in \mathcal{S}$  satisfies the Gibbs condition with respect to  $\Phi$  if and only if  $\psi$  satisfies the KMS condition with respect to  $\alpha_t^\Phi$ . Then we have

**COROLLARY 4.5.** *Let  $\mathcal{Q}$ ,  $\Phi$ ,  $\alpha_t = \alpha_t^\Phi$  and  $\varphi$  be as above, and let  $\psi \in \mathcal{S}$  with  $\psi < \varphi$ . Then the Gibbs condition for  $\psi$  with respect to  $\Phi$  is equivalent to each of the conditions (i)–(vi) in Theorem 4.4, and these conditions imply the following:*

(vii) *for each  $\Lambda \subset L$ ,  $\mathcal{Q}_{\Lambda^c} = \overline{\bigcup_{X \subset \Lambda^c} \mathcal{Q}_X}$  is weakly sufficient for  $\{\varphi, \psi\}$ ;*

(viii) *for each  $\Lambda \subset L$ ,  $S(\psi | \varphi) = S_{\mathcal{Q}_{\Lambda^c}}(\psi | \varphi)$ .*

*Further if  $\bigcup_{\Lambda \subset L} \pi_\varphi(\mathcal{Q}_\Lambda) \Omega_\varphi$  is a core for the modular operator  $\Delta_{\Omega_\varphi}$  associated with  $\Omega_\varphi$ , the condition (vii) conversely implies the Gibbs condition for  $\psi$  with respect  $\Phi$ .*

The main part of the corollary is immediate from Theorem 4.4 and the fact that  $\mathcal{B}_\varphi$  is identical to  $\bigcap_{\Lambda \subset L} \pi_\varphi(\mathcal{Q}_{\Lambda^c})''$  the algebra of observables at infinity. The last part follows by [6, Lemma 3].

We finally give some notes on the translationally invariant case of  $L = \mathbf{Z}^d$ . Let  $\tau$  be the automorphism group of translations on  $\mathbf{Z}^d$ . Let  $\Phi$  be a  $\tau$ -invariant interaction satisfying  $\sum_{\Lambda \ni 0} e^{r|\Lambda|} \|\Phi(\Lambda)\| < +\infty$  for some  $r > 0$ . A  $\tau$ -invariant state  $\varphi$  is said to be equilibrium with respect to  $\Phi$  if the following variational equality holds (see [17, 25]):  $P(\Phi) = s(\varphi) - \varphi(A_\Phi)$  where  $s(\varphi)$  is the mean entropy of  $\varphi$  and

$$P(\Phi) = \lim_{\substack{\Lambda \rightarrow \infty \\ (\text{van Hove})}} |\Lambda|^{-1} \log \text{tr}_\Lambda(e^{-H_\Phi(\Lambda)}),$$

$$A_\Phi = \sum_{\Lambda \ni 0} |\Lambda|^{-1} \Phi(\Lambda).$$

Then the equilibrium condition with respect to  $\Phi$ , the Gibbs condition with respect to  $\Phi$  and the KMS condition with respect to  $\alpha_t^\Phi$  are all equivalent for  $\tau$ -invariant states of  $\mathcal{Q}$  (cf. [3, 8, 20]). Let  $\varphi, \psi \in \mathcal{S}$  be

$\tau$ -invariant. Since  $\{\mathcal{Q}, \mathbf{Z}^d, \tau\}$  is asymptotically abelian, we obtain  $S(\psi | \varphi) = I(\nu | \mu)$  by Corollary 3.3 (2) where  $\mu$  and  $\nu$  are the ergodic decomposition measures of  $\varphi$  and  $\psi$ . If  $\varphi$  is equilibrium and  $\psi < \varphi$  (or more weakly  $\psi$  has the normal extension to  $\pi_\varphi(\mathcal{Q})''$ ), then it can be proved that  $\nu \ll \mu$ , so that  $\psi$  is automatically equilibrium because  $\mu$  is supported on the set of equilibrium states.

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