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CHARACTERISTIC CLASSES FOR SPHERICAL FIBRATIONS WITH FIBRE-PRESERVING FREE GROUP ACTIONS

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Let $G(H)$ be the monoid of H equivariant self maps of $S(nV)$, the unit sphere of n copies of a finite dimension orthogonal representation V of a finite group H, stabilized over n in an appropriate way. Let $SG(H)$ be the submonid of $G(H)$ consisting of all degree 1 maps. If H_1 is a subgroup of H there is a natural forgetful map $SG(H) \to SG(H_1)$ and if Z is the center of H there is a natural action map $BZ \times SG(H) \to SG(H)$ induced by the natural action of Z on H. The main results of this paper are the calculations of the Hopf algebra structures of $H_*(SG(\mathbf{Z}/p^n), \mathbf{Z}/p)$ and $H_*(BSG(\mathbf{Z}/p^n), \mathbf{Z}/p)$ for all n and all primes p, the calculations in homology of forgetful maps induced by the natural inclusions $\mathbf{Z}/p^{n-1} \to \mathbf{Z}/p^n$ and, for $H = \mathbf{Z}/2$, the calculation of the action map $H_*(\mathbf{R}P^\infty, \mathbf{Z}/2) \otimes H_*(BSG(\mathbf{Z}/2), \mathbf{Z}/2) \to H_*(BSG(\mathbf{Z}/2), \mathbf{Z}/2)$.

Introduction. In this paper we study oriented spherical fibrations modeled on an orthogonal representation V of a finite group H, a theory first studied by Segal [**25**], and Becker and Schultz [**2**]. More precisely we consider fibrations with a fibre-preserving action of H on the total space such that the fibre is H equivariantly homotopy equivalent to the unit sphere $S(V)$ of V. We stabilize these fibrations by forming the fibrewise join with the trivial $S(V)$ bundle. As we have specific geometric applications in mind, we concentrate on the case where $S(V)$ is *H-free*.

Well known results of Barrett, Gugenheim, and Moore [**1**], May [**15**], Stasheff [**29**], and Waner [**35**], reduce this question to the study of the homotopy-type of the classifying space of the submonoid $SG(H)$ of degree 1 maps in the space

$$G(H) = \varinjlim_{n} \mathrm{Map}_H(S(nV), S(nV))$$

of stable equivariant self-maps of $S(nV)$. This is the model studied by Becker and Schultz [**2**]. They showed for any compact Lie group H, that $G(H)$ is homotopy equivalent to $Q(BH^\zeta)$, where $Q(X)$ is the union over n of $\Omega^n\Sigma^n X$, and BH^ζ is the Thom space of the vector bundle ζ over BH associated to the adjoint representation of H on its Lie algebra. Notice that if H is finite then $\zeta = 0$, and $BH^\zeta = BH^+$, the disjoint union of BH

with a distinguished base point. Becker and Schultz also show that $SG(H)$, the degree 1 component of $G(H)$, is, in fact, an infinite loop space under the composition product.

Of course when H is the trivial group one obtains the classically important space G which has been the cornerstone for much important work in the study of smooth, piecewise linear and topological manifolds. Milgram **[18]**, computed the Hopf algebra structure, under composition product, of $H_*(SG, \mathbf{Z}/p)$ for all primes p. The Hopf algebra structure of $H_*(BSG, \mathbf{Z}/p)$ was obtained by Madsen **[11]**, for the prime 2 and independently by May **[6]**, and Tsuchiya **[33]**, for odd primes. These computations were all obtained by relating the composition product in SG to the loop sum and composition product in $Q(S^0)$. $Q(S^0)$ was, in turn, approximated by finite models where the loop sum and composition product were interpreted directly in terms of homomorphisms of the symmetric groups.

Additional structure is present in the equivariant case as the classifying space, BC, of the center C of H acts naturally on both $SG(H)$ and $BSG(H)$. In fact this action, when $H = \mathbf{Z}/2$, provided the initial motivation for this paper as well as our joint work with Haynes Miller **[12]**, and **[13]**. More precisely we show that when $H = \mathbf{Z}/2$ this action is geometrically represented by the "tensoring of a real line bundle with a $\mathbf{Z}/2$-equivariant spherical fibration." This information is used in **[14]**, to investigate the question of which unoriented PL cobordism classes admit mapping tori and projective bundle representatives.

Further additional structure comes from group inclusions $H_1 \subset H_2$ which induce transfer maps t: $SG(H_2) \to SG(H_1)$ and Bt: $BSG(H_2) \to BSG(H_1)$ which are infinite loop maps.

The main results of this paper, which are stated in Chapter I, sections two through four, are the determination of the Hopf algebras, with respect to the composition product, of $H_*(SG(\mathbf{Z}/p^n), \mathbf{Z}/p)$ and $H_*(BSG(\mathbf{Z}/p^n), \mathbf{Z}/p)$ for all primes p, the determination of the action maps $H_*(B\mathbf{Z}/2, \mathbf{Z}/2) \otimes H_*(SG(\mathbf{Z}/2), \mathbf{Z}/2) \to H_*(SG(\mathbf{Z}/2), \mathbf{Z}/2)$ and $H_*(B\mathbf{Z}/2, \mathbf{Z}/2) \otimes H_*(BSG(\mathbf{Z}/2), \mathbf{Z}/2) \to H_*(BSG(\mathbf{Z}/2), \mathbf{Z}/2)$ and the determination of the transfer maps in homology associated to the inclusions of $1 \subset \mathbf{Z}/p^{n-1} \subset \mathbf{Z}/p^n$. We remark that in a closely related paper with Haynes Miller **[13]**, we computed $H_*(SG(S^1), \mathbf{Z}/p)$ and $H_*(BSG(S^1), \mathbf{Z}/p)$ as well as the transfer maps associated to the inclusions $\mathbf{Z}/p^n \subset S^1$, and obtained results quite similar in spirit to those obtained here.

Our study of $G(H)$ began because we were interested in computing characteristic classes for certain natural PL constructions. Hence before explaining how we prove the results of this paper we briefly state, without proof, some relevant geometric facts that led us to study $G(H)$. The construction of geometric representatives for generators for PL bordism (see **[14]**) should (but does not at present) procede by means of projective bundles as it does in the smooth category. Moreover, finite group actions on PL manifolds should have the same computational characteristic class underpinnings that smooth actions do. In the semi-free setting the relevant bundle classifying spaces may be described as follows.

First there exist equivariant classifying spaces for stable piecewise linear and topological block bundles which admit block preserving H actions which are free off the zero section. Further there are natural inclusions $BSO \to BSPL(H) \to BSTOP(H) \to BSG(H)$ with the evident fibres $G/PL(H)$, $G/TOP(H)$ and so on. Secondly there are natural action maps $BC \times BSPL(H) \to BSPL(H)$ and $BC \times BSTOP(H) \to BSTOP(H)$ obtained by twisting by the center which are compatible with the natural inclusions mentioned above. For example when $H = C = \mathbf{Z}/2(S^1)$ the action map on the $BSO(BU)$ level classifies tensor product with a real (complex) line bundle. Thirdly there are unstable versions of points one and two.

It is evident that the results of this paper play the same role in the equivariant category as the results of **[18]**, **[11]**, **[6]** and **[33]** on $H_*(G)$ and $H_*(BG)$ played in the fundamental calculations of Brumfiel, Madsen, and Milgram **[5]**, of $H_*(BPL)$ and PL cobordism. In future papers we intend to obtain calculational control of $BSPL(H)$, $BSTOP(H)$, their associated action maps and investigate the geometric consequences of these calculations.

To prove our results we follow the same general procedure used in the classical, nonequivariant, case. In Chapters II, III, and IV we prove the theorems necessary to reduce all computations of homology groups, Pontrjagin products, homology operations, action maps and transfer maps for $SG(H)$ and $BSG(H)$ to computations in certain finite models. Before explaining this reduction, we mention that in our companion paper with Haynes Miller **[12]**, we carry out the computations in the correct finite models which yield the results stated in Chapter I. The computations are described in Chapter V.

The methods we use apply the E_∞-ring space machinery of May [**16**], and [**17**]. We take this opportunity to point out that current work of Hauschild, May, and Waner [**9**], on equivariant infinite loop space theory gives the general E_∞-ring space theory for based equivariant spaces. Their machinery is in a setting of much greater generality than the free and semi-free cases we consider here. We do give a complete description of the machinery necessary to make all our required homological calculations. This is done not only for the convenience of the reader but also in order to identify the free model and to describe the action of the center which has important geometrical consequences (see Chapter IV and [**14**]).

To explain the reduction of our problem to finite models, we begin by considering a space studied by Segal [**25**], [**27**]. Segal examined

$$F(H) = \varinjlim_{m,n} \mathrm{Map}_H(t(mV \oplus R^n), t(mV \oplus R^n))$$

the space of based equivariant self-maps of the one point compactification $t(mV \oplus R^n)$ of $mV \oplus R^n$ stabilized by forming the limit over equivariant inclusions of orthogonal representations. Inclusion and radical extension gives a map from the Becker-Schultz space to the Segal space $\rho\colon G(H) \to F(H)$. Unlike the classical case ($H = \mathrm{id}$), in general $G(H)$ does not fill out a component of $F(H)$. Nonetheless, ρ is an infinite loop map, with respect to the composition product, and in Chapter III we express $G(H)$ as a factor of $F(H)$.

Segal also considered the semi-free Burnside space of H, $SA(H)$, a group completion of the monoid structure induced by disjoint union on the classifying space of a small category, $S\mathcal{A}(H)$, equivalent to the category of finite H sets and their equivariant automorphisms. In fact Segal constructed a natural configuration map $\phi\colon BS\mathcal{A}(H) \to F(H)$ and thereby obtained the decomposition $F(H) \simeq Q(S^0) \times Q(BH^+)$ as spaces.

We next require a result of Hauschild, May and Waner [**9**], which using another model for $BS\mathcal{A}(H)$, K^H, considers the E_∞-ring compatibility of the Segal decomposition. More precisely, in Chapter II we consider the E_∞ operad pair $(\mathcal{K}^H, \mathfrak{L}^H)$ discovered by Steiner [**30**], and show it acts on K^H and $F(H)$ so as to make both space E_∞-ring spaces and ϕ an E_∞-ring map. Independently Segal [**27**], and Hauschild [**8**], have shown ϕ is a group completion. The $\mathfrak{L}^H$ structure on $F(H)$ and $G(H)$ correspond to the composition product.

Thus to obtain the desired results concerning $SG(H)$, we need only compute the $\mathfrak{L}^H$ product structure on the correct factor of K^H. We isolate this factor in Chapter III and the Appendix.

In Chapter IV we define the center action map for all finite groups H. We then restrict our attention to $\mathbf{Z}/2$ where we prove the homological action map is the same as the geometric twist map of [**14**] and §4.3.

Finally we note that there are more general versions of based equivariant function spaces, where the H action is not assumed to be semi-free, and that many of our results, especially 2.1, 2.2, and 4.1, hold for these more general spaces with no essential change in the proofs. In particular, the center action map is well-defined for all these spaces and homologically described in [**12**].

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CHAPTER I

1.1. In this section we define the Becker-Schultz and Segal equivariant function spaces, related transfer maps and the $\mathbf{Z}/2$ and S^1 equivariant J homomorphisms which are our fundamental objects of interest.

Let H be a finite group and let V be a fixed even dimensional real representation of H such that H acts freely on $V - \{0\}$. Further, suppose V has an H invariant metric. Let $\mathbf{R}$ be the trivial 1-dimensional real representation of H.

DEFINITION 1.1.1. $G(H) = \varinjlim_n \mathrm{Map}_H[S(nV), S(nV)]$, is the space of H-equivariant self-maps of the unit sphere $S(nV)$ of nV stabilized.

DEFINITION 1.1.2. $F(H) = \varinjlim_{m,n}[t(mV \oplus \mathbf{R}^n), t(mV \oplus \mathbf{R}^n)]$ is the space of H-equivariant pointed self-maps of the one point compactification $t(mV \oplus \mathbf{R}^n)$ of $(mV \oplus \mathbf{R}^n)$ stabilized.

REMARKS.

1. $S(nV)$ is H-free.
2. To form the limits above one uses the suspension maps

$$\sigma_G(n_1, n_2)\colon \mathrm{Map}_H(S(n_1V), S(n_1V)) \to \mathrm{Map}_H(S(n_2V), S(n_2V))$$

and

$$\sigma_F(m_1, n_1, m_2, n_2)\colon \mathrm{Map}_H(t(m_1 \oplus \mathbf{R}^{n_1}), t(m_1V \oplus \mathbf{R}^{n_1})) \\ \to \mathrm{Map}_H(t(m_2V \oplus \mathbf{R}^{n_2}), t(m_2V \oplus \mathbf{R}^{n_2}))$$

defined by the formula

$$\sigma_G(n_1, n_2)f = f \wedge \mathrm{id}_{(n_2-n_1)V},$$

$$\sigma_F(m_1, n_1, m_2, n_2)f = f \wedge \mathrm{id}_{(m_2-m_1)V\oplus(n_2-n_1)\mathbf{R}}$$

where we use the H-equivariant homeomorphisms and identifications

$$S(n_1V) \wedge S((n_2 - n_1)V) \cong S(n_2v)$$

and

$$t(m_1V \oplus \mathbf{R}^{n_1}) \wedge t((m_2 - m_1)V \oplus \mathbf{R}^{n_2-n_1}) \cong t(m_2V \oplus \mathbf{R}^{n_2})$$

which follows as V has an H invariant metric.

$G(H)$ and $F(H)$ were studied by Becker and Schultz [**2**], and Segal [**25**], respectively. Composition of maps gives each space an infinite loop space structure and the natural map

(1.1.3) $$\rho\colon G(H) \to F(H)$$

induced by the inclusion and radial extension from $S(mV)$ to $t(mV \oplus \mathbf{R}^n)$ is an infinite loop space map with respect to the composition product (see 2.2.7 and 2.3.1).

Let $H_1 \hookrightarrow H_2$ be an inclusion of a subgroup. Representations of H_2 restrict to give representations of H_1 and any H_2 equivariant map f: $S(nV_{H_2}) \to S(nV_{H_2})$ or f: $t(mV_{H_2} \oplus \mathbf{R}^n) \to t(mV_{H_2} \oplus \mathbf{R}^n)$ is trivially H_1 equivalent. Thus we obtain the canonical "forgetful maps"

(1.1.4) $$t\colon G(H_2) \to G(H_1), \\ t\colon F(H_2) \to F(H_1)$$

which are also infinite loop space maps with respect to the composition product.

We note that Segal and Becker and Schultz showed $F(H)$ and $G(H)$ are homotopy equivalent to $Q(S^0) \times Q(BH^+)$ and $Q(BH^+)$ respectively and furthermore, they identified the forgetful maps with the associated transfer maps. The following commutative diagrams summarize the situation.

(1.1.5)
$$\begin{array}{ccc} G(H_2) & \stackrel{\sim}{\to} & Q(BH_2^+) \\ \downarrow \text{forget} & & \downarrow t \\ G(H_1) & \stackrel{\sim}{\to} & Q(BH_1^+) \end{array}$$

and

(1.16)
$$\begin{array}{ccc} F(H_2) & \stackrel{\sim}{\to} & QS^0 \times Q(BH_2^+) \\ \downarrow \text{forget} & & \downarrow t \\ F(H_1) & \stackrel{\sim}{\to} & QS^0 \times Q(BH_1^+). \end{array}$$

The precise relation between 1.1.5 and 1.1.6 is given in Chapter III.

We are primarily interested in the $+1$ component of $G(H)$ which we call $SG(H)$ and are primarily interested in the case where H is a cyclic group of prime power order. Of course, $SG(\text{id})$ is the classical space SG studied by many people. In the classical case the J homomorphism J: $SO \to SG$ played a pivotal role at the prime 2. Since a linear orthogonal map commutes with the antipodal action $x \to -x$ on the unit sphere the J homomorphism lifts to a map

(1.1.7)
$$\begin{array}{ccc} & \overset{J_{\mathbf{Z}/2}}{\nearrow} & SG(\mathbf{Z}/2) \\ & & \downarrow \text{forget} \\ SO & \stackrel{J}{\to} & SG \end{array}$$

which is compatible with the forgetful map. We shall see that $J_{\mathbf{Z}/2}$ can *not* lift to a $\mathbf{Z}/4$ equivariant J homomorphism for homological grounds in the next section.

REMARK. In this paper we restrict our attention to finite groups whereas Becker and Schultz have defined $G(H)$ for all compact Lie groups H. For $H = S^1$ there is a complex J homomorphism J_{S^1}: $U \to SG(S^1)$ compatible with the following diagrams:

(1.1.8)
$$\begin{array}{ccc} U & \stackrel{J_{S^1}}{\to} & SG(S^1) \\ & & \downarrow \\ \downarrow & & SG(\mathbf{Z}/2^n) \\ & & \downarrow \\ SO & \stackrel{J_{\mathbf{Z}/2}}{\to} & SG(\mathbf{Z}/2) \\ & \searrow J & \downarrow \\ & & SG \end{array}$$

and, for p odd

$$(1.1.9)\qquad \begin{array}{ccc} U & \xrightarrow{J_{S^1}} & SG(S^1) \\ & & \downarrow \\ \downarrow & & SG(\mathbf{Z}/p^n) \\ & & \downarrow \\ SO & \xrightarrow{J} & SG \end{array}$$

See [13] for an analysis of J_{S^1} and its compatibility with 1.1.8 and 1.1.9.

1.2. The next three sections state the main results of this paper. We postpone all proofs of theorems in these sections to Chapter V. We begin with the characteristic classes for spherical fibrations with fibre-preserving free $\mathbf{Z}/p^n$ actions. The reader familiar with the classical results of Milgram [18], May [6], Tsuchiya [33], Madsen [11], and Brumfiel, Madsen, and Milgram [5], will note the extreme similarity with our equivariant results. We assume the reader is familiar with the definitions, structure, and properties of the Dyer-Lashof algebra R (for details, see [6]). To fix our notation, we recall:

(a) $\tilde{H}_*(B\mathbf{Z}/2, \mathbf{Z}/2) \cong P[e_r \mid r > 0]$;

(b) $\tilde{H}_*(B\mathbf{Z}/2^n, \mathbf{Z}/2) \cong E[e_{2r+1} \mid r \geq 0] \otimes P[e_{2r} \mid r > 0]$ for $n > 1$;

(c) $\tilde{H}_*(B\mathbf{Z}/p^n, \mathbf{Z}/p) \cong E[e_{2r+1} \mid r \geq 0] \otimes P[e_{2r} \mid r > 0]$ for p odd and all $n > 0$; and

THEOREM 1.2.1 [7]. *As a Hopf algebra under the loop product*

$$H_*(Q_0(B\mathbf{Z}/p^{n^+}), \mathbf{Z}/p) \cong \Lambda(x_{I,\alpha})$$

where $\Lambda(x_{I,\alpha})$ *is the free commutative algebra on* $X_{I,\alpha} = Q^I(e_\alpha) * (\chi(e_0))^{p^{l(I)}}$ *where* $Q^I(e_\alpha)$ *is admissible of positive excess.*

Here $*$ is the loop sum and χ: $Q(B\mathbf{Z}/p^{n^+}) \to Q(B\mathbf{Z}/p^{n^+})$ is the canonical anti-automorphism.

Recall, 1.1.5, that Becker and Schultz showed $Q(B\mathbf{Z}/p^{n^+})$ and $G(\mathbf{Z}/p^n)$ are homotopy equivalent. In Chapter III we prove that $SG(\mathbf{Z}/p^n)$ may be identified as an infinite loop space under composition product with the factor $[1] \times Q_0(B\mathbf{Z}/p^{n^+}) \hookrightarrow Q_1(S^0) \times Q_0(B\mathbf{Z}/p^{n^+})$ in the Segal model for $SF(\mathbf{Z}/p^n)$.

DEFINITION 1.2.2. $\overline{X}_{I,\alpha} = [1] * X_{I,\alpha}$.

Recall

THEOREM 1.2.3. *As a Hopf algebra under the composition product* $\tilde{H}_*(SO, \mathbf{Z}/2) \cong E[e_r \mid r > 0]$ *where*

(a) $\dim e_r = r$;

(b) $\psi(e_r) = \Sigma_{i+j=r} e_i \otimes e_j$ (*ψ is the coalgebra map*).

Recall the $\mathbf{Z}/2$ equivariant J homomorphism, $J_{\mathbf{Z}/2}$, 1.1.7, maps SO into $SG(\mathbf{Z}/2)$.

THEOREM 1.2.4.

(a) $J_{\mathbf{Z}/2*}$: $H_*(SO, \mathbf{Z}/2) \to H_*(SG(\mathbf{Z}/2), \mathbf{Z}/2)$ *is an injection of Hopf algebras.*

(b) *As a Hopf algebra under the composition product*

$$H_*(SG(\mathbf{Z}/2), \mathbf{Z}/2) \cong H_*(SO, \mathbf{Z}/2) \otimes P\left[J_{\mathbf{Z}/2}(e_r) * J_{\mathbf{Z}/2}(e_r) * [-1], \bar{X}_{I,\alpha}\right]$$

where (I, α) is an admissible sequence of positive length and positive excess. The co-algebra structure on the polynomial subalgebra is induced by the Dyer-Lashof algebra.

It is possible to explicitly determine $J_{\mathbf{Z}/2}(e_r)$ in $H_*(\mathrm{SG}(\mathbf{Z}/2), \mathbf{Z}/2)$.

THEOREM 1.2.5 **[13]**. $J_{\mathbf{Z}/2}(e_r) = (e_r \circ (e_0 * [-1])) * ([1] * \chi(e_0)) = \Sigma_{i+j=r} \chi(e_i) * Q^j(e_0) * [1] * \chi(e_0)$.

For later computations it is convenient to change to a basis of $H_*(SG(\mathbf{Z}/2), \mathbf{Z}/2)$ that is more natural with respect to the $\mathbf{Z}/2$ J-homomorphism. More precisely, letting

$$\bar{\bar{X}}_{I,\alpha} = Q^I\left(J_{\mathbf{Z}/2}(e_\alpha)\right) * \left(\chi(e_0)\right)^{2^{l(I)}} * [1],$$

we have

THEOREM 1.2.6. *As a Hopf algebra under the composition product*

$$H_*(SG(\mathbf{Z}/2), \mathbf{Z}/2) \cong H_*(SO, \mathbf{Z}/2) \otimes P\left[J_{\mathbf{Z}/2}(e_r) * J_{\mathbf{Z}/2}(e_r) * [-1], \bar{\bar{X}}_{I,\alpha}\right]$$

where (I, α) is an admissible sequence of positive length and positive excess.

We now consider $SG(\mathbf{Z}/2^n)$ for $n > 1$.

Consider the composition

$$U \xrightarrow{J_{S^1}} SG(S^1) \xrightarrow{t_n} SG(\mathbf{Z}/2^n).$$

By results of [**13**], if $\bar{a}_r \in H_{2r+1}(U, \mathbf{Z}/2)$ is the standard generator, then

$$(t_n \circ J_{S^1})_* \bar{a}_r = \sum_s e_{2s+1} * \chi e_{2(r-s)} * [1]$$
$$+ \sum_t Q^{2t+1}(e_{2t}) * (\chi(e_{r-2t}))^{*2} * [1].$$

DEFINITION 1.2.7. For $n > 1$ let

$$J_n(e_{2r+1}) = (t_n \circ J_{S^1})_* \bar{a}_r$$

and

$$\bar{e}_{2r} = e_{2r} * [1] * \chi(e_0).$$

The following theorem shows there is no $\mathbf{Z}/2^n$ equivariant J homomorphism for $n > 1$.

THEOREM 1.2.9. *As a Hopf algebra under the composition product*

$$H_*(SG(\mathbf{Z}/2^n), \mathbf{Z}/2) \cong H_*(U, \mathbf{Z}/2) \otimes P(\bar{e}_{2r})$$
$$\otimes P[(J_n(e_{2r+1}) * J_n(e_{2r+1}) * [-1]), \bar{X}_{I,\alpha}].$$

Here $n > 1$ and (I, α) is an admissible sequence of positive length and positive excess.

Again, the coalgebra structure is induced from the Dyer-Lashof algebra.

THEOREM 1.2.10. *Let p be an odd prime. As a Hopf algebra under the composition product*

$$H_*(SG(\mathbf{Z}/p^n), \mathbf{Z}/p) \cong H_*(Q_0(B\mathbf{Z}/p^{n^+}), \mathbf{Z}/p).$$

That is, the composition and loop Hopf algebra structures are abstractly isomorphic.

As in the classical case there does not exist a map inducing the isomorphism in 1.2.10.

We now turn to the classifying space level. First let $p = 2$.

DEFINITION 1.2.11. $B(\mathbf{Z}/2^n) = P[s(\overline{X}_{I,\alpha})]$ is the primitively generated Hopf algebra on $s(\overline{X}_{I,\alpha})$, where degree $s(\overline{X}_{I,\alpha}) =$ degree $\overline{X}_{I,\alpha} + 1$, (I, α) is an admissible sequence with $e(I, \alpha) \geq 2$ if $l(I, \alpha) \geq 2$ and

(a) for $n = 1$, $e(I, \alpha) \geq 1$ if $l(I, \alpha) = 1$;

(b) for $n > 1$, $e(I, 2\alpha + 1) \geq 1$ if $l(I, 2\alpha + 1) = 1$, $e(I, 2\alpha) \geq 2$ if $l(I, 2\alpha) = 1$;

(c) for $n > 1$ the classes $\overline{X}_{\phi,2\alpha} = e_{2\alpha} * \chi(e_0) * 1$ are in $B(\mathbf{Z}/2^n)$.

THEOREM 1.2.12. *As a Hopf algebra*

$$H_*(BSG(\mathbf{Z}/2), \mathbf{Z}/2) \cong H_*(BSO, \mathbf{Z}/2) \otimes E\big[s\big(J_{\mathbf{Z}/2}(e_r) * J_{\mathbf{Z}/2}(e_r) * [-1]\big)\big] \otimes B(\mathbf{Z}/2).$$

THEOREM 1.2.13. *Let $n > 1$, then as a Hopf algebra*

$$H_*(BSG(\mathbf{Z}/2^n), \mathbf{Z}/2) \cong H_*(BU, \mathbf{Z}/2) \otimes E\big[s\big(J_{\mathbf{Z}/2}(e_{2r+1}) * J_{\mathbf{Z}/2}(e_{2r+1}) * [-1]\big)\big] \otimes B(\mathbf{Z}/2^n).$$

Now let p be odd.

THEOREM 1.2.14. *For p odd and all $n \geq 1$*

$$H_*(BSG(\mathbf{Z}/p^n), \mathbf{Z}/p) \cong H_*(BSO, \mathbf{Z}/p) \otimes S$$

as Hopf algebras where S is a primitively generated free commutative algebra.

1.3. In this section we give the determinations of the forgetful maps t: $SG(H_2) \to SG(H_1)$ for the spaces in the previous section. Segal and Becker and Schultz have identified the forgetful map up to homotopy with the transfer map t: $Q(BH_2^+) \to Q(BH_1^+)$ which is an infinite loop map with respect to the loop sum structure. As the forgetful map is naturally an infinite loop space with respect to the composition product structure, it behaves well in homology with respect to both products and their associated Dyer-Lashof operations. Thus it suffices to determine t: $SG(H_2) \to SG(H_1)$ on the cells of BH_2^+.

THEOREM 1.3.1. *The transfers*

$$t(2^n, 2^{n-1})\colon H_*(SG(\mathbf{Z}/2^n), \mathbf{Z}/2) \to H_*(SG(\mathbf{Z}/2^{n-1}), \mathbf{Z}/2)$$

are completely determined by the following formulae.

(a) $t(2,1)e_i = Q^i[1]$;
(b) $t(4,2)e_{2i} = Q^i e_i$, $t(4,2)e_{2i+1} = \Sigma_{j=0}^{i} e_{2j+1} * e_{2(i-j)}$;
(c) *for* $n > 2$,

$$t(2^n, 2^{n-1})e_{4i} = Q^{2i}e_{2i},$$

$$t(2^n, 2^{n-1})e_{4i+1} = Q^{2i+1}e_{2i} + \sum_{j=0}^{2i} e_{2j} * e_{4i-2j+1},$$

$$t(2^n, 2^{n-1})e_{4i+2} = 0,$$

$$t(2^n, 2^{n-1})e_{4i+3} = \sum_{j=0}^{2i} e_{2j} * e_{4i-2j+3}.$$

COROLLARY 1.3.2. *The transfers*

$$t(2^n, 2^{n-1}) \colon H_*(SG(\mathbf{Z}/2^n), \mathbf{Z}/2) \to H_*(SG(\mathbf{Z}/2^{n-1}), \mathbf{Z}/2)$$

are not surjective for $n > 1$.

Corollary 1.3.2 gives another proof that the classical J homomorphism may not be lifted to $SG(\mathbf{Z}/2^n)$ for $n > 1$.

Theorem 1.3.1 is sufficient to explicitly calculate all transfers induced by the inclusions $\mathbf{Z}/2^m - \mathbf{Z}/2^n$ for all $m < n$. We state one explicit result.

THEOREM 1.3.3. *The transfers*

$$t(2^n, 1) \colon H_*(SG(\mathbf{Z}/2^n), \mathbf{Z}/2) \to H_*(SG, \mathbf{Z}/2)$$

are completely determined by the following formulae.
(a) $t(2,1)e_i = Q^i(1)$;
(b) $t(4,1)e_{2i} = Q^iQ^i(1) = (Q^i(1))^{*2}$,
$t(4,1)e_{2i+1} = Q^{i+1}Q^i(1) + \Sigma_{j=0}^{i} Q^{2j+1}(1) * Q^{2(i-j)}(1)$;
(c) *for* $n > 2$,

$$t(2^n, 1)(e_{2^{n-1}\cdot M}) = (Q^M(1))^{*2^{n-1}},$$

$$t(2^n, 1)(e_{2N}) = 0 \quad \text{if } 2^{n-2} \nmid N;$$

$$t(2^n, 1)(e_{2d+1}) = \sum (Q^{j_1}(1))^{*2^{n-1}} * (Q^{j_2}(1))^{*2^{n-3}} * \cdots * (Q^{j_l}(1))^{*2} * \left[Q^{2k+1}(1) * Q^{2t}(1) + Q^{k+t+1}Q^{k+t}(1)\right]$$

where the sum runs over all $j_1, \ldots, j_l$, k, t *so that*

(1) $$\sum_{r=1}^{n-2} 2^{n-2-r} \cdot j_r \leq 2d;$$

(2) $$k, t \geq 0; \qquad t = d - \sum_{r=1}^{n-2} 2^{n-3-r} j_r - k.$$

One obtains analogous results for odd primes.

THEOREM 1.3.4. *The transfers*

$$t(p^n, p^{n-1}): H_*(SG(\mathbf{Z}/p^n), \mathbf{Z}/p) \to H_*(SG(\mathbf{Z}/p^{n-1}), \mathbf{Z}/p)$$

for p an odd prime are completely determined by the following formulae.

(a) $t(p,1)(e_{2s(p-1)}) = (-1)^s Q^s(1)$,
$t(p,1)(e_{2s(p-1)-1}) = (-1)^s \beta Q^s(1)$,
$t(p,1)(e_i) = 0$ *otherwise*;

(b) *for* $n > 1$,

$$t(p^n, p^{n-1})(e_{p(2m)}) = Q^{2m}(e_{2m}) = (e_{2m})^{*p},$$

$$t(p^n, p^{n-1})(e_{p(2m+2d+1)})$$
$$= \sum \alpha(2m_1 + 2d + 1, m_2, \ldots, m_p) e_{2m_1+2d+1} * e_{2m_2} * \cdots * e_{2m_p},$$

where

(i) $0 \leq d < (p-1)/2$.

(ii) *The sum runs over all nonnegative* $m_1, \ldots, m_p$ *such that* $\sum_{i=1}^{p} m_i = m$.

(iii) *The coefficient* $\alpha(j_1, \ldots, j_p)$ *is the number, taken* mod p, *of permutations* σ *of* $1, 2, \ldots, p$ *such that* $(j_1, j_2, \ldots, j_p) = (j_{\sigma(1)}, j_{\sigma(2)}, \ldots, j_{\sigma(p)})$. *And finally*

$$t(p^n, p^{n-1})(e_i) = 0 \quad \textit{otherwise}.$$

THEOREM 1.3.5. *The transfers*

$$t(p^n, 1): H_*(SG(\mathbf{Z}/p^n), \mathbf{Z}/p) \to H_*(SG, \mathbf{Z}/p)$$

for p an odd prime are completely determined by the following formulae.

(a) $t(p,1)e_{2s(p-1)} = (-1)^s Q^s(1)$,
$t(p,1)e_{2s(p-1)-1} = (-1)^s \beta Q^s(1)$,
$t(p,1)e_i = 0$ *otherwise*;

(b) $t(p^2,1)(e_{2s(p-1)p}) = (-1)^s Q^{2s}Q^s(1) = (-1)^s(Q^s(1))^{*p}$,
$t(p^2,1)(e_{2M}) = 0$ *if* $p(p-1) \nmid M$,
$t(p^2,1)(e_{2N-1}) = 0$ *if* $p-1 \nmid N$,

$$t(p^2,1)(e_{2d(p-1)-1}) = (-1)^d \sum \alpha(\cdot, s_2, \ldots, s_p)\beta Q^{s_1}(1) * Q^{s_2}(1) * \cdots * Q^{s_p}(1)$$

where the sum runs over all nonnegative $s_1, \ldots, s_p$ *such that* $\Sigma_{i=1}^p s_i = d$ *and* $\alpha(\cdot, s_2, \ldots, s_p)$ *is defined as in* 1.3.4;

(c) *for* $n > 2$,

$$t(p^n,1)e_{2s(p-1)p^{n-1}} = (-1)^s(Q^s(1))^{*p^{n-1}},$$

$$t(p^n,1)e_{2M} = 0 \quad \text{if } (p-1)p^{n-1} \nmid M,$$

$$t(p^n,1)e_{2N-1} = 0 \quad \text{if } (p-1) \nmid N,$$

$$\begin{aligned} t(p^n,1)e_{2d(p-1)-1} &= (-1)^d \sum (\text{coeff})\beta Q^{s_{n-1,1}}(1) * Q^{s_{n-1,2}}(1) * \cdots * Q^{s_{n-1,p}}(1) \\ &\quad * (Q^{s_{n-2,2}}(1) * \cdots * Q^{s_{n-2,p}}(1))^{*p} \\ &\quad * \cdots * (Q^{s_{1,2}}(1) * \cdots * Q^{s_{1,p}}(1))^{*p^{n-2}}, \end{aligned}$$

where the sum runs over $s_{n-1,1}$ *and all* $s_{a,b}$, $1 \le a \le n-1$, $2 \le b \le p$ *such that* $\Sigma_{c=1}^{n-1}[p^{c-1}\Sigma_{i=2}^p s_{n-c,i}] = d - s_{n-1,1}$ *and the coefficient*, coeff, *is the product* $\prod_{c=1}^{n-1} \alpha(\cdot, s_{n-c,2}, \ldots, s_{n-c,p})$.

1.4. In this section we examine the homological implications of the geometric fact that a real line bundle may be tensored with a $\mathbf{Z}/2$ equivariant spherical fibration. Thus we let $H = \mathbf{Z}/2$ and concentrate our attention on $G(\mathbf{Z}/2)$ and $BG(\mathbf{Z}/2)$. In fact the original motivation for this paper was to obtain Theorem 1.4.8 which is used in [**14**] to study mapping tori and projective bundle constructions in unoriented *PL* cobordism.

We now consider $G(\mathbf{Z}/2)_{\pm 1}$ the ± 1 component of $G(\mathbf{Z}/2)$ consisting of stable $\mathbf{Z}/2$ equivariant $V_{\mathbf{Z}/2}$ spherical homotopy equivalences. As mentioned in the introduction, [**1**], [**15**], and [**34**] imply $BG(\mathbf{Z}/2)_{\pm 1}$ classifies stable virtue spherical fibrations with free fibre-preserving involutions ($BSG(\mathbf{Z}/2)$ classifies the oriented theory).

Now given a real line bundle L over M and a spherical fibration over M with free involution, (ξ, t), one naturally obtains a new spherical fibration $L \otimes_t \xi$ over M which also has an involution induced by t.

DEFINITION 1.4.1. $L \otimes_t \xi$ is the space $\tilde{M} \times_{\mathbf{Z}/2} \xi$ where $\tilde{M}$ is the unique double cover of M associated to L. Denote by $\bar{t}$ the involution t induces on $L \otimes_t \xi$.

Hence by considering virtual bundles $L \otimes_t (\xi - N) = L \otimes_t \xi - L \otimes N$ (here $N = \dim \xi$), we obtain a map

$$\psi_G \colon \mathbf{R}P^\infty \times BG(\mathbf{Z}/2)_{\pm 1} \to BG(\mathbf{Z}/2)_{\pm 1}. \tag{1.4.2}$$

Of course if ξ is a vector bundle with the standard involution, we recover the standard tensor product so ψ_G covers the standard smooth map; that is, the following diagram commutes.

$$\begin{array}{ccc} \psi_G \colon \mathbf{R}P^\infty \times BG(\mathbf{Z}/2)_{\pm 1} & \to & BG(\mathbf{Z}/2)_{\pm 1} \\ \uparrow \mathrm{id} \times BJ_{\mathbf{Z}/2} & & \uparrow BJ_{\mathbf{Z}/2} \\ \psi_0 \colon \mathbf{R}P^\infty \times BO & \to & BO \end{array} \tag{1.4.3}$$

For ξ of virtual dimension zero, ψ_G and ψ_0 map $\mathbf{R}P^\infty \times$ base point to the base point. Thus we may loop the map in 1.4.3 to obtain

$$\begin{array}{ccc} \mathbf{R}P^\infty \times Q(\mathbf{R}P^{\infty +}) & \to & Q(\mathbf{R}P^{\infty +}) \\ \cup\!\!\uparrow & & \cup\!\!\uparrow \\ \Omega\psi_G \colon \mathbf{R}P^\infty \times \Omega BG(\mathbf{Z}/2)_{\pm 1} & \to & \Omega BG(\mathbf{Z}/2)_{\pm 1} \\ \uparrow \mathrm{id} \times J_{\mathbf{Z}/2} & & \uparrow J_{\mathbf{Z}/2} \\ \Omega\psi_0 \colon \mathbf{R}P^\infty \times \Omega BO & \to & \Omega BO \\ \cup\!\!\uparrow & & \cup\!\!\uparrow \\ \mathbf{R}P^\infty \times SO & \to & SO \end{array} \tag{1.4.4}$$

Let us fix the following notation.

(1) $\tilde{H}_*(SO, \mathbf{Z}/2) \cong E[x_r \mid r > 0]$ where x_r is represented by the image of $\mathbf{R}P^r \hookrightarrow SO$ given by sending a line l in $\mathbf{R}^n$ to reflection perpendicular to l times a fixed rotation.

(2) $\tilde{H}_*(BO, \mathbf{Z}/2) \cong P[z_s \mid s > 0]$ where z_s is represented by the image of $\mathbf{R}P^s \hookrightarrow \mathbf{R}P^\infty \hookrightarrow BO$ where the inclusion $\mathbf{R}P^\infty \hookrightarrow BO$ classifies the virtual non-trivial line bundle over $\mathbf{R}P^\infty$.

(3) Let $y_s \in \tilde{H}_*(BO, \mathbf{Z}/2)$ be the class represented by the image of $S^1 \wedge \mathbf{R}P^s_+ \to S^1 \wedge O_+ \to BO$ where the first inclusion is given by $S^1 \wedge x_s$ and the second map is the adjoint to $0 \sim \Omega BO$.

(4) Let e_t represent the generator of $H_t(\mathbf{R}P^\infty, \mathbf{Z}/2)$.

Of course, ψ_0 is approximated by the maps $\mathbf{R}P^\infty \times BO(N) \to BO$ which classify $L \otimes (E_N - \bar{N})$. The following theorems are proved in Chapter V.

THEOREM 1.4.5.
(a) $\psi_{0_*}(e_t \otimes z_s) = \sum_{s=p+q}\binom{t+p}{p} z_{t+p}\cdot\chi(z_q)$;
(b) $\psi_{0_*}(e_t \otimes y_s) = \binom{t+s}{t} y_{s+t}$;
(c) $y_s = \sum_{w=(i_1,\dots,i_r)} N(w) z_{i_1}\cdot z_{i_2}\cdot\cdots\cdot z_{i_r}$ *where* $N(w)$ *is the coefficient in the expansion* $S_w(\sigma) = N(w) w_{(\sum_{\alpha=1}^r i_\alpha)}$ + *decomposables in* $H_*(BO)$;
(d) $(\Omega\psi_0)_*(e_t, x_r) = \binom{t+r}{r} x_{t+r}$.

THEOREM 1.4.6. *The map* $\Omega\psi_G$ *is a homology pairing for both loop sum and composition product structures. More precisely, we have the following formulae.*
(a) $\Omega\psi_{G_*}(e_t, x * y) = \sum_{a+b=t}\Omega\psi_{G_*}(e_a, x) * \Omega\psi_{G_*}(e_b, y)$;
(b) $\Omega\psi_{G_*}(e_t, Q^r(x)) = \sum_{a\geq 0} Q^{r+a}(\Omega\psi_{G_*}((Sq^a_*)e_t, x))$;
(c) $\Omega\psi_{G_*}(e_t, x\cdot y) = \sum_{a+b=t}\Omega\psi_{G_*}(e_a, x)\cdot\Omega\psi_{G_*}(e_b, y)$;
(d) $\Omega\psi_{G_*}(e_t, \hat{Q}^r(x)) = \sum_{a\geq 0}\hat{Q}^{r+a}(\Omega\psi_{G_*}((Sq^a_*)e_t, x))$.

Theorems 1.4.5, 1.4.6 and the fact that $J_{\mathbf{Z}/2}$ is an infinite loop map determine $\Omega\psi_{G_*}$. In Chapter V we recall a weight valuation first used by May [6], and Tsuchiya [33], on $H_*(Q(S^0), \mathbf{Z}/p)$ and extend it to give a filtration on $H_*(Q(B\mathbf{Z}/p^{n^+}), \mathbf{Z}/p)$. We then have

THEOREM 1.4.7. *Modulo higher weight*

$$\Omega\psi_{G_*}\big(e_j, Q^I(J_{\mathbf{Z}/2}(e_r))\big) * \Big([1] * \big(\chi(e_0)^{2^{l(I)}}\big)\Big)$$
$$= \sum_T c(T) Q^{I+T}\big(J_{\mathbf{Z}/2}(e_{r+j-s})\big) * \Big([1] * (\chi(e_0))^{2^{l(I)}}\Big)$$

where
(1) *the sum runs over all* $T = (t_1,\dots,t_k)$ *with each* $t_l \geq 0$.
(2) $I + T = (i_1 + t_1, i_2 + t_2, \dots, i_k + t_k)$.
(3) $s = \sum_{l=1}^k t_l$.
(4) $c(T)$ *is the following product of binomial coefficients*:

$$(j - 2t_1, t_1)(j - t_1 - 2t_2, t_2)\cdots(j - t_1 - t_2 - \cdots - 2t_k, t_k)(r, j - s).$$

Here we use the notation $(a, b) = \binom{a+b}{a}$.

REMARK. 1.4.7 may be restated by replacing the statement "modulo higher weight" by the statement "modulo loop decomposables."

Recalling that σ_* is the homology suspension, we conclude this section with

THEOREM 1.4.8. *Modulo higher weight*

$$\psi_{G_*}\Big(e_j, \sigma_*\Big(Q^I\big(J_{\mathbf{Z}/2}(e_r)\big) * \Big([1] * \big(\chi(e_0)\big)^{2^{l(I)}}\Big)\Big)\Big)$$
$$= \sum_T c(T)\sigma_*\Big(Q^{I+T}\big(J_{\mathbf{Z}/2}(e_{r+j-s})\big) * \Big([1] * \big(\chi(e_0)\big)^{2^{l(I)}}\Big)\Big)$$

where T, $I+T$, s, *and* $c(T)$ *are as defined in* 1.4.7.

CHAPTER II

In this chapter we use the E_∞ ring space machinery of May and his school **[16]**, **[17]**, and **[9]** to construct a good E_∞ ring model for $F(H)$. The preliminary version of this chapter was revised after receiving a helpful letter from J. P. May whom we would like to thank for his advice and comments.

We begin by recalling a specific E_∞ operad pair. The additive operad $\mathcal{K}(\bar{j})$ was recently discovered by Richard Steiner **[30]**.

Let $\bar{j}$ be the finite right H-set $\{1, 2, \ldots, j\}$ and let W be any H invariant finite dimensional subspace of $(V \oplus \mathbf{R})_H = \bigcup_{m,n}(mV \oplus \mathbf{R}^n)$. Give $\bar{j} \times W$ the diagonal H action.

DEFINITION 2.1.1. **[30]** $\mathcal{K}(\bar{j}, W)$ is the space of homotopies $f_t\colon \bar{j} \times W \to W$, $0 \le t \le 1$, such that

(a) $f_t|_{y \times W}\colon y \times W \to W$ is a homeomorphism of $y \times W$ onto its image, which is open, for each $y \in \bar{j}$ and for all $0 \le t \le 1$;

(b) $f_0|_{y \times W}\colon y \times W \to W$ maps $(y, \vec{x})$ to $\vec{x}$ for all $x \in W$;

(c) f_1 is an embedding of $\bar{j} \times W$ into W.

Let $W \subset Z$ be H invariant finite dimensional subspaces of $(V \oplus \mathbf{R})_H$. We define a map

$$\mathcal{K}(\bar{j}, W, Z)\colon \mathcal{K}(j, W) \to \mathcal{K}(j, Z) \tag{2.1.2}$$

by the composition

$$y \times Z = y \times (W \oplus (W^\perp \cap Z)) \overset{f_t \oplus \mathrm{id}}{\to} W \oplus (W^\perp \cap Z) = Z$$

for all $y \in Y$. This allows us to pass to the limit of the $\mathcal{K}(\bar{j}, W)$'s.

DEFINITION 2.1.3 [**30**]. $\mathcal{K}(\bar{j}) = \varinjlim_W \mathcal{K}(\bar{j}, W)$ where W ranges over all H invariant finite dimensional subspaces of $(V \oplus \mathbf{R})_H$.

As $\bar{j} \times W$ has the diagonal H action H acts on $\mathcal{K}(\bar{j}, W)$ and thus on $\mathcal{K}(\bar{j})$ via conjugation.

THEOREM 2.1.4 [**30**]. *$\mathcal{K}(\bar{j})$ is an E_∞ operad.*

See [**30**] for full details. We merely note that the E_∞ operad structure is given by letting Σ_j permute the f_t^i's and by the mapping $\gamma_{\mathcal{K}}\colon \mathcal{K}(j) \times \mathcal{K}(i_1) \times \cdots \times \mathcal{K}(i_j) \to \mathcal{K}(i_1 + \cdots + i_j)$ given by

$$\begin{aligned}\gamma_{\mathcal{K}}&\left(\left(f_t^1,\ldots,f_t^j\right);\left(g_t^{11},\ldots,g^{1i_1}\right),\ldots,\left(g_t^{j1},\ldots,g_t^{ji_j}\right)\right)\\ &= \left(\left(f_t^1\cdot g_t^{11}\right),\ldots,\left(f_t^1\cdot g_t^{1i_1}\right),\ldots,\left(f_t^j\cdot g_t^{j1}\right),\ldots,\left(f_t^{j}\cdot^{ji_j}\right)\right).\end{aligned}$$

The multiplicative operad we consider was first used by Boardman and Vogt [**4**], and May [**16**].

THEOREM 2.1.5 [**4**], [**16**]. $\mathcal{L}(j) = \mathrm{Iso}((V \oplus \mathbf{R})_H^j, (V \oplus \mathbf{R})_H)$, *the space of isometries of $(V \oplus \mathbf{R})_H^j$ into $(V \oplus \mathbf{R})_H$ is an E_∞ operad.*

Again note that H acts on $\mathcal{L}(j)$ via conjugation.

See [**4**] or [**16**] for full details. Again, we merely recall that Σ_j acts freely on $\mathcal{L}(j)$ by permuting the factors of $(V \oplus \mathbf{R})_H^j$, and the structure map $\gamma_{\mathcal{L}}$ that makes $\mathcal{L}(j)$ an E_∞ operad $\gamma_{\mathcal{L}}\colon \mathcal{L}(j) \times \mathcal{L}(i_1) \times \cdots \times \mathcal{L}(i_j) \to \mathcal{L}(i_1 + \cdots + i_j)$ is given by $\gamma_{\mathcal{L}}(f; g_1,\ldots,g_j) = f \circ (g_1 \times \cdots \times g_j)$.

Notice now that $\mathcal{L}$ acts on $\mathcal{K}$ as follows. We define a map

$$\lambda\colon \mathcal{L}(j) \times \mathcal{K}(i_1) \times \cdots \times \mathcal{K}(i_j) \to \mathcal{K}(i_1 \cdots i_j) \tag{2.1.6}$$

by

$$\begin{aligned}\lambda&\left(f;\left(\left(g_t^{11},\ldots,g_t^{1i_1}\right),\ldots,\left(g_t^{j1},\ldots,g_t^{ji_j}\right)\right)\right)\\ &= \left\{f\circ\left(g_t^{1\alpha_1}\times g_t^{2\alpha_2}\times\cdots\times g_t^{j\alpha_j}\right)\circ f^{-1}\right\}_{1\le\alpha_r\le i_r;1\le r\le j}\end{aligned}$$

where we have lexicographically ordered the g_t^{ab}'s. By $f \circ (g^{1\alpha_1} \times \cdots \times g^{J\alpha_J}) \circ f^{-1}$ we mean the composition of maps given by

$$
\begin{aligned}
(V \oplus \mathbf{R})_H &= f\big((V \oplus \mathbf{R})_H^j\big) \oplus \big[f\big((V \oplus \mathbf{R})_H^j\big)\big]^{\perp} \\
&\overset{f^{-1}\oplus \mathrm{id}}{\to} (V \oplus \mathbf{R})_H^j \oplus \big[f\big((V \oplus \mathbf{R})_H^j\big)\big]^{\perp} \\
&\overset{(g^{1,\alpha_1}\times \cdots \times g^{J,\alpha_J})\oplus \mathrm{id}}{\to} (V \oplus \mathbf{R})_H^j \oplus \big[f\big((V \oplus \mathbf{R}j)_H^j\big)\big]^{\perp} \\
&\overset{f\oplus \mathrm{id}}{\to} f\big((V \oplus \mathbf{R})_H^j\big) \oplus \big[f\big((V \oplus \mathbf{R})_H^j\big)\big]^{\perp} = (V \oplus \mathbf{R})_H.
\end{aligned}
$$

This map is well defined as f is an isometry and hence a linear isomorphism onto its image. The resulting map is thus a homeomorphism onto an open set in $(V \oplus \mathbf{R})_H$ for each $0 \le t \le 1$ and $\alpha_1, \ldots, \alpha_j$. Furthermore, it is the identity for $t = 0$ and all factors have disjoint images for $t = 1$. Hence we obtain an element in $\mathcal{K}(i_1 \cdots i_j)$.

A long but direct check of the conditions in **[16]** or **[17]** shows

THEOREM 2.1.7. *$(\mathcal{K}, \mathcal{L}, \lambda)$ is an E_∞ operad pair.*

Proof: **[30]**.

2.2. In this section we summarize results we require from **[9]** to obtain an E_∞ ring version of the Segal configuration map. Let K be the realization of the $\mathcal{K}$ operad acting on the trivial H space in the standard monad construction. That is,

$$K = \coprod_j \mathcal{K}(\bar{j})/\Sigma_j \tag{2.2.1}$$

where we note K has the obvious H action.

Let W be any H invariant finite dimensional subspace of $(V \oplus \mathbf{R})_H$ and consider $F(tW, tW)$, the space of pointed self maps of the one point compactification of W. Note that $F(tW, tW)$ has a natural H action induced by conjugation.

We now define the configuration map

$$\psi_W \colon \mathcal{K}(\bar{j}, W) \to F(tW, tW) \tag{2.2.2}$$

as follows: Given $f_t \in \mathcal{K}_H(\bar{j}, W)$

$$\psi_W(f_t)\left(\vec{x}\right) = \begin{cases} \infty & \text{if } \vec{x} = \infty, \\ \infty & \text{if } \vec{x} \notin f_1(\bar{j} \times W), \\ \left(\pi \cdot f_1^{-1}\right)\vec{x} & \text{if } \vec{x} \in f_1(\bar{j} \times W), \end{cases}$$

where $\pi: \bar{j} \times W \to W$ is the natural projection. A trivial check shows ψ_W is compatible with taking limits as W ranges over all finite dimensional H invariant subspaces of $(V \oplus \mathbf{R})_H$ (see Remark 2 following 1.1.2 and 2.1.2) and thus we obtain the configuration map first considered by Segal

$$\psi: K \to \varinjlim_W F(tW, tW) = F. \tag{2.2.3}$$

Next we have the following result, due to Hauschild, May, and Waner.

THEOREM 2.2.4 [**9**]. *ψ is an E_∞ ring map of E_∞ ring spaces where the E_∞ ring structure is given by the operad* $(\mathcal{K}, \mathcal{L})$.

Proof. [**9**].

We further specialize 2.2.4 by considering the H-fixed point sets of 2.2.3 to obtain the configuration map we are interested in

$$\phi = \psi^H: K^H \to F^H = F(H). \tag{2.2.5}$$

COROLLARY 2.2.6 [**9**]. *ϕ: $K^H \to F(H)$ is an E_∞ ring map of E_∞ ring structure is given by the operad* $(\mathcal{K}^H, \mathcal{L}^H)$.

Proof of 2.2.4. We describe how $\mathcal{K}$ and $\mathcal{L}$ act on both K and F. The rest of the proof is straightforward and left to the reader.

(a) The action $\mathcal{L}(k) \times K^k \to K$ is induced by the maps

$$\mathcal{L}(k) \times \mathcal{K}(j_1, W_1) \times \cdots \times \mathcal{K}(j_k, W_k) \to \mathcal{K}\left(\bigtimes_{i=1}^{k} j_1, f\left(\bigtimes_{i=1}^{k} W_i \right)\right)$$

where given $f \in \operatorname{Iso}((V \oplus \mathbf{R})_H^k, (V \oplus \mathbf{R}))_H$ and $g_t^i \in \mathcal{K}(j_i, W_i)$ we obtain a map

$$\bigtimes_{i=1}^{k} j_i \times f\left(\bigtimes_{i=1}^{k} W_i \right) \overset{\mathrm{id} \times f^{-1}}{\to} \bigtimes_{i=1}^{k} j_i \times \bigtimes_{i=1}^{k} W_i$$

$$\overset{\text{shuffle}}{\to} \bigtimes_{i=1}^{k} (j_i \times W_i) \overset{\bigtimes_{i=1}^{k} g_t^i}{\to} \bigtimes_{i=1}^{k} W_i \overset{f}{\to} f\left(\bigtimes_{i=1}^{k} W_i \right)$$

which is easily verified to be an element of $\mathcal{K}(\times_{i=1}^{k} j_i, f(\times_{i=1}^{k} W_i))$. This is compatible with the H action and with passage to the limit.

(b) The action $\mathcal{L}(k) \times F^k \to F$ is similarly induced as follows. Let $f \in \mathrm{Iso}((V \oplus \mathbf{R})_H, (V \oplus \mathbf{R})_H)$ and $g_i \in F(tW_i, tW_i)$ for $1 \leq i \leq k$. Then define the self map $tf(\times_{i=1}^{k} W_i)$ by the composition

$$tf\left(\bigtimes_{i=1}^{k} W_i\right) \xrightarrow{f^{-1}} t\left(\bigtimes_{i=1}^{k} W_i\right) = tW_1 \wedge tW_2 \wedge \cdots \wedge tW_k$$
$$\xrightarrow{\bigtimes_{i=1}^{k} g_i} tW_1 \wedge tW_2 \wedge \cdots \wedge tW_k = t\left(\bigtimes_{i=1}^{k} W_i\right) \xrightarrow{f} tf\left(\bigtimes_{i=1}^{k} W_i\right).$$

Here $\wedge$ is the smash product. This is compatible with the H action and with passage to the limit.

(c) The action $\mathcal{K}(k) \times K^k \to K$ is induced from the natural maps

$$\mathcal{K}(k, W) \times \mathcal{K}(j_1, W) \times \cdots \times \mathcal{K}(j_k, W) \to \mathcal{K}\left(\sum_{i=1}^{k} j_i, W\right)$$

given by taking $f_t \in \mathcal{K}(k, W)$, $g_t^i \in \mathcal{K}(j_i, W)$ and constructing a homotopy $\sum_{i=1}^{k} j_i \times W \to W$ defined on the subspace $j_i \times W$ by $j_i \times W \xrightarrow{g_t^i} W \xrightarrow{f_t|i \times W} W$. This is compatible with the H action and with passage to the limit.

(d) The $\mathcal{K}(k) \times F^k \to F$ action is induced by maps

$$\mathcal{K}(k, W) \times \overbrace{F(tW, tW) \times \cdots \times F(tW, tW)}^{k \text{ copies}} \to F(tW, tW)$$

given by taking $f_t \in \mathcal{K}(k, W)$ and $g_i \in F(tW, tW)$ and defining a self map of tW by sending

$$\vec{x} \to \begin{cases} \infty & \text{if } \vec{x} = \infty, \\ \infty & \text{if } \vec{x} \notin \bigcup_{i=1}^{k} ((f_1 | i \times W) \circ g_i)(W), \\ [(f_1 | i \times W) \circ g_i]^{-1}(\vec{x}) & \text{otherwise.} \end{cases}$$

This is well-defined as the $f_1 | i \times W$'s are homeomorphisms onto their images which are disjoint for different values of i. Thus the images of $(f_1 | i \times W) \circ g_i(W)$ are disjoint for different values of i. This construction is also clearly compatible with the H action and with passage to the limit.

Furthermore it is clear that all four actions defined above commute with the configuration map ψ.

We close this section by noting that the action defined in part (b) above is easily modified to show $\mathfrak{L}$ also acts naturally on the H space $G = \varinjlim_W G(SW, SW)$, the space of free self maps of the unit sphere of W where W runs over all finite dimensional subspace of V_H, and $\mathfrak{L}^H$ thus acts naturally on $G^H = G(H)$. This implies

THEOREM 2.2.7. *The radial extension map*

$$\rho\colon G(H) \to F(H)$$

is an $\mathfrak{L}^H$ map.

2.3. We now identify the $\mathfrak{L}^H$ action on $F(H)$ and $G(H)$ with the composition product. We say an $\mathfrak{L}^H$ space structure on a space X refines a given H-space structure μ on X if for each point $p \in \mathfrak{L}^H(k)$ the following diagram commutes up to homotopy:

$$\begin{array}{ccc} X^k & \xrightarrow{k \text{ fold product by } \mu} & X \\ \swarrow \text{id} & & \nearrow \gamma_{\mathfrak{L}^H} \\ p \times X^k & \to & \mathfrak{L}^H(k) \times X^k \end{array}$$

The spaces $SG(H)$ and $SF(H)$ of degree 1 maps are endowed under $\mathfrak{L}^H$ with an infinite loop space structure. Recall the standard smash product gives the composition product infinite loop space structure.

THEOREM 2.3.1. *The $\mathfrak{L}^H$ structure on $G(H)$ and $F(H)$ refines the composition product structure. Hence ρ: $SG(H) \to SF(H)$ is an infinite loop map with respect to the composition product.*

Proof. The first statement for $F(H)$ is immediate from the proof 2.2.4 part b. Minor modification gives the first statement for $G(H)$. The second statement follows immediately from the commutative diagram:

$$\begin{array}{ccc} \mathfrak{L}^H(k) \times G(H)^k & \to & G(H) \\ \downarrow \text{id} \times \rho^k & & \downarrow \rho \\ \mathfrak{L}^H(k) \times F(H)^K & \to & F(H) \end{array}$$

CHAPTER III

In this chapter we relate the configuration map $\phi\colon K^H \to F(H)$ to the radial extension map $\rho\colon G(H) \to F(H)$.

Using equivariant transversality Segal [27], and Hauschild [8], independently proved the following theorem.

THEOREM 3.1.1. *The configuration map* $\phi\colon K^H \to F(H)$ *is a group completion.*

COROLLARY 3.1.2. *The map* $\Phi_*\colon H_*(K^H) \to H_*(F(H))$ *becomes an isomorphism after inverting* $H_0(K^H)$ *and for field coefficients* Φ_* *is a Hopf algebra isomorphism under both the loop sum* ($\mathcal{K}^H$ *structure*) *and the composition product* ($\mathfrak{L}^H$ *structure*) *Pontrjagin product. Furthermore, the following diagrams commute* (*again with field coefficients*):

$$\begin{array}{ccc} H_*(K^H) & \xrightarrow{Q^i} & H_*(K^H) \\ \downarrow \Phi_* & & \downarrow \Phi_* \\ H_*(F(H)) & \xrightarrow{Q^i} & H_*(F(H)) \end{array}$$

and

$$\begin{array}{ccc} H_*(K^H) & \xrightarrow{\hat{Q}^i} & H_*(K^H) \\ \downarrow \Phi_* & & \downarrow \Phi_* \\ H_*(F(H)) & \xrightarrow{\hat{Q}^i} & H_*(F(H)) \end{array}$$

So far we have discussed mainly $F(H)$, the space of semi-free or based H-equivariant maps for the simple reason that it is, as we have seen, well approximated by K^H. However, for geometric reasons, we are really interested in $G(H)$, the space of *free* H-equivariant maps. In this section we explain how $G(H)$ sits in $F(H)$ under ρ.

THEOREM 3.1.3. *The Becker-Schultz and Segal homotopy equivalences of* 1.1.5 *and* 1.1.6 *are related by the commutative diagram*

$$
(3.1.4) \qquad
\begin{array}{ccccc}
G(H) & \overset{\gamma_{BS}}{\rightarrow} & Q(BH^+) & & \\
 & & & \searrow^{j} & \\
\downarrow \rho & & \downarrow i & & [1] \times Q(BH^+) \\
 & & & \nearrow_{k} & \\
F(H) & \overset{\gamma_S}{\rightarrow} & Q(S^0) \times Q(BH^+) & &
\end{array}
$$

where γ_{BS} *and* γ_S *are the Becker-Schultz and Segal equivalences*, j *the inclusion on the second factor*, k *the standard inclusion, and* $i = k \circ j$. *Recall* $[1] \in Q_1(S^0)$ *is the limit of* $t(\mathbf{R}^n) \overset{\text{id}}{\rightarrow} t(\mathbf{R}^n)$.

Although 3.1.3 describes ρ quite well, the theorem is not sufficient for our purposes. More precisely, although a group completion of K^H is well-known to be homotopy equivalent, as an infinite loop space under loop sum, to $Q(S^0) \times Q(BH^+)$ we must check that this equivalence is compatible with the Segal equivariant transversality map γ_S. The following diagram summarizes the situation.

$$
(3.1.5) \qquad
\begin{array}{ccccc}
G(H) & \overset{\rho}{\rightarrow} & F(H) & \overset{\phi}{\leftarrow} & K^H \\
\downarrow \gamma_{BS} & & \downarrow \gamma_S & & \downarrow \text{group complete} \\
Q(BH^+) & \overset{i}{\rightarrow} & Q(S^0) \times Q(BH^+) & \overset{\text{id}}{\rightarrow} & Q(S^0) \times Q(BH^+)
\end{array}
$$

THEOREM 3.1.6. *The diagram* 3.1.5 *commutes up to homotopy.*

In the appendix we prove 3.1.3 and 3.1.6.

CHAPTER IV

4.1. We begin this chapter by describing the action map on K^H and thus on a group completion $Q(S^0) \times Q(BH^+)$. Recall from **[12]**, there is an action map

$$(4.1.1) \qquad \bar{\lambda}\colon B(C) \times K^H \rightarrow K^H$$

induced by the homomorphism $\alpha\colon C \times H \rightarrow H$ defined on the group level by $\alpha(\theta, h) = \theta \cdot h$ (here C is center of H).

THEOREM 4.1.2. *$\bar{\lambda}$ is described by the composition*

$$BC \times Q(S^0) \times Q(BH^+) \xrightarrow{\text{shuffle}} Q(S^0) \times BC \times Q(BH^+)$$
$$\xrightarrow{\text{id} \times \wedge} Q(S^0) \times Q((BC \times BH)^+) \xrightarrow{\text{id} \times Q(B\alpha)} Q(S^0) \times Q(BH^+).$$

Proof. $\bar{\lambda}$ is a map of $\mathcal{K}^H$ spaces. Furthermore, $\bar{\lambda}$ is the infinite loop extension of the two maps

$$BC \times S^0 \times \text{pt} \xrightarrow{\text{pr}_2} S^0 \times \text{pt} \hookrightarrow Q(S^0) \times Q(BH^+)$$

and

$$BC \times \text{pt} \times BH \xrightarrow{\text{shuffle}} \text{pt} \times BC \times BH \xrightarrow{\text{id} \times B\alpha} \text{pt} \times BH \hookrightarrow Q(S^0) \times Q(BH^+).$$

The first composition reflects the fact that, by definition, the action of C on the trivial H-set $\{1\}$ is trivial, while the second composition reflects the fact that α gives the action of C on the free H-set $\{H\}$.

4.2. In this section we identify the maps $\Omega\psi_G$ and $\bar{\lambda}$. Recall $SG(V) = \varinjlim_n \text{Map}_H(S(nV), S(nV))_{\text{degree } 1}$, the degree $+ 1$ component of $G(V)$. We wish to describe the geometric twist map

(4.2.1) $$\psi_G\colon BC \times BSG(H) \to BSG(H)$$

which classifies $L \otimes_\tau \xi - L \otimes \dim \xi$. ψ_G loops down to

(4.2.2) $$\Omega\psi_G\colon BC \times SG(H) \to SG(H)$$

and $SG(H) \simeq Q_0(BH^+)$ [2].

THEOREM 4.2.3. *The following diagram commutes up to homotopy*:

$$\begin{array}{ccc} BC \times SG(H) & \xrightarrow{\Omega\psi_G} & SG(H) = Q_0(BH^+) \\ \downarrow = & & \nearrow Q(B\alpha) \\ BC \times Q_0(BH^+) & \xrightarrow{\wedge} & Q((BC \times BH)^+) \end{array}$$

THEOREM 4.2.4. *$\Omega\psi_G$ extends to $\Omega\psi_F$: $BC \times F(H) \to F(H)$ which commutes with the $\mathcal{K}^H$ loop space structure on $F(H)$. Furthermore, the following diagram commutes*:

$$(4.2.5)\qquad \begin{array}{ccc} BC \times SG(H) & \overset{\Omega\psi_G}{\to} & SG(H) \\ \downarrow \mathrm{id} \times \rho & & \downarrow \rho \\ BC \times F(H) & \overset{\Omega\psi_F}{\to} & F(H) \end{array}$$

Given the identifications of Chapter III, we may rewrite 4.2.5 as

$$(4.2.6)\qquad \begin{array}{ccc} BC \times ([1] \times Q_0(BH^+)) & \overset{\Omega\psi_G}{\to} & [1] \times Q_0(BH^+) \\ \downarrow & & \downarrow \\ BC \times Q(S^0) \times Q(BH^+) & \overset{\Omega\psi_F}{\to} & Q(S^0) \times Q(BH^+) \end{array}$$

Theorem 4.2.3 follows from 4.2.4 and the following lemma.

LEMMA 4.2.7. *The following diagram commutes up to homotopy*:

$$\begin{array}{ccc} BC \times Q(S^0) \times Q(BH^+) & \overset{\Omega\psi_F}{\to} & Q(S^0) \times Q(BH^+) \\ \downarrow \mathrm{shuffle} & & \uparrow \mathrm{id} \times Q(B\alpha) \\ Q(S^0) \times BC \times Q(BH^+) & \overset{\mathrm{id} \times \wedge}{\to} & Q(S^0) \times Q((BC \times BH)^+) \end{array}$$

In turn, to prove 4.2.7 we need only show

LEMMA 4.2.8.
(a) $\Omega\psi_F$ *restricted to* $BC \times S^0 \times \mathrm{pt}$ *is given by the composition*

$$BC \times S^0 \times \mathrm{pt} \overset{\mathrm{pr}_2}{\to} S^0 \times \mathrm{pt} \hookrightarrow Q(S^0) \times Q(BH^+).$$

(b) $\Omega\psi$ *restricted to* $BC \times \mathrm{pt} \times BH$ *is given by the composition*

$$BC \times BH \overset{\mathrm{shuffle}}{\to} \mathrm{pt} \times BC \times BH \overset{\mathrm{id} \times B\alpha}{\to} \mathrm{pt} \times BH \hookrightarrow Q(S^0) \times Q(BH^+).$$

We begin the proofs of 4.2.4 and 4.2.8 by giving an explicit model for $\Omega\Psi_G$. Let $(EC)_k = C * C * \cdots * C$ be the $k+1$ fold join of C and $(BC)_k = (EC)_k/C$. Since C is finite, the H bundle $\xi = V \times_Z (EC)_k$ over $(BC)_k$ has finite order, say M_k; that is, $M_k\xi \overset{\mathcal{K}}{\simeq} M_k(V \times (BC)_k)$ as a H bundle. This gives a C equivariant map ϕ: $(EC)_k \to \mathrm{Isom}_H(M_kV, M_kV)$

with $\mathcal{K}([x, h]) = (\phi(y)x, [y])$ for (x, y) in $V \times (EC)_k$. Here C acts by premultiplication on $\mathrm{Isom}_H(-, -)$.

Given N we get, by direct sum, a map $N\phi\colon (EC)_k \to \mathrm{Isom}_H(M_k(NV), M_k(NV))$ which induces a C-equivariant map

$$\widehat{N\phi}\colon (EC)_k \to \mathrm{Isom}_H\big(S(M_k(NV)), S(M_k(NV))\big).$$

Given $f \in \mathrm{Map}_H(S(NV), S(NV))$ and $\alpha \in (EC)_k$ we consider the composition

$$\begin{aligned} S(M_k(NV)) &\overset{\widehat{N\phi}(\alpha)}{\to} S(M_k(NV)) = S(NV) * S(M_k - 1)(NV) \\ &\overset{f * \mathrm{id}}{\to} S(NV) * S((M_k - 1)(NV)) = S(M_k(NV)) \\ &\overset{[\widehat{N\phi}(\alpha)]^{-1}}{\to} S(M_k(NV)) \end{aligned}$$

which is well-defined as $\widehat{N\phi}(\alpha)$ is an isometry. Furthermore, as C is the center of H, this composition is independent of the coset of $\alpha \in (EC)_k$ under the C action. Thus we have

DEFINITION 4.2.9. We define the map

$$\begin{aligned} \lambda^1_{kN}\colon (BC)_k &\times \mathrm{Map}_H(S(NV), S(NV)) \\ &\to \mathrm{Map}_H\big(S(M_k(NV)), S(M_k(NV))\big) \end{aligned}$$

by the formula

$$\lambda^1_{kN}([\alpha], f) = [\widehat{N\phi}(\alpha)]^{-1} \circ (f * \mathrm{id}) \circ [\widehat{N\phi}(\alpha)].$$

Passing to the limit over N gives

DEFINITION 4.2.10.

$$\lambda^1_k = \varinjlim_N \lambda^1_{kN}\colon (BC)_k \times G(H) \to G(H).$$

Thus by making choices we may pass to the limit over k to obtain

DEFINITION 4.2.11.

$$\lambda^1 = \varinjlim_k \lambda^1_k\colon BC \times G(H) \to G(H)$$

which induces $S\lambda^1\colon BC \times SG(H) \to SG(H)$.

In §4.3 we prove

THEOREM 4.2.12. $S\lambda^1 = \Omega\psi_G$.

Assuming 4.2.12 we procede with the proofs of 4.2.4 and 4.2.8. We define $N\phi^1\colon (EC)_k \to \mathrm{Isom}_H(M_k(N(\mathbf{R} \oplus V)), M_k(N(\mathbf{R} \oplus V)))$ by the assignment $\alpha \to (N\phi)(\alpha)$ on $M_k(NV)$ and $\alpha \to \mathrm{id}$ on $M_k(N\mathbf{R})$. Note that $N\phi^1$ is C equivariant. Thus in a manner similar to 4.2.9 and 4.2.10, we obtain

DEFINITION 4.2.13. The map

$$\lambda_k^{11}\colon (BC)_k \times F(H) \to F(H)$$

is induced by the maps

$$\lambda_{k,N}^{11}([\alpha], f) = [t(N\phi^1(\alpha))]^{-1} \circ (f \wedge \mathrm{id}) \circ [t(N\phi^1(\alpha))]$$

on the finite levels.

Notice, by 4.2.11, 4.2.12 and 4.2.13, that 4.2.5 commutes. Also note that for $g \in \mathrm{Map}_H(t(N\mathbf{R}), t(N\mathbf{R})) = \mathrm{Map}(t(N\mathbf{R}), t(N\mathbf{R}))$ that the suspension g,

$$g \wedge \mathrm{id} \in \mathrm{Map}_H(t(N(\mathbf{R} \oplus V)), t(N(\mathbf{R} \oplus V))),$$

has the property $\lambda_{kN}^{11}([\alpha], g \wedge \mathrm{id}) = (g \wedge \mathrm{id}) \wedge \mathrm{id}$ since "C does not twist $g \wedge \mathrm{id}$ at all." This uses the fact that the H action on $N\mathbf{R}$ is trivial. Thus, under the identification $F(H) \simeq Q(S^0) \times Q(BH^+)$ the following diagram commutes:

$$\begin{array}{llcl} & (BC)_k \times Q(S^0) \times \mathrm{pt} & \hookrightarrow & (BC)_k \times Q(S^0) \times Q(BH^+) \\ (4.2.14) & \quad\downarrow \mathrm{pr}_2 & & \quad\downarrow \lambda_k^1 \\ & Q(S^0) \times \mathrm{pt} & \hookrightarrow & Q(S^0) \times Q(BH^+) \end{array}$$

Along with 4.2.12 this proves 4.2.8a.

PROPOSITION 4.2.15. *The following diagram commutes up to homotopy*

$$\begin{array}{ccc} (BC)_k \times \mathcal{K}^H(j) \times F(H)^{(j)} & \xrightarrow{\mathrm{id} \times \gamma_{\mathcal{K}^H}} & (BC)_k \times F(H) \\ \downarrow (\mathrm{shuffle}) \circ (\Delta^j \times \mathrm{id} \times \mathrm{id}) & & \\ \mathcal{K}^H(j) \times [(BC)_k \times F(H)]^{(j)} & & \downarrow \lambda_k^{11} \\ \downarrow \mathrm{id} \times (\lambda_k^{11})^{(j)} & & \\ & & F(H) \\ \mathcal{K}^H(j) \times F(H)^{(j)} & \xrightarrow[\gamma_{\mathcal{K}^H}]{} & \end{array}$$

Proof. [12.2.6].

Of course, we may stabilize 4.2.15 over k and obtain a similar commutative diagram for λ^{11}. This finishes the proof of 4.2.4. We conclude this section by proving 4.2.8b. We must check the commutivity of the diagram up to homotopy:

$$\begin{array}{ccc}
BC \times \mathrm{pt} \times BH & \overset{\mathrm{id}\times i}{\rightarrow} & BC \times F(H) \\
\downarrow \text{shuffle} & & \lambda^{11} \\
\mathrm{pt} \times BC \times BH & & \\
\downarrow \mathrm{id} \times B\alpha & & \downarrow \\
\mathrm{pt} \times BH & \overset{i}{\rightarrow} & F(H)
\end{array}$$

The map

$$i \colon BH = \varinjlim_{N} \mathrm{Map}_H\big(\{H\}, N(\mathbf{R} \oplus V) - N(\mathbf{R} \oplus V)^H\big)/H \rightarrow F(H)$$

is given by the configuration map which sends

$$f \in \mathrm{Map}_H\big(\{H\}, N(\mathbf{R} \oplus V) - N(\mathbf{R} \oplus V)^H\big)$$

to the $\frac{1}{2}\varepsilon_f$-ball map associated to $f(H) \subset N(\mathbf{R} \oplus V)$ where ε_f is the minimum distance between elements of $f(H)$. It is easy to check, as λ^{11}_{kN} maps the image of i_N to the image of $i_{M_k N}$, that we have a commutative diagram

(4.2.16)

$$\begin{array}{ccc}
(BC)_k \times \mathrm{Map}_H\big(\{H\}, N(\mathbf{R} \oplus V) - N(\mathbf{R} \oplus V)^H\big) & \overset{\lambda^{11}_{kN}}{\rightarrow} & \mathrm{Map}_H\big(\{H\}, M_k N(\mathbf{R} \oplus V) - M_k N(\mathbf{R} \oplus V)^H\big) \\
\downarrow \mathrm{id} \times \varepsilon \text{ ball map} & & \downarrow \varepsilon \text{ ball map} \\
(BC)_k \times \mathrm{Map}_H(t(N(\mathbf{R} \oplus V)), t(N(\mathbf{R} \oplus V))) & \overset{\lambda^{11}_{kN}}{\rightarrow} & \mathrm{Map}_H(t(M_k N(\mathbf{R} \oplus V)), t(M_k N(\mathbf{R} \oplus V)))
\end{array}$$

which is compatible with limits. This implies

(4.2.17)

$$\begin{array}{ccc}
(BC)_k \times BH & \overset{\lambda^{11}_k}{\rightarrow} & BH \\
\downarrow \mathrm{id} \times i & & \downarrow i \\
(BC)_k \times F(H) & \overset{\lambda^{11}_k}{\rightarrow} & F(H)
\end{array}$$

also commutes. Thus it suffices to note:

PROPOSITION 4.2.18. *On the level of fundamental groups* $\hat{\lambda}_k^{11}$ *is precisely the map* $\alpha: C \times H \to H$ *given by* $\alpha(\theta, h) = \theta \cdot h$.

Proof. This is a simple consequence of lifting the map λ_k^{11} to the universal cover $(EC)_k \times EH \to EH$ and looking over a point to see the action $C \times H \to H$.

This completes the proof of 4.2.8b and therefore 4.2.3.

4.3. We conclude this chapter by proving 4.2.12. That is, we must deloop ψ_G: $BC \times BSG(H) \to BSG(H)$. We first require a model for ψ_G itself. Let $\eta_N \to BSG(NV)$ be the universal $SG(NV)$ bundle. We may choose an open cover $\{U_\alpha\}$ trivializing η_N and thus obtain cocycles $g_{\alpha,\beta}$: $\{U_\alpha \cap U_\beta \to SG(NV)\}$ for η_N. Now form the bundles ξ_α over $BC \times U_\alpha$ where $\xi_\alpha = SG(NU) \times_C EC \times U_\alpha$. As C is the center of H we may glue $\xi_\alpha | U_\alpha \cap U_\beta$ to $\xi_\beta | U_\alpha \cap U_\beta$ by the self-map $[x, y, z] \to [g_{\alpha\beta}(z), x, y, z]$ on $SG(NV) \times_C (EC)_k \times (U_\alpha \cap U_\beta)$. This defines a twisted bundle $\tilde{\eta}_N \to BC \times BSG(NV)$ and so defines a map $\bar{\lambda}_N$: $BC \times BSG(NV) \to BSG(NV)$.

In order to stabilize we take the trivial $SG(NV)$ bundle $[NV] = SG(NV) \times BSG(NV)$ and twist it to give $\widetilde{[NV]}_N$ over $BC \times BSG(NV)$. Now if we classify the difference $\tilde{\eta}_N - \widetilde{[NV]}_N$, we may stabilize to yield the geometric twist map ψ_G: $BC \times BSG(H) \to BSG(H)$. Now as ψ_G sends $BC \times$ pt to pt we may deloop ψ_G. Consider the composition

$$BC \times (S^1 \wedge SG(NV)^+) \hookrightarrow BC \times BSG(NV) \xrightarrow{\bar{\lambda}_N} BSG(NV).$$

$S^1 \wedge SG(NV)^+$ is the union of an upper cone $C_+ SG(NV)$ and a lower cone $C_- SG(NV)$ and $\tilde{\eta}|_{S^1 \wedge SG(NV)^+}$ may be described by the clutching function $C_+ SG(NV) \cap C_- SG(NV) = SG(NV) \xrightarrow{\text{id}} SG(NV)$. Hence we may describe $\tilde{\eta}_N | BC \times S^1 \times SG(NV)$ by clutching $\xi_+ = S(NV) \times_C EC \times C_+ SG(NV)$ over $BC \times C_+ SG(NV)$ to $\xi_- = S(NV) \times_C EC \times C_-(NV)$ over $BC \times C_- SG(NV)$ via $[x, y, z] \to [z \circ x, y, z]$ over $BC \times SG(NV)$.

But over $(BC)_k$ we have a trivialization

$$\{NV \oplus (M_k - 1)NV \times_C (EC)_k\} \cong \{NV \oplus (M_k - 1)NV \times (BC)_k\}$$

of H bundles where M_k is sufficiently large. Hence over $(BC)_k \times BSG(NV)$ we have

$$(4.3.1)\quad [\widetilde{NV}]_N \otimes [(M_k - 1)NV]_n \sim SG(NV \oplus (M_k - 1)NV) \times BC_k \times BSG(NV)$$

where $\otimes$ means fibre-wise joint and $\sim$ means homeomorphism. Thus a model for

$$\tilde{\eta}_N - [\widetilde{NV}]_N | (BC)_k \times BSG(nV)$$

is

$$\tilde{\eta}_N | (BC)_k \otimes [\widetilde{(M_k - 1)NV}]_N.$$

Thus we have proved

THEOREM 4.3.2. *$\tilde{\eta}_N - [\widetilde{NV}]_N$ restricted to $(BC)_k \times [S^1 \wedge SG(NV)]$ may be obtained by identifying*

$$\left\{\begin{matrix} SG(M_kNV) \times (EC)_k \times C_+ SG(NV) \\ \downarrow \\ (EC)_k \times C_+ SG(NV) \end{matrix}\right\}$$

and

$$\left\{\begin{matrix} SG(M_kNV) \times (EC)_k \times C_- SG(NV) \\ \downarrow \\ (EC)_k \times C_- SG(NV) \end{matrix}\right\}$$

along $(EC)_k \times SG(NV)$ via the identification

$$(x, y, w) \sim \left(\left(w * \mathrm{id}|_{S(M_k - 1)NV}\right) \circ x, y, w\right)$$

and then dividing out by the free action of C given by $\sigma \cdot (x, y, w) = (\sigma \cdot x, \sigma^{-1} y, w)$ on each piece.

Here the action of C on $SG(M_kNV)$ is given by precomposition. To prove 4.2.12 we merely reinterpret 4.3.2.

THEOREM 4.3.3. *$\tilde{\eta}_N - [\widetilde{NV}]_N$ restricted to $(BC)_k \times [S^1 \wedge SG(NV)]$ may be obtained by identifying*

$$\left\{\begin{matrix} SG(M_kNV) \times (BC)_k \times C_+ SG(NV) \\ \downarrow \\ (BC)_k \times C_+ SG(NV) \end{matrix}\right\}$$

and

$$\left\{\begin{array}{c} SG(M_kNV) \times (BC)_k \times C_- SG(NV) \\ \downarrow \\ (BC)_k \times C_- SG(NV) \end{array}\right\}$$

along $(BC)_k \times SG(NV)$ *via the identification*

$$(x, [y], w) \sim \Big([N\phi(y)]^{-1} \circ \big(w \times \mathrm{id}|_{S(M_k-1)NV}\big) \circ x \circ [\widehat{N\phi}(y)], [y], w\Big).$$

Proof. $M_k(NV) \times_C (EC)_k$ is a trivial H bundle.

As a self-map on $SG(M_kNV) \times (BC)_k \times C_+ SG(NV)$ given by

$$(x, [y], w) \to \big([\widehat{N\phi}(y)] \circ x \circ [\widehat{N\phi}(y)]^{-1}, [y], w\big)$$

is an identification over $(BC)_k \times C_+ SG(NV)$, we may modify 4.3.3 by replacing the last identification by

$$(x, [y], w) \sim \Big([\widehat{N\phi}(y)]^{-1} \circ \big(w \times \mathrm{id}|_{S(M_k-1)NV}\big) \circ [\widehat{N\phi}(y)] \circ x, [y], w\Big).$$

That is, $\tilde{\eta}_N - [\widetilde{NV}]_N$ restricted to $(BC)_k \times (S^1 \wedge SG(NV)^+)$ pulls back from $S^1 \wedge [(BC)_k \times SG(NV)]^+$ and is obtained from the clutching map λ^1_{kN}: $(BC)_k \times SG(NV) \to SG(M_kNV)$ of 4.2.9. Thus λ^1_{kN} is the desired adjoint of

$$(BC)_k^+ \wedge S^1 \wedge SG(NV)^+ \to (BC)_k^+ \wedge BSG(H) \xrightarrow{\psi_G} BSG(H).$$

This establishes 4.2.12.

CHAPTER V

5.1. Let $H = \mathbf{Z}/p^n$ for some prime p and integer n. It remains to prove the results of Chapter I. $SG(H)$ and $SF(H)$, the spaces of degree 1 maps in $G(H)$ and $F(H)$ respectively, are infinite loop spaces with respect to the composition product. To compute

$$H_*(SG(H), \mathbf{Z}/p),\ H_*(BSG(H), \mathbf{Z}/p)$$

and all related transfer maps it suffices, by the results of Chapter III to compute

$$H_*(S\overline{G}(H), \mathbf{Z}/p),\ H_*(BS\overline{G}(H), \mathbf{Z}/p),$$

and all the related transfer maps where $S\overline{G}(H)$ is the factor identified with $[\mathbf{1}] \times Q_0(BH^+) \subset K^H$. The specific calculations on $[\mathbf{1}] \times Q_0(BH^+)$ we require were carried out in [**12**]. Summarizing the facts we need from [**6**], [**22**], and [**12**] to compute we recall:

1. The composition product and composition Dyer-Lashof operations, $\circ$ and $\tilde{Q}^i$, as well as the loop sum and loop Dyer-Lashof operations, $*$ and Q^i, are related by the usual Cartan and Adem formulae.

2. The loop and composition products are related by the mixed Cartan formula and the distributivity formula $(x * y) \circ z = \Sigma \pm (x \circ z') * (y \circ z'')$ where $\Delta z = \Sigma z' \otimes z''$.

3. We have the following May formula, which we state, for convenience, when $p = 2$:

$$Q^s(y) \circ z = \sum Q^{s+1}\left(y \circ Sq_*^i z\right).$$

See [**6**] for the odd prime analog.

4. The composition product $\circ$ on $K^H \simeq Q(S^0) \times Q(BH^+)$ extends the diagonal transfer map $\#\colon Q(B(H \times H)^+) \to Q(BH^+)$ induced by the diagonal inclusion $\Delta\colon H \to H \times H$. A result of Schultz, [**22**] (see also [**13**]), allows the computation of $\#$ on the cells of BH^+ as follows:

THEOREM 5.1.1 [**22**].

$$\#(e_r, e_s) = \sum_{i=0}^{r} e_{r-i} \wedge t\left(\mu_*(\chi e_i \otimes e_s)\right),$$

where χ is the canonical antiautomorphism, μ is the group multiplication map $M\colon H \times H \to H$, and t is the transfer induced by the inclusion $1 \hookrightarrow H$.

We remark that 5.1.1 is true for all abelian compact Lie groups ([**13**]).

5. The loop sum, loop Dyer-Lashof, and transfer calculations of [**12**] apply.

We note that the results of §1.3 follow immediately from Chapter III and [**12**]. Now the results of [**12**] on the transfers, 5.1.1, and the remarks 1 through 5 above make the remaining computations in this section purely formal. To make these calculations we require a weight filtration a la May [**6**], and Tsuchiya [**33**]. Let $n \geq 1$.

DEFINITION 5.1.2. [**13**]. The weight valuation

$$w\colon H_*\left(Q\left(B\mathbf{Z}/p^{n^+}\right), \mathbf{Z}/p\right) \to N$$

where N is the set of nonnegative integers is the smallest function with the stated domain and range satisfying

$$w(x * y) \geq w(x) + w(y),$$
$$w(x + y) \geq \min\{w(x), w(y)\},$$
$$w(Q^I(e_r)) \geq p^{l(I)+1} \quad \text{for } e(I) > r \geq 0,\ l(I) > 0,$$
$$w(e_r) = p \text{ for } r > 0,$$
$$w(e_0) = 0$$

where I is an admissible sequence.

Let $\overline{X}_{I,\alpha} = Q^I(e_\alpha)$ where we allow $l(I) = 0$ in which case $\overline{X}_{I,\alpha} = e_\alpha$. We now state our main technical result (compare with [**18**], [**6**], and [**33**]).

LEMMA 5.1.3. *In any* $(H_*(K^{\mathbf{Z}/p^n}), \mathbf{Z}/p)$, $n \geq 1$,

$$\overline{X}_{I,\alpha} \circ \overline{X}_{J,\beta} = \overline{X}_{I,\alpha} * \overline{X}_{J,\beta}$$

modulo elements of higher weight where:

(a) *Either* $l(I)$ *or* $l(J)$ *is greater than zero if* $p = 2$.

(b) *If* $p > 2$ *there is no restriction on* $l(I)$ *or* $l(J)$.

We require the following two computational lemmas. We may simultaneously consider $\mathbf{Z}/p^n$ for all p and $n \geq 1$.

LEMMA 5.1.4. $w(Q^I(e_\alpha) \circ Q^J(e_\beta)) \geq p^{l(I)+l(J)+2}$.

Proof. First assume α, $\beta > 0$. By May's formula

$$Q^I(e_\alpha) \circ Q^J(e_\beta) = \sum_{I',J',\alpha',\beta'} Q^{I'}Q^{J'}(e_{\alpha'} \circ e_{\beta'})$$

where $l(I) = l(I')$, $l(J) = l(J')$ and α', $\beta' > 0$. But 5.1.1, 1.3.3 and 1.3.5 clearly show $w(e_{\alpha'} \circ e_{\beta'}) \geq p^2$. Thus the lemma follows for $\alpha, \beta > 0$. The other cases are similar and left to the reader.

LEMMA 5.1.5. $w(Q^I(e_\alpha) \circ \chi(e_0^{p^m})) \geq p^{l(I)+1+m+1}$.

Proof. $Q^I(e_\alpha) \circ \chi(e_0^{p^m}) = Q^I(e_\alpha \circ \chi(e_0^{p^m})) = \chi Q^I(e_\alpha \circ e_0^{p^m}) = \chi Q^I(e_\alpha \circ (p^m e_0)) = \chi(Q^{I'}(e_\alpha \circ e_0))^{*p^m}$ by the Cartan formula and distributivity. As χ preserves weight the lemma follows.

Proof of 5.1.3. Expanding

$$\left(Q^I(e_\alpha) * \left([1] * \chi\left(e_0^{p^{l(I)}}\right)\right)\right) \circ \left(Q^J(e_\beta) * \left([1] * \chi\left(e_0^{p^{l(J)}}\right)\right)\right)$$

we obtain

$$\sum_{I',J',\alpha',\beta'} \left(Q^{I'}(e_{\alpha'}) \circ Q^{J'}(e_{\beta'})\right) * \left(Q^{I''}(e_{\alpha''}) \circ \left([1] * \chi\left(e_0^{p^{l(J)}}\right)\right)\right)$$

$$* \left(Q^{J''}(e_{\beta''}) \circ \left([1] * \chi\left(e_0^{p^{l(I)}}\right)\right) * \left([1] * \chi\left(e^{p^{l(I)+l(J)}-p^{l(I)}-p^{l(I)}}\right)\right)\right)$$

where $l(I') = l(I'') = l(I)$, $l(J) = l(J'') = l(J)$ and $\alpha' > 0$ if $\alpha > 0$, $\beta' > 0$ if $\beta > 0$. But the two middle terms may be rewritten as

$$Q^{I'''}(e_{\alpha'''}) * \left(Q^{I''''}(e_{\alpha''''}) \circ \chi\left(e_0^{p^{l(J)}}\right)\right)$$

and

$$Q^{J'''}(e_{\beta'''}) * \left(Q^{J''''}(e_{\beta''''}) \circ \chi\left(e_0^{p^{l(I)}}\right)\right)$$

where 5.1.4 and 5.1.5 imply each nontrivial $\circ$ product term has weight $\geq p^{l(I)+l(J)+2}$. The lemma follows immediately from the observation that $p^{l_1+l_2+2} \geq p^{l_1+1} + p^{l_2+1}$ and equality holds *only* for $p = 2$, $l_1 = l_2 = 0$.

5.1.3 immediately implies 1.2.10. Now consider $p = 2$. First let $n = 1$. 1.2.5 is precisely Corollary 5.3.a of [**13**]. We note, in passing, that it is easy to check that $J_{\mathbf{Z}/2}(e_r)^{\circ 2} = 0$ by direct computation:

$$\begin{aligned} J_{\mathbf{Z}/2}(e_r)^{\circ 2} = \sum &\left(e_{r-i} \circ (e_0 * [-1]) \circ e_{r-j} \circ (e_0 * [-1])\right) \\ &* \left((e_i \circ (e_0 * [-1])) \circ ([1] * \chi(e_0))\right) \\ &* \left(([1] * \chi(e_0)) \circ (e_j \circ (e_0 * [-1]))\right) * [1] \end{aligned}$$

by repeated distributivity. As in [**18**] we note that $i = j$ in the above sum by symmetry as we are working mod 2. But 5.1.3 and 1.3.1.a imply the sum reduces to $\Sigma Q^{r-i}(e_{r-i}) * \chi e_i * \chi e_i * [1] = \Sigma(\chi(e_{r-i}) * e_i)^{*2} * [1] = 0$.

Proof of 1.2.4 *and* 1.2.6. Part (a) of 1.2.4 is trivial. We note in passing that the computation above showing $J_{\mathbf{Z}/2}(e_r)^{\circ 2} = 0$ and the fact that the algebra map commutes with both products would be sufficient, even without 5.3a of [**13**], to formally show $E[J_{\mathbf{Z}/2}(e_r) \mid r > 0]$ is a sub-Hopf algebra of $H_*(SG(\mathbf{Z}/2), \mathbf{Z}/2)$. Now consider the exact sequence

$$0 \to H_*(SO, \mathbf{Z}/2) \xrightarrow{J_{\mathbf{Z}/2*}} H_*(SG(\mathbf{Z}/2), \mathbf{Z}/2)$$

$$\to H_*(SG(\mathbf{Z}/2), \mathbf{Z}/2)//H_*(SO, \mathbf{Z}/2) \to 0.$$

As we have identified the image of $J_{\mathbf{Z}/2}(e_r)$ with $e_r * [-1]$ modulo higher weight (5.1.3), it follows that $H_*(SG(\mathbf{Z}/2), \mathbf{Z}/2)//H_*(SO, \mathbf{Z}/2)$ is a polynomial subalgebra on the stated generators of 1.2.4.b. The weight filtration, 5.1.3, and 1.2.4.b then imply 1.2.6.

Now let $n > 1$. While it is not true that $\bar{e}_{2r} \circ \bar{e}_{2r}$ is equal to $\bar{e}_{2r} * \bar{e}_{2r}$ modulo higher weight, we do have the following:

LEMMA 5.1.6. *$e_{2r} \circ e_{2r} = e_{2r} * e_{2r} + \Sigma_{i<2r} Q^{4r-i}(e_i) * ([1] \circ \chi(e_0)^4)$ in $H_*(SG(\mathbf{Z}/2^n), \mathbf{Z}/2)$.*

Proof.

$$(e_{2r} * [1] * \chi(e_0))^{\circ 2} = \sum_i (e_{2r-i} \circ e_{2r-i}) * (e_i \circ ([1] * \chi(e_0)))^{*2}$$
$$* ([1] * \chi e_0 * \chi e_0 * e_0^2)$$

by mod 2 symmetry. This simplifies to

$$\sum (e_{2r-i} \circ e_{2r-i}) * (e_{i-j})^{*2} * (\chi(e_i \circ e_0))^{*2} * ([1] * \chi e_0 * \chi e_0 * e_0^2).$$

Thus there are three terms of possible lowest weight, namely

$$(e_{2r} \circ e_{2r}) * ([1] * \chi(e_0^2)), \quad e_{2r} * e_{2r} * ([1] * \chi(e_0^2)) \quad \text{and}$$
$$(\chi(e_{2r} \circ e_0))^{*2} * (e_0^2 * [-1]).$$

5.1.1 and 1.3.3 clearly show, however, that the only terms of weight $< 2^3 = 8$ in $e_{2r} \circ e_{2r}$ and $e_{2r} \circ e_0$ are of the form $Q^{4r-i}(e_i)$ for $i < 2r$.

5.1.6 and a slight generalization of 5.1.3 yield

COROLLARY 5.1.7. *$\bar{e}_{2r}$ is a polynomial generator in $H_*(SG(\mathbf{Z}/2^n), \mathbf{Z}/2)$ for $n > 1$.*

The proof of 1.2.9 now follows from 4.4.c and 5.1.c of **[13]**, and from 5.1.3 and 5.1.7.

We now turn to the Hopf algebra structure of the classifying spaces $BSG(\mathbf{Z}/p^n)$ by studying the Eilenberg-Moore classifying space spectral sequence as in **[6]**, **[5]** and **[13]**.

$$(5.1.8) \quad \mathrm{Tor}_{s,t}^{H_*(SG(\mathbf{Z}/p^n), \mathbf{Z}/p)}(\mathbf{Z}/p, \mathbf{Z}/p) \Rightarrow H_*(BSG(\mathbf{Z}/p^n), \mathbf{Z}/p).$$

This is a first quadrant homology spectral sequence of Hopf algebras. Consider the following cases:

Case 1. $p = 2$, $n = 1$. Then $H_*(SO, \mathbf{Z}/2)$ in $H_*(SG(\mathbf{Z}/2), \mathbf{Z}/2)$ yields generators in E^2, namely the divided powers of the suspension $\sigma(J_{\mathbf{Z}/2}(e_r))$ of $J_{\mathbf{Z}/2}(e_r)$. As $J_{\mathbf{Z}/2}$ maps SO onto these elements and the spectral sequence

$$\mathrm{Tor}_{s,t}^{H_*(SO,\mathbf{Z}/2)}(\mathbf{Z}/2, \mathbf{Z}/2) \Rightarrow H_*(BSO, \mathbf{Z}/2)$$

collapses at E^2, we conclude that these generators are permanent cycles. As there are no further generators with $s > 1$, 5.1.8 also collapses at E^2.

Case 2. $p = 2$ and $n \geq 2$ or p odd. Then t_n: J_{S^1}: $U \to SG(S^1) \to SG(\mathbf{Z}/p^n)$ includes $H_*(U, \mathbf{Z}/p)$ in $H_*(SG(\mathbf{Z}/p^n), \mathbf{Z}/p)$ and yields generators in E^2 of 5.1.8, namely the divided powers of the suspensions $\sigma((t_n \circ J_{S^1})_* \bar{a}_r)$ of $(t_n \circ J_{S^1})_* \bar{a}_r$ (recall $(t_n \circ J_{S^1})_* \bar{a}_r = J_n(e_{2r+1})$ when $p = 2$). As $t_n \circ J_{S^1}$ maps U onto these elements and the spectral sequence

$$\mathrm{Tor}_{s,t}^{H_*(U,\mathbf{Z}/p)}(\mathbf{Z}/p, \mathbf{Z}/p) \Rightarrow H_*(BU, \mathbf{Z}/p)$$

collapses at E^2 we again conclude that these generators are permanent cycles. Again, when $p = 2$, there are no further generators with $s > 1$, so the spectral sequence 5.1.8 collapses at E^2. For p odd, each odd generator in $H_*(SG(\mathbf{Z}/p^n), \mathbf{Z}/p)$ yields a divided power algebra in E^2. Just as in the non-equivariant case, **[6]**, they are connected by a universal differential, **[6]**,

$$d^{p-1}\gamma_{p+j}(\sigma x) = -\left(\sigma\beta\tilde{Q}_1 x\right)\gamma_j(\sigma x).$$

(Recall that if $2s = |x| + 1$, then $\tilde{Q}_1 x = \omega \tilde{Q}^s x$ for some unit $\omega \in \mathbf{Z}/p$.) To compute $\tilde{Q}_1$ we have

LEMMA 5.1.9. *For p odd*

$$\tilde{Q}_1\left(Q^I(e_r) * [1] * (\chi e_0)^{p^{l(I)}}\right) \equiv Q_1 Q^I(e_r) * [1] * (\chi e_0)^{p^{l(I)+1}}$$

modulo higher weight in $H_*(SG(\mathbf{Z}/p^n), \mathbf{Z}/p)$ *provided that* I *is admissible with* $e(I) > r$ *and* $l(I) > 0$.

Assuming the lemma, which we prove later, the E^p term has the form

$$E^p \cong \Gamma\left[\sigma\left((t_n \circ J_{S^1})_* \bar{a}_r\right) : r \geq 0\right]$$
$$\otimes E\left[\sigma\left((t_n \circ J_{S^1})_* \beta Q^s \bar{a}_r\right) : s > r > 0\right] \otimes D$$

where D is a free commutative algebra truncated at height p. As no further differentials are possible, $E^p = E^\infty$.

5.1.9 and the equivariant J homomorphism solve the extension problems when p is odd and thus yields 1.2.14.

For $p = 2$ we require several lemmas to take the place of 5.1.9. However, 5.1.10 through 5.1.15 do, along with the equivariant J homomorphism, solve the extension problems at 2 and thus yield 1.2.12 and 1.2.13.

LEMMA 5.1.10. *In* $H_*(SG(\mathbf{Z}/2), \mathbf{Z}/2)$

$$\tilde{Q}^{r+1}(Q^r(e_0) * [1] * \chi e_0^2) \equiv Q^{r+1}Q^r(e_0) * [1] * \chi(e_0^3)$$

modulo higher weight.

Proof. By May's formula, $Q^r(e_0) = Q^r[1] \circ e_0$. Thus

$$\tilde{Q}^{r+1}(Q^r e_0) = \tilde{Q}^{r+1}(Q^r(1) \circ e_0) = \sum_{s=1}^{r+1} \tilde{Q}^r(1) \circ Q^s(e_0)$$

but evidently the only non-zero terms in the sum are $\tilde{Q}^{r+1}Q^r(1) \circ Q^0 e_0$ and $\tilde{Q}^r Q^r(1) \circ Q^1 e_0$. As $Q^0 e_0 = e_0 * e_0$ and $\tilde{Q}^{r+1}Q^r(1)$ is odd dimensional, the first term is zero by distributivity. As $\tilde{Q}^r Q^r(1) \circ Q^1(e_0) = Q^r(1) \circ Q^r(1) \circ Q^1(e_0)$ and as $Q^r(1) \circ Q^r(1) = Q^r(1) * Q^r(1)$, [**18**], distributivity again implies the second is zero. Now apply the mixed Cartan formula to $\tilde{Q}^{r+1}(Q^r(e_0) * [1] * (\chi e_0^2))$.

LEMMA 5.1.11. *In* $H_*(SG(\mathbf{Z}/2), \mathbf{Z}/2)$ *with* $r > \alpha$

$$\tilde{Q}^{r+\alpha+1}(Q^r(e_\alpha) * [1] * \chi e_0^2) = Q^{r+\alpha+1}Q^r(e_\alpha) * [1] * \chi e_0^3$$

modulo higher weight.

Proof. Using May's formula, [**6**], $\tilde{Q}^r(e_\alpha)$ may be written as a sum $\Sigma\, c_s(Q^{r+s}(1) \circ e_{\alpha-s})$ where c_s is some binomial coefficient. But

$$\tilde{Q}^{r+\alpha+1}(Q^{r+s}(1) \circ e_{\alpha-s}) = \tilde{Q}^{r+s+1}Q^{r+s}(1) \circ \tilde{Q}^{\alpha-s}(e_{\alpha-s}) + \tilde{Q}^{r+s}Q^{r+s}(1) \circ \tilde{Q}^{\alpha-s+1}(e_{\alpha-s}).$$

As $n = 1$, $\tilde{Q}^{\alpha-s} e_{\alpha-s} = e_{\alpha-s} * e_{\alpha-s}$; thus as in 5.1.10 the above sum is zero. Applying the mixed Cartan formula to $\tilde{Q}^{r+\alpha+1}(Q^r e_\alpha * [1] * \chi e_0^2)$ gives the result.

LEMMA 5.1.12. *Let $n > 1$, then in $H_*(SG(\mathbf{Z}/2^n), \mathbf{Z}/2)$ with $r > \alpha$,*

$$\tilde{Q}^{r+\alpha+1}(Q^r(e_\alpha) * [1] * \chi e_0^2) = Q^{r+\alpha+1}Q^r(e_\alpha) * [1] * \chi e_0^3 + \sum Q^{S_1}Q^{S_2}(e_\beta) * [1] * \chi e_0^3$$

modulo higher weight where the sum runs over certain $(S_1, S_2, \beta) \neq (r + \alpha + 1, r, \alpha)$.

Proof. As $Q^r(e_\alpha) = \Sigma_s Q^{r+s}(1) \circ e_{\alpha-s}$ for certain s we have

$$\begin{aligned} &\tilde{Q}^{r+\alpha+1}(Q^r(e_\alpha)) \\ &= \sum_s (\tilde{Q}^{r+s+1}Q^{r+s}(1) \circ \tilde{Q}^{\alpha-s}(e_{\alpha-s})) + \tilde{Q}^{r+s}Q^{r+s}(1) \circ \tilde{Q}^{\alpha-s+1}(e_{\alpha-s}) \\ &= \sum_s \tilde{Q}^{r+s+1}Q^{r+s}(1) \circ \tilde{Q}^{\alpha-s}(e_{\alpha-s}) \end{aligned}$$

as before. Again, $\tilde{Q}^{r+s+1}Q^{r+s}(1) = Q^{2r+2s+1}(1)$ modulo higher weight and thus the terms of lowest weight in the sum are $\Sigma_s Q^{2r+2s+1}(1) \circ \tilde{Q}^{\alpha-s}(e_{\alpha-s})$. Unfortunately, $\tilde{Q}^{\alpha-s}(e_{\alpha-s})$ is not a loop sum square (as $n > 1$) but 5.1.3 and 1.3.3 imply the elements of lowest weight in the sum above are $\Sigma Q^{2r+2s+1+q}(Q^{\alpha-s+t+k-q+w}(e_{\alpha-s-t-k-w}))$ where the sum runs over certain s, q, t, k, and w. But the partial sum of the terms where $s = t = k = w = 0$ is equal to $Q^{2r+1}(1) \circ (e_\alpha * e_\alpha)$ which is zero as before. Once again the mixed Cartan formula gives the desired result.

LEMMA 5.1.13. *In $H_*(SG(\mathbf{Z}/p^n), \mathbf{Z}/p)$ for all p and all n if $l(I) \geq 2$, and $e(I) > \alpha$ then $\tilde{Q}_1(Q^I(e_\alpha) * [1] * \chi e_p^{p^{l(I)}}) \equiv Q_1Q^I(e_\alpha) * [1] * \chi(e_0^{p^{l(I)+1}})$ modulo higher weight.*

Proof. A slight generalization of [**5**, 8.3.i] or [**6**, 1.7] shows $Q^I(e_\alpha) = \Sigma Q^{i_1}(1) \circ \cdots \circ Q^{i_k}(1) \circ e_\beta$, where $\beta > 0$ if and only if $\alpha > 0$ and $l(I) = k$. The rest of the proof follows exactly as in [**5**, 8.3.iii] or [**6**, 5.7].

Proof of 5.1.9. 5.1.13 proves 5.1.9 if $l(I) \geq 2$. Minor modification of the arguments of 5.1.10 and 5.1.11 for p odd as in [**6**] handle $l(I) = 1$. We leave the details to the reader.

LEMMA 5.1.14. *Let $n = 1, p = 2$. Then*

$$\tilde{Q}_1(J_{\mathbf{Z}/2}(e_r) * J_{\mathbf{Z}/2}(e_r) * [-1]) = 0.$$

Proof. Apply the mixed Cartan formula to obtain

$$\sum_{\substack{i+j=0 \\ s_0+s_1+s_2=1}} \tilde{Q}_{s_0}(J_{\mathbf{Z}/2}(e_i)) * J_{\mathbf{Z}/2}(e_i) * \tilde{Q}_{s_1}([-1])$$

$$* Q_{s_2}(J_{\mathbf{Z}/2}(e_j) * J_{\mathbf{Z}/2}(e_j) \circ [-1])$$

by symmetry. But $J_{\mathbf{Z}/2}(e_j) * J_{\mathbf{Z}/2}(e_j) \circ [-1] = 0$.

LEMMA 5.1.15. *Let $n > 1, p = 2$. Then*

$$\tilde{Q}_1(J_n(e_{2r+1}) * J_n(e_{2r+1}) * [-1]) = 0.$$

Proof. [**13**, 7.10].

5.2. We conclude this chapter by considering the geometric twist action map defined in Chapter IV. From Chapter IV we know the following diagram commutes:

$$(5.2.1)\qquad \begin{array}{ccc} C \times K^H & \xrightarrow{\bar{\lambda}} & K^H \\ \downarrow \text{id} \times \Phi & & \downarrow \Phi \\ C \times F(H) & \xrightarrow{\lambda^{11}} & F(H) \\ \uparrow \text{id} \times \rho & & \uparrow \rho \\ C \times SG(H) & \xrightarrow{\lambda^1 = \Omega\psi_G} & SG(H) \end{array}$$

Classifying this diagram we obtain

$$(5.2.2)\qquad \begin{array}{ccc} BC \times BK^H & \rightarrow & BK^H \\ \downarrow \text{id} \times B\Phi & & \downarrow B\Phi \\ BC \times BF(H) & \rightarrow & BF(H) \\ \uparrow \text{id} \times \rho & & \uparrow \text{id} \times \rho \\ BC \times BSG(H) & \xrightarrow{\psi_G} & BSG(H) \end{array}$$

Now let $H = \mathbf{Z}/2$. As 2.3.1 identifies the $\mathfrak{L}^H$ structure with the composition product structure, diagram 5.2.2 for $H = \mathbf{Z}/2$ is compatible with diagram 1.4.3. Thus we may loop 5.2.2 for $H = \mathbf{Z}/2$ to obtain a diagram compatible with 1.4.4. Thus [12.2.6] implies 1.4.6. 1.4.7 follows from the properties of the $\mathbf{Z}/2$ equivariant J homomorphism once we prove 1.4.5. Finally, results of Thomason and Wilson [**32**], implies the homology suspension commutes with the action map which in turn yields 1.4.8.

Proof of 1.4.5. This result is standard. It follows from the following well-known lemma which may be found in **[31]**.

LEMMA 5.2.3.

(a) $H^*(BO,\mathbf{Z}/2) \cong P[w_i \mid i \geq 1]$ *where* w_i *are the Stiefel-Whitney classes.*

(b) $H^*(BO,\mathbf{Z}/2)$ *has an additive basis consisting of* $s_w(\sigma)$ *as* w *runs over partitions* $w = (i_1,\ldots,i_r)$ *with coproduct* $\psi(s_w(\sigma)) = \Sigma_{w=w_1\amalg w_2} s_{w_1}(\sigma) \otimes s_{w_2}(\sigma)$.

(c) $s_i(\sigma)$ *is dual to* z_i *defined in* §1.4.

(d) *The Pontrjagin product* $z_{i_1} \circ \cdots \circ z_{i_r}$ *in* $H_*(BO,\mathbf{Z}/2)$ *is dual to* $s_{(i_1,\ldots,i_r)}(\sigma)$ *in the* $\{s_w(\sigma)\}$ *basis.*

LEMMA 5.2.4. *The map*

$$S^1 \times \mathbf{R}P^\infty \to S^1 \wedge (\mathbf{R}P^\infty_+) \overset{S^1\wedge i}{\hookrightarrow} S^1 \wedge 0_+ \hookrightarrow BO$$

classifies the virtual bundle $(\hat{L} - 1) \otimes L$ *over* $S^1 \times \mathbf{R}P^\infty = \mathbf{R}P^1 \times \mathbf{R}P^\infty$. *Here* $\hat{L} \to \mathbf{R}P^1$, $L \to \mathbf{R}P^\infty$ *are the canonical nontrivial line bundles.*

Proof. The virtual bundle $E^n - \bar{n}$ over $S^1 \wedge O(n)_+$ induced by the inclusion $S^1 \wedge O(n)_+ \hookrightarrow BO$ is defined by using the clutching function $O(n) \overset{\text{id}}{\to} O(n)$ to define the bundle $E^n \to S^1 \wedge O(n)_+$. Now i_n: $\mathbf{R}P^{n-1} \hookrightarrow O(n)$ is given by sending a line l in $\mathbf{R}^n$ to the map determined by

$$\left\{\begin{array}{lll} \vec{x} \to & \vec{x} & \text{for } x \in l \\ \vec{x} \to & -\vec{x} & \text{for } x \in l^\perp . \end{array}\right\}$$

Hence the bundle $E^n|_{S^1\wedge \mathbf{R}P^{n-1}_+}$ is defined by the clutching map i_n.

Thus the pull-back of E^n to $S^1 \times \mathbf{R}P^{n-1}$ is described as follows. First clutch the complementary bundle Q, $\{l \in \mathbf{R}P^{n-1} \to l^\perp \subset \mathbf{R}^n\}$, by the identity map and clutch the subplane bundle L, $\{l \in \mathbf{R}P^{n-1} \to l \subset \mathbf{R}^n\}$, by the antipodal map. Then the pull-back of E^n to $S^1 \times \mathbf{R}P^{n-1}$ is $1 \otimes Q \oplus \hat{L} \oplus L$ where $\hat{L}$ is the "Möbius band" bundle over S^1. Hence the pull-back of the universal bundle ξ over BO of the composition $S^1 \times \mathbf{R}P^{n-1} \to S^1 \wedge \mathbf{R}P^{n-1} \to S^1 \wedge O_{(n)_+} \to BO_{(n)} \to BO$ is $1 \otimes Q + \hat{L} \otimes L - 1 \otimes \bar{n}$. But over $\mathbf{R}P^{n-1}$ we have the equation $\bar{n} = Q \otimes L$. Thus

$$\begin{aligned} 1 \otimes Q + \hat{L} \otimes L - 1 \otimes \bar{n} &= 1 \otimes (Q - \bar{n}) + \hat{L} \otimes L \\ &= \hat{L} \otimes L - 1 \otimes L = (\hat{L} - 1) \otimes L. \end{aligned}$$

We may now prove 1.4.5.c.

LEMMA 5.2.5.

$$y_s = \sum_{w=(i_1,\dots,i_r)} N(w) z_{i_1} \circ \cdots \circ z_{i_r}, \qquad \sum_{j=1}^{n} i_j = s.$$

Proof. Recall for $w = (i_1,\dots,i_r)$ we may write $s_w(\sigma) \in H^{|w|}(BO, \mathbf{Z}/2)$ as $s_w(\sigma) = N(w) w_{|w|} +$decomposables. Using 4.1.6 we calculate as follows

$$\left(j^* \circ S^1 \wedge i_s^*\right) w_*(\xi) = w_*\left(\hat{L} \otimes L - 1 \otimes L\right)$$

$$= \frac{1 + w_1(L) + w_1(\hat{L})}{1 + w_1(L)} = 1 + w_1(\hat{L}) \otimes \frac{1}{1 + w_1(L)}$$

where w_* is the total Stiefel-Whitney class. That is, for $i \geq 1, w_i(\xi) \to w_1(L) \otimes w_1(L)^{i-1}$ but $j^* \circ (S^1 \wedge i_s)^*$ sends decomposables to zero. A calculation shows $j^* \circ (S^1 \wedge i_s)^* s_W(\sigma) = N(w)$. The result follows from 5.2.3.

LEMMA 5.2.6. *The following diagram commutes.*

$$\begin{array}{ccc}
\mathbf{R}P_+^\infty \wedge BO & \overset{\tilde{L}\otimes\xi}{\to} & BO \\
\uparrow \mathrm{id} \wedge (S^1 \wedge i) & & \\
\mathbf{R}P_+^\infty \wedge \left(S^1 \wedge (\mathbf{R}P_+^\infty)\right) & & \\
\| & & \uparrow j(S^1 \wedge i) \\
S^1 \wedge (\mathbf{R}P_+^\infty) \wedge (\mathbf{R}P_+^\infty) & & \\
\| & & \\
S^1 \wedge (\mathbf{R}P^\infty \times \mathbf{R}P^\infty)_+ & \overset{S^1\wedge(L\otimes L)}{\to} & S^1 \wedge \mathbf{R}P_+^\infty
\end{array}$$

Proof. Obvious.

COROLLARY 5.2.7.

$$\psi_{0_*}\left(e_p, \left(j \circ (S^1 \wedge i)\right)_*\left(S^1 \wedge e_q\right)\right) = \left(j \circ (S^1 \wedge i)\right)_*\left(S^1 \wedge (p, q) e_{p+q}\right).$$

Proof. Follows from 5.2.6 as $\mathbf{R}P^\infty \times \mathbf{R}P^\infty \overset{\tilde{L}\otimes L}{\to} \mathbf{R}P^\infty$ sends $e_p \otimes e_q$ to $(p, q) e_{p+q}$. 5.2.7 implies 1.4.5(b).

The commutative diagram

(5.2.8)

$$\begin{array}{ccccc}
\mathbf{R}P^\infty \times \mathbf{R}P^\infty & & \xrightarrow{\tilde{L}\otimes L} & & BO \\
\uparrow \text{id} \times \text{inc} & & \nearrow \tilde{L}\times(L-1) & & \nwarrow L_1 - L_2 \\
\mathbf{R}P^\infty \times \mathbf{R}P^\infty & \xrightarrow{\Delta} & (\mathbf{R}P^\infty \times \mathbf{R}P^\infty) \times (\mathbf{R}P^\infty \times \mathbf{R}P^\infty) & \xrightarrow{\otimes \times \pi_2} & \mathbf{R}P^\infty \times \mathbf{R}P^\infty
\end{array}$$

yields 1.4.5a since in homology

$$\begin{array}{ccccc}
e_p \otimes z_r & & \longrightarrow & & \psi_{0_*}(e_p \otimes z_r) \\
\uparrow & & & & \nwarrow \\
e_p \otimes e_r \to & \displaystyle\sum_{\substack{p=p_1+p_2\\ r=r_1+r_2}} (e_{p_1} \otimes e_{r_1}) \otimes (e_{p_2} \otimes e_{r_2}) & \to & \displaystyle\sum_{r=r_1+r_2} (p, r_1) e_{p+r_1} \otimes e_{r_2} &
\end{array}$$

as the right-hand vertical map is zero unless $p_2 = 0$.

Before we prove 1.4.5d we note that the equation $L \otimes (\xi_1 \oplus \xi_2) = (L \otimes \xi_1) \oplus (L \otimes \xi_2)$ implies

LEMMA 5.2.9.
(a) $\psi_{0_*}(e_n, x \circ y) = \sum_{i+j=n} \psi_{0_*}(e_i, x) \circ \psi_{0_*}(e_j, y)$.
(b) $\Omega\psi_{0_*}(e_n, x \circ y) = \sum_{i+j=n} \Omega\psi_{0_*}(e_i, x) \circ \Omega\psi_{0_*}(e_j, y)$.

Proof. The commutative diagram.

$$\begin{array}{ccccc}
\mathbf{R}P^\infty \wedge [BO \times BO]_+ & \overset{\text{id}\times(\oplus)}{\to} & \mathbf{R}P^\infty \wedge BO_+ & & \\
\downarrow \Delta & & & \searrow & \\
(\mathbf{R}P^\infty \wedge BO_+) \times (\mathbf{R}P^\infty \wedge BO_+) & \overset{\psi_0 \times \psi_0}{\to} & BO \times BO & \overset{\oplus}{\to} & BO
\end{array}$$

represents $L \otimes (\xi_1 \oplus \xi_2) = (L \otimes \xi_1) \oplus (L \otimes \xi_2)$. This diagram loops down to give

$$\begin{array}{ccccc}
\mathbf{R}P^\infty \times 0 \times 0 & \overset{\text{id}\times\text{product}}{\to} & \mathbf{R}P^\infty \times 0 & & \\
\downarrow \Delta & & & \searrow \Omega\psi_0 & \\
\mathbf{R}P^\infty \times 0 \times \mathbf{R}P^\infty \times 0 & \overset{\Omega\psi_0 \times \Omega\psi_0}{\to} & 0 \times 0 & \overset{\text{product}}{\to} & 0
\end{array}$$

The lemma follows.

Finally 1.4.5.d follows from the next lemma.

LEMMA 5.2.10. *The following diagram commutes up to homotopy.*

$$\begin{array}{ccc}
\mathbf{R}P^{\infty} \times \mathbf{R}P^{\infty} & \overset{\tilde{L}\otimes L}{\rightarrow} & \mathbf{R}P^{\infty} \\
\downarrow \mathrm{id} \times i & & \downarrow i \\
\mathbf{R}P^{\infty} \times 0 & \rightarrow & 0
\end{array}$$

Proof. The diagram in 5.2.6 yields the following commutative diagrams

$$\begin{array}{cccccc}
 & \mathbf{R}P^{\infty}_{+} \wedge \mathbf{R}P^{\infty}_{+} & \overset{\otimes}{\rightarrow} & \mathbf{R}P^{\infty}_{+} & & \\
\mathrm{id} \times i \swarrow & \downarrow \mathrm{id} \wedge i & & \downarrow i & \searrow i & \\
\mathbf{R}P^{\infty}_{+} \wedge 0_{+} & \mathbf{R}P^{\infty}_{+} \wedge \Omega(S^1 \wedge \mathbf{R}P^{\infty}_{+}) & \overset{\otimes}{\rightarrow} & \Omega(S^1 \wedge \mathbf{R}P^{\infty}_{+}) & & 0_{+} \\
 & \downarrow & & \downarrow \Omega(S^1 \wedge i) & & \\
\searrow \sim & \mathbf{R}P^{\infty}_{+} \wedge \Omega(S^1 \wedge 0_{+}) & & \Omega(S^1 \wedge 0_{+}) & \nearrow \sim & \\
 & \downarrow & & \downarrow \Omega_j & & \\
 & \mathbf{R}P^{\infty}_{+} \wedge \Omega BO & \overset{\Omega\psi_0}{\rightarrow} & \Omega BO & & \\
 & \uparrow & & \uparrow \wr\wr & & \\
 & \mathbf{R}P^{\infty}_{+} \wedge 0 & \overset{\Omega\psi_0}{\rightarrow} & 0 & &
\end{array}$$

and hence

$$\begin{array}{ccccc}
 & \mathbf{R}P^{\infty}_{+} \wedge \mathbf{R}P^{\infty}_{+} & \overset{\otimes}{\rightarrow} & \mathbf{R}P^{\infty}_{+} & \\
\swarrow \mathrm{id} \times i & & & & \searrow i \\
\mathbf{R}P^{\infty}_{+} \wedge 0_{+} & & & & 0_{+} \\
\searrow \sim & & & & \swarrow \sim \\
 & \mathbf{R}P^{\infty}_{+} \wedge 0_{+} & \overset{\Omega\psi_0}{\rightarrow} & 0_{+} &
\end{array}$$

which gives the lemma up to homotopy.

APPENDIX

We will prove Theorems 3.1.3 and 3.1.6. Actually we prove these theorems using a map $\gamma'_{BS}\colon G(H) \rightarrow Q(BH^{+})$ which is a variant of the Becker-Schultz equivalence γ_{BS}. γ'_{BS} was used in [**13**]. It is a homotopy equivalence that respects composition product and is natural with respect to transfers. For details of the comparison between γ_{BS} and γ'_{BS} we refer the reader to [**13**].

We begin by recalling Segal's equivalence γ_S.

Given a map $M \xrightarrow{\varphi} \mathrm{Map}_H(t(nV \oplus mR))$ of a smooth compact, closed manifold M, Segal produces a map $M \to Q(S^0) \times Q(BH^+)$. Given φ this is done by first deforming φ so that the induced H equivariant map $H \times t(nV \oplus mR) \xrightarrow{\phi} t(nV \oplus mR)$ is transverse to $\vec{0} \in (nV \oplus mR) \subset t(nV \oplus mR)$. Then $\phi^{-1}(0) \subset M \times (nV \oplus mR)$ is an H-equivariant submanifold of the H-equivariant manifold $M \times (nV \oplus mR)$ with an H-equivariant trivialization of its normal bundle with $(nV \oplus mR) \times M$. As such the isotopy groups of $\phi^{-1}(0)$ are constant and $\phi^{-1}(0)$ may be canonically expressed as the union of disjoint closed H-equivariant submanifolds $A = \phi^{-1}(0) \cap (0 \times mR)$ in the fixed part and $B = \phi^{-1}(0) \cap ((nV - \{0\}) \times mR)$ in the free part of $nV \oplus mR$ since the isotopy groups of $nV \oplus mR$ are id and H as H acts freely in $S(V)$.

(A1) $$\phi^{-1}(0) = A \coprod B.$$

Furthermore, the equivariant trivialization of the normal bundle of $\phi^{-1}(0)$ in $nV \oplus mR$, yield trivializations

(A2) $$\nu(A \hookrightarrow (0 \times mR) \times M) \overset{\alpha}{\approx} R^m \times A,$$

$$\nu(B/H \hookrightarrow ((mV - \{0\})/H \times mR) \times M) \overset{\beta}{\approx} R^{m+n\dim V} \times B$$

of the normal bundle of A (respectively B) in the fixed point set (respectively the quotient of the free H action).

The pair $(A \hookrightarrow (0 \times mR) \times M, \alpha)$ describes a stable map $S^N \wedge \mathrm{M} \to S^N$; that is, a map $M \to Q(S^0)$; the pair $(B \hookrightarrow ((nV - \{0\})/H \times mR) \times M, \beta)$ describe a stable map $S^N \wedge M \to S^N \wedge ((nV - \{0\}/H) \times mR)^+$; that is, a map $M \to Q(BH^+)$ where we use the classifying map $(nV - (0)/H) \times mR \to BH$ of the free action on $(nV \oplus mR - (0 \times mR))$. Together these give the desired map $M \to Q(S^0) \times Q(BH^+)$.

We remark that when $m = 0$ γ_S clearly maps $F(H)$ into $[\mathbf{1}] \times Q(BH^+)$. We next describe γ'_{BS}.

Given a map $M \xrightarrow{\psi} \mathrm{Map}_H(S(nV))$ of a smooth compact closed manifold M, we associate a map $M \to Q(BH^+)$ as follows:

Deform ψ so that the induced H-equivariant map $\hat{\psi}$: $M \times S(nV) \to M \times S(nV)$ has

$$(\hat{\psi}, \mathrm{pr}_2)\colon M \times S(NV) \to S(NV) \times S(NV)$$

transverse to the diagonal $\Delta\colon S(NV) \to S(NV) \times S(NV)$. Then the H-equivariant submanifold $P = (\hat{\psi}, \mathrm{pr}_2)^{-1}(\mathrm{im}\,\Delta) \subset M \times S(NV) \subset M \times NV$ has normal bundle H-equivariantly trivialized by the bundle maps covering

$$\begin{array}{c} \nu(P \hookrightarrow M \times NV) \cong \nu(P \hookrightarrow M \times S(NV)) \oplus \overline{R} \\ \downarrow \\ \nu\left(S(NV) \overset{\Delta}{\hookrightarrow} S(NV) \times S(NV)\right) \oplus \overline{R} \\ \downarrow \text{via}(\mathrm{pr}_2 \times \mathrm{id}) \\ N\overline{V} = T_*(S(NV)) \oplus \overline{R} \end{array}$$

(since the normal bundle of $S(NV) \hookrightarrow S(NV) * S(NV)$ is isomorphic to the tangent bundle $T_*S(NV)$ of $S(NV)$).

This H-equivariant trivialization yields a trivialization of the normal bundle $\nu(P/H \hookrightarrow M \times (NV - \{0\}/H))$ and thus a stable map $S^N \wedge M \to S^N \wedge (NV - \{0\}/H)^+$. This describes a map $M \to Q(BH^+)$ where we use the classifying map $NV - \{0\}/H \to BH$ of the free action of H on $NV - \{0\}$.

If W is an even-finite dimensional subspace of nV we define a map

(A3) $$\mathbb{S}\colon \mathrm{Map}_H(S(W)) \to \mathrm{Map}_H(t(W))$$

by the formulae:

$$\mathbb{S}(g)(\infty) = \infty$$

and

$$\mathbb{S}(g)(\lambda\hat{x}) = \begin{cases} -\lambda\hat{x} & \text{if } 0 \le \lambda \le 1/2, \\ -1/2\hat{x} + (\lambda - 1/2)g(\hat{x}) & \text{if } \lambda \ge 1/2. \end{cases}$$

REMARKS.

(a) $g\colon S(W) \to S(W)$ and $\|\hat{x}\| = 1$.

(b) $\mathbb{S}(g)$ restricted to the disk of radius $1/2$ is just the antipodal map.

(c) $\mathbb{S}(g)$ extends continuously to tW as $\|\mathbb{S}(g)(\lambda\hat{x}) - \lambda g(\hat{x})\| = \| - \frac{1}{2}(\hat{x} + g(\hat{x}))\| \le 1$.

Thus we may define a homotopy

(A4) $$\mathbb{S}_t(g)(\lambda\hat{x}) = (1-t)\lambda g(\hat{x}) + t\mathbb{S}(g)(\lambda\hat{x})$$

between $\mathbb{S}_0(g) = \rho g$ and $\mathbb{S}_1(g) = \mathbb{S}(g)$.

An easy computation shows

$$\text{(A5)} \qquad \mathbb{S}(g)^{-1}(0) = \{0\} \coprod \{\hat{x} \mid g(\hat{x}) = \hat{x}\}.$$

Then given $\hat{\psi}$, P and the trivialization above (recall the description of γ'_{BS}) $\phi(\mathbb{S}(\hat{\psi}))$: $M \to \mathrm{Map}_H(t(W))$ is given by the adjoint of $(m, \lambda\hat{x}) \to (m, \mathbb{S}(\hat{\psi}(m))(\lambda\hat{x}))$. By (A5) we conclude

$$\text{(A6)} \qquad \phi(\mathbb{S}\hat{\psi})^{-1}\left(M \times \vec{0}\right) = M \times 0 \coprod P.$$

The framings are compatible and thus induce the following commutative diagram

$$\text{(A7)} \qquad \begin{array}{ccccc} \mathrm{Map}(S(W)) & & \xrightarrow{\rho} & & \mathrm{Map}(t(W)) \\ \downarrow \gamma_{BS} & & & & \downarrow \gamma_S \\ Q(BH^+) & \to & [1] \times Q(BH^+) & \to & QS^0 \times Q(BH^+) \end{array}$$

As W is even dimensional (A7) is compatible with limits (see [**13**]). This proves 3.1.3.

To prove 3.1.6 we first recall from [**9**] the identification of K^H. As pointed out by May

$$\text{(A8)} \qquad (\mathcal{K}(j)/\Sigma_j)^H \simeq \coprod_{\varphi} \mathcal{K}(j)^{\varphi}/\Sigma_{\varphi}$$

where

(a) φ runs through conjugacy classes of homomorphisms φ: $H \to \Sigma_j$.

(b) $\mathcal{K}(j)^{\varphi}$ is the subset of $\mathcal{K}(j)$ consisting of all points k such that $hk = k\varphi(h)$ for all h in H.

(c) Σ_{φ} is the subgroup of Σ_j consisting of all elements σ such that $\sigma\varphi(h) = \varphi(h)\sigma$ for every h in H.

(d) As $\mathcal{K}(j) = \varinjlim_W \mathcal{K}(j, W)$ is the limit of semi-free H actions on finite dimensional subspaces of $(V \oplus \mathbf{R})_H$, $\mathcal{K}(j)^{\varphi}$ is empty unless φ corresponds to a semi-free action. If φ does correspond to a semi-free action then $\mathcal{K}(j)^{\varphi}$ is contractible and $\mathcal{K}(j)^{\varphi}/\Sigma_{\varphi}$ is a $K(\pi, 1)$.

Hence we obtain the decomposition

$$K^H \cong \coprod_j \coprod_{\varphi_j} \left(\mathcal{K}(j)^{\varphi_j}/\Sigma_{\varphi_j}\right) \cong \coprod_n B\Sigma_n \times \coprod_m B\left(\Sigma_m \int H\right)$$

where the $(\mathcal{K}^H, \mathcal{L}^H)$ operad action on K^H is described on $\amalg_n B\Sigma_n \times \amalg_m B(\Sigma_m \int H)$ and hence on a group completion $Q(S^0) \times Q(BH^+)$ by the maps of 2.2.4 restricted to the $K(\pi, 1)$ level.

As $K^H \xrightarrow{\phi} F(H) \xrightarrow{\gamma_S} Q(S^0) \times Q(BH^+)$ is clearly a $\mathcal{K}^H$ map to prove 3.1.6 it suffices to check

(a) The composite

$$S^0 \wedge BH^+ \hookrightarrow K^H \xrightarrow{\phi} F(H) \xrightarrow{\gamma_S} Q(S^0) \times Q(BH^+)$$

is homotopic to the standard inclusion.

(b) More generally, the composites

$$B\Sigma_\alpha \times B\left(\Sigma_\beta \int H\right) \hookrightarrow K^H \xrightarrow{\phi} F(H) \xrightarrow{\gamma_S} Q(S^0) \times Q(BH^+)$$

are homotopic to the standard inclusion.

(a) is proved by starting with an approximation to BH, say a compact smooth manifold M and a map $M \xrightarrow{\tau} BH$ which is N-connected. This map may be replaced by a smooth map

$$M \xrightarrow{f} (nV - \{0\})/H \times mR$$

for n large. (Note that the right-hand side is a smooth manifold.) We may regard f as a map $M \to \{$configurations of one free H-orbit in $(nV - \{0\}) \times mR\}$. That is, for $x \in M$, we have $f(x) = (x_1, x_2, \ldots, x_{|H|})$ where $x_1, \ldots, x_{|H|}$ is a free H-orbit of $nV \oplus mR$. The map $\phi \circ f$: $M \to \mathrm{Map}_H(t(nV \oplus mR))$ is then defined by

$$f(x)(y) = \begin{cases} \dfrac{y - x_i}{\varepsilon^2 - |y - x_i|^2} & \text{if } y \in (\varepsilon\text{-ball about } x_i \text{ for some } i), \\ \infty & \text{otherwise,} \end{cases}$$

where $f(x) = \mathrm{orbit}(x_1, x_2, \ldots, x_{|H|})$, and $\varepsilon = 1/2$ (minimum distance between the x_i's). (We have chosen an H-invariant metric on $nV \oplus mR$.)

Now the composite $\gamma_S \circ \phi \circ f$: $M \to Q(S^0) \times Q(BH)$ is easy to describe since $\phi \circ f(x)$ is already transverse to $0 \in nV \oplus mR$. By our prescription above, $\phi \circ f$ induces a map

$$\Psi: M \times t(nV \oplus mR) \to t(nV \oplus mR),$$
$$(x, y) \to (\phi \circ f)(x)(y)$$

with

$$\Psi^{-1}(0) = \{(x, z) \mid z \text{ is in the free } H\text{-set } f(x) = (x_1, x_2, \ldots, x_{|H|})\}$$
$$= \pi^{-1}\left(\text{graph of } M \xrightarrow{f} (nV - \{0\}/H) \times mR\right)$$

where π: $(M \times (nV - \{0\}) \times mR) \to M \times (nV - \{0\}/H) \times mR$ is the projection pr_2 which induces the isomorphism

$$\nu(\Psi^{-1}(0) \hookrightarrow M \times (nV - \{0\}) \times mR) \xrightarrow{\approx} (nV \oplus mR) \times \Psi^{-1}(0).$$

Hence, $\gamma_S \circ \phi \circ f$ is described by $M \to [0] \in Q(S^0)$ and $M \to Q(BH^+)$. The latter is given by $M_\alpha = \{\text{graph of } M \xrightarrow{f} (nV - \{0\}/H) \times mR\}$ with the isomorphism

$$\nu(M \hookrightarrow M \times (nV - \{0\}/H) \times mR)$$
$$\simeq (\mathrm{pr}_2)^*\left(T_*((nV - \{0\}/H) \times mR)\right)$$

coming from the projection pr_2 which defines a map

$$M \to Q((nV - \{0\}/H) \times mR) \to Q(BH^+)$$

by classifying the free H-action on $(nV - (0)/H) \times mR$. That is, this second map $M \to Q(BH^+)$ is merely $M \xrightarrow{f} BH \xrightarrow{i} Q(BH^+)$ where i is the standard inclusion. This proves (a).

The proof of (b) is similar. Briefly, given g: $M \to B\Sigma_\alpha \times B(\Sigma_\beta \int H)$ which is N-connected, we replace g by a smooth map

$$f\colon M \to \begin{Bmatrix} \text{configurations of} \\ \text{points in } 0 \times mR \end{Bmatrix} \times \begin{Bmatrix} \text{configurations of free } H\text{-orbits} \\ \text{in } (nV - \{0\}) \times mR \end{Bmatrix}$$

and then proceed as before to get property (b).

This concludes the proof of Theorem 3.1.6.

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