

# Pacific Journal of Mathematics

**SUBSETS OF  $\omega_\omega$  AND GENERALIZED METRIC SPACES**

DENNIS K. BURKE AND SHELDON DAVIS

## SUBSETS OF ${}^\omega\omega$ AND GENERALIZED METRIC SPACES

D. K. BURKE AND S. W. DAVIS

In this note we use subsets of  ${}^\omega\omega$  and certain set-theoretic statements about such sets to solve, up to consistency, several questions concerning generalized metric spaces. These results are improvements of some of Michael's work using the continuum hypothesis (hereafter CH) and some earlier work by the authors using CH and  $P(c)$ .

**1. Introduction.** As usual,  ${}^\omega\omega$  denotes the set of all functions from the natural numbers ( $\omega$ ) into the natural numbers. An ordinal is equal to the set of all smaller ordinals, a cardinal is an initial ordinal, and  $c = 2^\omega$ .

**DEFINITION 1.1.** If  $f, g \in {}^\omega\omega$ , we write

$f <_* g$  iff there is  $k \in \omega$  such that  $f(n) < g(n)$  for all  $n \geq k$ , and

$f =_* g$  iff there is  $k \in \omega$  such that  $f(n) = g(n)$  for all  $n \geq k$ .

**DEFINITION 1.2.** Suppose  $\kappa$  is a cardinal. A set  $S \subseteq {}^\omega\omega$  is called a  $\kappa$ -scale iff  $S = \{g_\alpha: \alpha < \kappa\}$  where  $g_\alpha <_* g_\beta$  whenever  $\alpha < \beta < \kappa$  and if  $f \in {}^\omega\omega$ , then there is  $\alpha < \kappa$  such that  $f <_* g_\alpha$ .

$\text{BF}(\kappa)$  is the statement: "If  $F \subseteq {}^\omega\omega$  and  $|F| < \kappa$ , then there is  $g \in {}^\omega\omega$  such that  $f <_* g$  for all  $f \in F$ ."

We shall be mainly interested in using as additional axioms with ZFC the statements "There exists an  $\omega_1$ -scale" and  $\text{BF}(c)$ . We state without proof the following theorems regarding the consistency, independence, and strengths of these axioms.

**THEOREM 1.3.**

- (a) [**Ha**] CH implies that there exists an  $\omega_1$ -scale.
- (b)  $\text{BF}(c)$  implies there exists a  $c$ -scale.
- (c) [**He**] It is consistent that there exists an  $\omega_1$ -scale and CH is false (due to R. M. Solovay).
- (d) [**Bo**], [**Ro**<sub>2</sub>]  $\text{CH} \Rightarrow \text{MA} \Rightarrow P(c) \Rightarrow \text{BF}(c)$ .
- (e) [**So**]  $\text{BF}(c)$  and not  $P(c)$  is consistent.
- (f) [**He**] There exists a  $\kappa$ -scale implies  $\text{BF}(\text{cf}\kappa)$ .

REMARK. In the notation used by Vaughan and van Douwen [Va], the statement  $\text{BF}(\kappa)$  is equivalent to “ $b \geq \kappa$ ”. In this notation  $b$  is the smallest cardinal  $\kappa$  such that there exists a family  $F \subseteq {}^\omega\omega$  with  $|F| = \kappa$  and there is no  $g \in {}^\omega\omega$  with  $f <_* g$  for all  $f \in F$ .

**2. Scaling old problems.** In this section we use the existence of  $\omega_1$ -scales as an alternative to CH in the improvement of existing results on several old problems. The construction techniques which we shall be using are based upon the following lemmas. Analogous results using CH are Lemmas 3.1 and 3.5 of [M<sub>2</sub>]. It is pointed out in [M<sub>2</sub>] that the statements given there are actually equivalent to CH, and it is asked if CH is essential to the existence of examples of the type we shall describe (Question 7.1 of [M<sub>2</sub>]). Thus our results may be considered as an answer to this question.

LEMMA 2.1. (*Assume there exists an  $\omega_1$ -scale.*) Let  $\{\tau_\alpha: \alpha < \omega_1\}$  be a family of first countable topologies on a set  $X$  and let  $D$  be a countable subset of  $X$  such that  $D$  is not the intersection of any countable subcollection from  $\bigcup_{\alpha < \omega_1} \tau_\alpha$ . There exists an uncountable set  $Z$  such that  $D \subseteq Z \subseteq X$  and whenever  $D \subseteq W \subseteq X$  and  $W \in \bigcup_{\alpha < \omega_1} \tau_\alpha$ , then  $|Z \setminus W| \leq \omega$ . (Hence  $Z$  is Lindelöf with respect to every  $\tau_\alpha$ .)

*Proof.* Let  $S = \{f_\alpha: \alpha < \omega_1\}$  denote an  $\omega_1$ -scale. If  $D = \{z_n: n \in \omega\}$  and  $\beta < \omega_1$ , let  $\{U(z_n, k, \beta): k \in \omega\}$  be a decreasing open base at  $z_n$  with respect to  $\tau_\beta$ . Now for any  $\alpha < \omega_1$  and  $h \in {}^\omega\omega$ , let  $W(h, \beta) = \bigcup_{n \in \omega} U(z_n, h(n), \beta)$  and  $G_{\alpha\beta} = \bigcap \{W(h, \beta): h =_* f_\alpha\}$ . Note that  $D \subseteq G_{\alpha\beta}$ ,  $G_{\alpha\beta}$  is a  $G_\delta$ -set with respect to  $\tau_\beta$ , and  $G_{\alpha\beta} \supseteq G_{\gamma\beta}$  if  $\alpha < \gamma < \omega_1$ . For each  $\gamma$ , the set  $\bigcap_{\beta \leq \gamma} G_{\gamma\beta}$  is uncountable; so it is possible to choose a set  $\{x_\gamma: \gamma < \omega_1\}$  such that  $x_\gamma \in \bigcap_{\beta \leq \gamma} G_{\gamma\beta} \setminus D$  and  $x_\gamma \neq x_\theta$  if  $\gamma \neq \theta$ . Let  $Z = \{x_\gamma: \gamma < \omega_1\} \cup D$ . To see that  $Z$  satisfies the desired property, suppose there is  $\theta < \omega_1$  and  $W \in \tau_\theta$  with  $D \subseteq W$ . Choose  $h \in {}^\omega\omega$  such that  $W(h, \theta) \subseteq W$  and find  $f_\alpha \in S$  such that  $h <_* f_\alpha$  and  $\theta \leq \alpha$ . It is clear that for any  $\gamma \geq \alpha$ ,

$$\bigcap_{\beta \leq \gamma} G_{\gamma\beta} \subseteq \bigcap_{\beta \leq \alpha} G_{\alpha\beta} \subseteq G_{\alpha\theta} \subseteq W(h, \theta) \subseteq W,$$

so that  $\{x_\gamma: \alpha \leq \gamma < \omega_1\} \subseteq W$ . Hence  $|Z \setminus W| \leq \omega$ .

REMARK. The hypothesis of the above lemma is somewhat stronger than necessary. Examination of the proof indicates that the topologies  $\tau_\beta$  need not be first countable, but only that the points of  $D$  have countable character with respect to  $\tau_\beta$  for each  $\beta < \omega_1$ .

For the uses made in the sequel, we find it convenient to have the following weaker form of Lemma 2.1 when there is only one topology given on the set  $X$ .

**LEMMA 2.2.** (*Assume there exists an  $\omega_1$ -scale.*) *If  $X$  is a first countable  $T_1$ -space and  $D$  is a countable, non- $G_\delta$ -subset of  $X$ , then there exists an uncountable set  $Z$  such that  $D \subseteq Z \subseteq X$  and whenever  $W$  is an open subset of  $X$  containing  $D$ , then  $|Z \setminus W| \leq \omega$ .*

**EXAMPLE 2.3.** (*Assume there exists an  $\omega_1$ -scale.*) There is a regular Lindelöf space  $Y$  whose product with the irrationals is not normal.

*Proof.* Let  $Y$  be the set  $Z$  whose existence is given by 2.2 when  $D = Q$  and  $X = \mathbf{R}$ , and give  $Y$  the subspace topology from the Michael line  $[\mathbf{M}_1]$ . Michael constructed this space using CH (and the analogue of 2.2) in  $[\mathbf{M}_2]$  and  $[\mathbf{M}_1]$ , and the verification of the properties claimed is given in  $[\mathbf{M}_2]$ .

To the authors' knowledge, it is unknown whether an example of this type can be constructed using no axioms beyond ZFC; see Problem 5.6  $[\mathbf{Pr}]$ .

The remaining examples in this section are built upon  $H$ , Heath's "bow-tie space"  $[\mathbf{Hh}]$ . The points of  $H$  are the points of the plane. For a point  $x = (x_1, x_2)$ , we let  $B(x, n)$  be the set given by

$$B(x, n) = \left\{ y \in \mathbf{R} \times \mathbf{R} : \|x - y\| < \frac{1}{n} \text{ and } \left| \frac{y_2 - x_2}{y_1 - x_1} \right| < \tan \frac{1}{2n} \right\}.$$

Geometrically,  $B(x, n)$  is a horizontal bow-tie centered at  $x$ . As  $n$  increases, both the radius and the vertex angle decrease.

It is known that the topology generated by using  $\{B(x, n) : n \in N, x \in H\}$  is a semimetrizable topology on  $H$ . We note that it is known that  $H$  is a Baire space, and that Lindelöf semimetric spaces are hereditarily Lindelöf. We define a "bow-tie"  $B$  to have orientation  $\alpha$  if  $B$  can be obtained from  $B(x, n)$ , for some  $x \in H, n \in N$ , by a counter-clockwise rotation through the angle  $\alpha$ . In this case, we write  $B = B(x, n, \alpha)$ . In particular,  $B(x, n) = B(x, n, 0)$  for each  $x \in H, n \in N$ .

**EXAMPLE 2.4.** (*Assume there exists an  $\omega_1$ -scale.*) There exists a Lindelöf semimetrizable space having no countable network.

*Proof.* Let  $\tau_1$  be the topology on  $H$  generated by  $\{B(x, n, 0): x \in H, n \in N\}$  and  $\tau_2$  be the topology on  $H$  generated by  $\{B(x, n, \pi/2): x \in H, n \in N\}$ . Let  $Z$  be the set whose existence is given by 2.1 with  $X = H$  and  $D = Q \times Q$ . Now let  $Z_1 = (Z, \tau_1)$  and  $Z_2 = (Z, \tau_2)$ . Then both  $Z_1$  and  $Z_2$  are Lindelöf semimetric spaces. However  $\{(x, x): x \in Z\}$  is a closed discrete subset of  $Z_1 \times Z_2$ . Hence  $Z_1 \times Z_2$ , and consequently at least one of  $Z_1$  and  $Z_2$ , cannot have a countable network.

For our next example we would like to have a set  $Z \subseteq H$  which is Lindelöf with respect to all possible orientations. Since 2.1 allows only  $\omega_1$  topologies, and we are admitting the possibility that  $\omega_1 < c$ , we use the following lemma.

LEMMA 2.5. *For  $\alpha \in [0, \pi)$ , let  $\tau_\alpha$  be the topology generated on  $H$  by  $\{B(x, n, \alpha): x \in H, n \in N\}$ . If  $Z \subseteq H$  is Lindelöf with respect to  $\tau_\alpha$  for each rational  $\alpha$ , then  $Z$  is Lindelöf with respect to  $\tau_\alpha$  for every  $\alpha$ .*

*Proof.* Only the center point poses a problem, but if  $B$  is a bow-tie in  $Z$  with any orientation, then there is some rational  $\alpha$  so that  $B$  is open with respect to  $\tau_\alpha$ . Now since  $Z$  is hereditarily Lindelöf with respect to  $\tau_\alpha$  for each rational  $\alpha$ , the result follows.

EXAMPLE 2.6. (Assume there exists an  $\omega_1$ -scale.) There is a nonmetrizable semimetrizable space which has a locally connected, perfectly normal compactification.

*Proof.* The desired example is given in Examples 3.2 and 3.3 of [BD<sub>1</sub>] using CH. The need for CH is to obtain the set which is called  $X$  in [BD<sub>1</sub>]. This set can be obtained by using an  $\omega_1$ -scale from Lemma 2.1 by letting  $\tau_\alpha$  be the topology generated by  $\{B(x, n, \alpha): x \in H, n \in N\}$  for each rational  $\alpha$ ,  $X = H$  (in the notation of 2.1) and  $D = Q \times Q$ . Then the set  $Z$  (in the notation of 2.1) is the desired set. The remainder of the construction now follows from Lemma 2.5 and the work in [BD<sub>1</sub>] by first building a first countable compactification of  $H$  and then taking a quotient to get the compactification we are seeking.

It is not known if there exist ZFC examples which have the properties of those given above. However, it can be shown that the type of construction used here, as well as in [M<sub>2</sub>] and [Ve], cannot be done if  $\text{BF}(\omega_2)$  is assumed. Rothberger [Ro<sub>1</sub>] has shown that under  $\text{BF}(\omega_2)$  any subset  $A$  of

the real numbers of cardinality  $\omega_1$  has the property that every countable subset of  $A$  is a relative  $G_\delta$ . In fact, the setting of the real numbers is not needed for this theorem. The following result is well known (to at least van Douwen and some others).

**THEOREM 2.7.** (*Assume  $\text{BF}(\omega_2)$ .*) *If  $A$  is a subset of a first countable  $T_1$  space and  $|A| \leq \omega_1$ , then every countable subset of  $A$  is a relative  $G_\delta$ .*

In particular, one easily sees that the type of set given by 2.1, as is used in 2.3, 2.4 and 2.6, will not exist if  $\text{BF}(\omega_2)$  is assumed.

**3. Applications of  $\text{BF}(\kappa)$ .** It is often useful to specify a sequence converging to a given limit point of a set. We now turn our discussion to the problem of producing such sequences in spaces which may not be Fréchet, and show that in some cases one may actually obtain the Fréchet condition from sequential.

The following may be thought of as a topological characterization of  $\text{BF}(\kappa)$ .

**THEOREM 3.1.**  *$\text{BF}(\kappa)$  if and only if whenever  $D$  is a conditionally compact subset of a regular space  $X$ ,  $x_n$  is a limit of a sequence in  $D$  for each  $n \in \omega$ , and  $x_n \rightarrow x$  where  $\psi(x, X) < \kappa$ , then  $x$  is a limit of a sequence in  $D$ .*

*Proof.* Assume  $\text{BF}(\kappa)$ . Recall that  $\psi(x, X) < \kappa$  implies there is a cardinal  $\eta < \kappa$  and an open collection  $\{U_\alpha: \alpha < \eta\}$  such that  $\{x\} = \bigcap_{\alpha < \eta} U_\alpha$ . Since  $X$  is regular, we may also assume  $\{x\} = \bigcap_{\alpha < \eta} \bar{U}_\alpha$ . For each  $n \in \omega$ , choose a sequence  $\langle x_{n,k}: k \in \omega \rangle$  in  $D$  such that  $x_{n,k} \rightarrow x_n$ . For each  $U_\alpha$ , choose a function  $f_\alpha \in {}^\omega\omega$  such that whenever  $x_n \in U_\alpha$  and  $k \geq f_\alpha(n)$ , then  $x_{n,k} \in U_\alpha$ . Apply  $\text{BF}(\kappa)$  to obtain  $g \in {}^\omega\omega$  such that  $f_\alpha <_* g$  for each  $\alpha < \eta$ . Now it is easy to see that the sequence  $s = \langle x_{n,g(n)}: n \in \omega \rangle$  is eventually in every  $U_\alpha$ . Hence every cluster point of  $s$  is in every  $\bar{U}_\alpha$ . Thus, since  $D$  is conditionally compact,  $x$  is the unique cluster point of every subsequence of  $s$ , and  $s$  must converge to  $x$ .

Conversely, suppose  $\text{BF}(\kappa)$  fails; then  $b < \kappa$ . Choose  $\{f_\alpha: \alpha < b\} \subseteq {}^\omega\omega$  such that for each  $\alpha$ ,  $f_\alpha$  is nondecreasing,  $f_\alpha <_* f_\beta$  whenever  $\alpha < \beta$ , and there is no  $g \in {}^\omega\omega$  such that  $f_\alpha <_* g$  for every  $\alpha < b$ . Initially, we follow the construction of the space  $X$  given in Example 3 of [BvD]. Let  $X = b \cup (\omega \times \omega)$ , and topologize  $X$  as follows: Points of  $\omega \times \omega$  are

isolated. If  $\alpha \in b \setminus \{0\}$ ,  $\eta < \alpha$ ,  $m \in \omega$ , the set  $U(\alpha, \eta, m) = (\eta, \alpha] \cup \{(k, n) : k \geq m, f_\eta(k) < n \leq f_\alpha(k)\}$  is a basic neighborhood of  $\alpha$ . If  $\alpha = 0$ ,  $m \in \omega$ , the set

$$U(0, m) = \{0\} \cup \{(k, n) : k \geq m, n \leq f_0(k)\}$$

is a basic neighborhood of 0. Now for each  $n \in \omega$ , let  $A_n = \{n\} \times \omega$ , and choose distinct points  $x_n \notin X$ . We let  $Y = X \cup \{x_n : n \in \omega\}$  with  $X$  open in  $Y$  and basic neighborhoods of  $x_n$  will be cofinite subsets of  $A_n$  along with  $x_n$ . Note that  $Y$  is a locally compact  $T_2$  space, and  $\omega \times \omega$  is a dense conditionally compact subset of  $Y$  (see Lemma 4 of [BvD]). Let  $Z$  be the one-point compactification of  $Y$ , i.e.,  $Z = Y \cup \{p\}$ . Now, since  $|Z| = b$ ,  $\psi(p, Z) < \kappa$ . It is easy to see that  $x_n \rightarrow p$ , and, since  $\omega \times \omega$  is conditionally compact in  $Y$ , no sequence in  $\omega \times \omega$  can converge to  $p$ . This completes the proof.

A similar argument verifies the following result in which the conditionally compact assumption is removed by replacing the pseudocharacter restriction with a character restriction.

**THEOREM 3.2.** *Assume  $\text{BF}(\kappa)$ . If  $X$  is sequential and  $\chi(X) < \kappa$ , then  $X$  is Fréchet.*

*Proof.* Suppose  $A \subseteq X$ . We will show that if, for each  $n \in \omega$ ,  $\langle x_{n,k} : k \in \omega \rangle$  is a sequence in  $A$  converging to a point  $x_n$ , and the sequence  $\langle x_n : n \in \omega \rangle$  converges to a point  $x$ , then there is a sequence in  $A$  which converges to  $x$ . Since closure is obtained recursively from sequential closures, this suffices for the result.

Choose a local base  $\{U_\alpha : \alpha < \theta\}$  at  $x$  where  $\theta < \kappa$ . For each  $U_\alpha$ , we may choose a function  $f_\alpha \in {}^\omega\omega$  such that, when  $x_n \in U_\alpha$ ,  $k \geq f_\alpha(n)$  implies  $x_{n,k} \in U_\alpha$ . Apply  $\text{BF}(\kappa)$  to obtain  $g \in {}^\omega\omega$  such that  $f_\alpha <_* g$  for every  $\alpha < \theta$ . Now the sequence  $\langle x_{n,g(n)} : n \in \omega \rangle$  is eventually in every  $U_\alpha$  and, hence, converges to  $x$ .

We now apply 3.1 and 3.2 to obtain improvements of certain results given in [BD<sub>2</sub>]. For definitions of symmetrizable and weakly first countable spaces, see [Ar<sub>1</sub>] or [BD<sub>2</sub>].

**THEOREM 3.3** *Assume  $\text{BF}(c)$ . Every regular symmetrizable space with a dense conditionally compact subset is first countable (hence, separable [St<sub>2</sub>]).*

*Proof.* Suppose  $X$  is regular and symmetrizable, and  $D$  is a dense conditionally compact subspace of  $X$ . By Lemma 3.4 of [BD<sub>2</sub>] and the fact that, in a space with a dense conditionally compact subset, points which are  $G_\delta$ 's have countable character, it suffices to show that any point which is the limit of a sequence of sequential limit points of  $D$  must itself be a sequential limit point of  $D$ . This follows from 3.1.

**THEOREM 3.4.** *Assume BF( $c$ ). If  $X$  is an uncountable  $T_2$  compact weakly first countable space, and  $|X| \leq c$ , then  $|X| = c$ .*

*Proof.* Suppose  $\omega < |X| < c$ . By 3.1,  $X$  must be Fréchet, and it is well known that a  $T_2$  Fréchet weakly first countable space is first countable. Now from an old result in [AU], every uncountable, first countable, compact  $T_2$  space has cardinality at least  $c$ , we have a contradiction.

Malýhin has announced in [Ma] that using forcing there are  $T_2$  compact weakly first countable spaces with cardinality strictly between  $\omega$  and  $c$ .

It is an open question whether every regular feebly compact symmetrizable space must be separable. Stephenson [St<sub>2</sub>] has shown that this question has an affirmative answer for spaces which have a dense set of isolated points. We use BF( $c$ ) to give an affirmative answer for spaces with a dense set of points of countable character. The crux of the argument is contained in the following lemma.

**LEMMA 3.5.** *Assume BF( $c$ ). If  $X$  is regular, feebly compact, sequential,  $\psi(x, X) < c$  for each  $x \in X$ , and there is a subset  $M \subseteq X$  with each point of  $M$  having a countable local base, then every limit point of  $M$  is a sequential limit point of  $M$ .*

*Proof.* We show that any point which is a limit of a sequence of sequential limit points of  $M$  must be a sequential limit point of  $M$ . Since  $X$  is sequential, this gives the result. Let  $x \in X$  and  $\langle x_n: n \in \omega \rangle$  be a sequence in  $X$  such that  $x_n \rightarrow x$  and for each  $n \in \omega$  there is a sequence  $\langle x_{n,k}: k \in \omega \rangle$  in  $M$  with  $x_{n,k} \rightarrow x_n$ . Choose a collection  $\{U_\alpha: \alpha < \beta\}$  of open subsets of  $X$  such that  $\beta < c$  and  $\{x\} = \bigcap_{\alpha < \beta} U_\alpha = \bigcap_{\alpha < \beta} \bar{U}_\alpha$ . For each  $z \in M$ , let  $\{V(z, n): n \in \omega\}$  be a decreasing local base at  $z$ .

For each  $\alpha < \beta$ , choose  $f_\alpha \in {}^\omega\omega$  such that, when  $x_n \in U_\alpha$ ,  $k \geq f_\alpha(n)$  implies  $x_{n,k} \in U_\alpha$ . Apply BF( $c$ ) to  $\{f_\alpha: \alpha < \beta\}$  to obtain  $f \in {}^\omega\omega$  with  $f_\alpha <_* f$  for each  $\alpha$ . For each  $\alpha < \beta$ , choose  $g_\alpha \in {}^\omega\omega$  such that, when



$x_{n,f(n)} \in U_\alpha$ ,  $k \geq g_\alpha(n)$  implies  $V(x_{n,f(n)}, k) \subseteq U_\alpha$ . Apply  $\text{BF}(c)$  to  $\{g_\alpha: \alpha < \beta\}$  to obtain  $g \in {}^\omega\omega$  such that  $g_\alpha <_* g$  for each  $\alpha$ . Consider the sequence  $\langle V(x_{n,f(n)}, g(n)): n \in \omega \rangle$  of open sets. By the feeble compactness, the tails of this sequence must form a countable open filterbase with unique cluster point  $x$ . Hence the filterbase converges to  $x$ , and so must the sequence  $\langle x_{n,f(n)}: n \in \omega \rangle$ .

**THEOREM 3.6.** *Assume  $\text{BF}(c)$ . Every regular feebly compact symmetrizable space with a dense set of points of countable character is first countable (and hence separable [ $\text{St}_2$ ]).*

*Proof.* Suppose  $X$  is regular and symmetrizable, and  $M$  is a dense subset of  $X$  with each point of  $M$  having countable character. By Lemma 3.4 of [**BD**<sub>2</sub>] and the fact that, in a feebly compact space, points which are  $G_\delta$ 's have countable character, it suffices to show that every limit point of  $M$  is a sequential limit of  $M$ . This follows from 3.5.

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Received June 4, 1982 and in revised form September 15, 1982. This work was partially supported by a Miami University Summer Research Grant.

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The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$132.00 a year (6 Vol., 12 issues). Special rate: \$66.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Old back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

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The *Pacific Journal of Mathematics* ISSN 0030-8730 is published monthly by the Pacific Journal of Mathematics at P.O. Box 969, Carmel Valley, CA 93924. Application to mail at Second-class postage rates is pending at Carmel Valley, California, and additional mailing offices. Postmaster: Send address changes to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924.

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