RATIONAL SURGERY CALCULUS: EXTENSION OF KIRBY'S THEOREM

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**Every 3-dimensional closed orientable manifold may be constructed
by surgery on the 3-sphere as introduced by M. Dehn. We are concerned
in this paper with answering the question: when do two surgery schemes
actually produce the same 3-manifold?**

Kirby [4] described certain geometric moves — a calculus of links —
and showed that two "framed links" in S^3 produce homeomorphic
surgery manifolds if and only if they are related by these moves. But not
all surgeries are permitted in the Kirby calculus. Only those which
correspond to attaching 2-handles to the 4-ball are considered, since
Kirby's techniques are 4-dimensional.

EXAMPLE. Dehn considered the homology spheres determined by the
surgery data
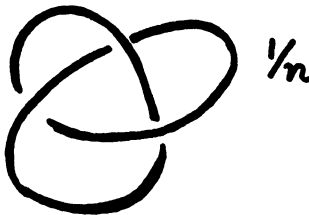
where n is an integer (notation of [6], also explained below.) Except for
$n = \pm 1$, these constructions are not considered in the Kirby calculus. In
continuations of this example we will show that these spaces are all
distinct. One can, of course, construct these manifolds using surgeries
permitted in the Kirby calculus, but one is obliged to use different knots
K_n for each n, which we will construct.

In this paper we prove a generalization of Kirby's theorem which
applies to the entire class of Dehn surgeries on S^3. Instead of Kirby's
calculus, we use the "rational" link calculus introduced (independently) in
[6]. Our main result states that different surgery data (= links with
rational coefficients) yield the same oriented manifold if and only if the

data are related by a finite sequence of moves of the calculus of [6]. The proof depends on Kirby's theorem, and on a theorem of Rourke and Fenn which provides the essential connection between the calculi [2].

Surgery data and their modification. The rational surgery calculus deals with *surgery data* (V, J), where $V = V_1 \cup \cdots \cup V_n$ is a disjoint collection of solid tori $(V_i \cong S^1 \times D^2)$ in S^3 and $J = J_1 \cup \cdots \cup J_n$ a collection of Jordan curves $J_i \subset \partial V_i$. We require only that J_i be essential (= nonseparating) in the torus ∂V_i. The Kirby calculus requires also that J_i be a longitude (i.e. homologous with the centerline in V_i). If every J_i is a longitude, we say that (V, J) is *integral*.

The *surgery manifold* is the space

$$M(V, J) = S^3 - \overset{\circ}{V} \underset{h}{+} V$$

obtained by removing the interior of the solid tori and sewing them back with any attaching homeomorphism

$$h: \partial V \to \partial(S^3 - \overset{\circ}{V}) = \partial V \quad \text{such that} \quad h(\text{meridian of } V_i) = J_i$$

"Meridian" means the curve $* \times S^1$ in $S^1 \times D^2$.

Then $M(V, J)$ is a well-defined closed 3-manifold, with a naturally preferred orientation, agreeing with the orientation for S^3, which we take as fixed throughout the discussion.

We now consider three ways of modifying surgery data (V, J) to obtain the data (V', J').

(I) *Homeomorphism.* Let $h: S^3 \to S^3$ be an orientation preserving homeomorphism and define $V' = h(V)$, $J' = h(J)$.

(II) *Twist.* Locate (if possible) an unknotted component, say V_1 of $V = V_1 \cup \cdots \cup V_n$. Then there is a thickened disk $D^2 \times I$ in $S^3 - \overset{\circ}{V}_1$ which is attached to ∂V_1 along $S^1 \times I$. The self-homeomorphism $(x, t) \to (xe^{2\pi i t}, t)$ of $D^2 \times I$ (in $\mathbf{C} \times I$), extended by the identity off of $D^2 \times I$, defines a "twist homeomorphism"

$$g: S^3 - \overset{\circ}{V}_1 \to S^3 - \overset{\circ}{V}_1;$$

let

$$V' = V_1 \cup gV_2 \cup \cdots \cup gV_n, \qquad J' = gJ_1 \cup gJ_2 \cup \cdots \cup gJ_n.$$

(III) *Trivial insertion (deletion).* Given $V = V_1 \cup \cdots \cup V_n$, let V_{n+1} by any solid torus disjoint from V and let $J_{n+1} \subset \partial V_{n+1}$ be a *meridian* of

V_{n+1}. Define

$$(V', J') = (V \cup V_{n+1}, J \cup J_{n+1}).$$

This is a trivial insertion. Its inverse is a trivial deletion, i.e. delete any V_i for which J_i is a meridian.

LEMMA 1. *If (V', J') results from (V, J) by a move of types* I, II *or* III, *then $M(V, J)$ and $M(V', J')$ are orientation-preserving homeomorphic.*

The proof is completely routine. In brief, here are the two theories:

RATIONAL CALCULUS.
Objects are surgery data (V, J).
Equivalence is the relation " $\leftrightarrow$ " generated by (I), (II), (III).

KIRBY CALCULUS.
Objects are integral surgery data.
Equivalence is the relation " $\underset{\partial}{\approx}$ " generated by the Kirby moves [4].

Statement of results. Let $\cong$ mean "orientation-preserving homeomorphic".

THEOREM 1. *Let (V, J) and (V', J') be surgery data in S^3: Then*

$$M(V, J) \cong M(V', J') \quad \text{if and only if} \quad (V, J) \leftrightarrow (V', J').$$

Lemma 1 proves the easy direction. To prove the converse we need three results.

KIRBY'S THEOREM [4]. *If (V, J) and (V', J') are integral surgery data in S^3, then*

$$M(V, J) \cong M(V', J') \quad \text{if and only if} \quad (V, J) \underset{\partial}{\approx} (V', J').$$

LEMMA 2. (*Fenn-Rourke* [2], *or see* [4].) *If (V, J) and (V', J') are integral, then*

$$(V, J) \underset{\partial}{\approx} (V', J') \quad \text{implies} \quad (V, J) \leftrightarrow (V', J').$$

LEMMA 3. *Given surgery data (V, J), there exist integral surgery data (V_I, J_I) such that $(V, J) \leftrightarrow (V_I, J_I)$.*

To complete the proof of Theorem 1, assume $M(V, J) \cong M(V', J')$. Then by Lemma 3, there are integral data with $(V, J) \leftrightarrow (V_I, J_I)$ and $(V', J') \leftrightarrow (V'_I, J'_I)$. By Lemma 1, $M(V_I, J_I) \cong M(V'_I, J'_I)$, and by Kirby and Lemma 2, we conclude that $(V_I, J_I) \leftrightarrow (V'_I, J'_I)$. It follows that $(V, J) \leftrightarrow (V', J')$ and Theorem 1 is proved.

COROLLARY. *Kirby's calculus equals the rational calculus, restricted to integral surgery data. That is, if (V, J) and (V', J') are integral then the following are equivalent*:

(a) $(V, J) \underset{\partial}{\approx} (V', J')$;

(b) $(V, J) \leftrightarrow (V', J')$;

(c) $M(V, J) \cong M(V', J')$.

To prove Lemma 3, it will be very useful to formulate surgery data in terms of links with rational labels, as in [6]. The surgery data (V, J) is described by the link $L = L_1 \cup \cdots \cup L_n$, where L_i is a central curve ($S^1 \times *$ in $S^1 \times D^2$) of V_i. The attachment curve $J_i \subset \partial V_i$ is completely determined up to isotopy by the fraction

$$r_i = p_i/q_i \qquad (= \infty \text{ if } q_i = 0)$$

where

$$[J_i] = q_i[L_i] \quad \text{in } H_1(V_i; \mathbf{Z}) \qquad \text{and} \qquad p_i = lk(J_i, L_i).$$

Although one must orient J_i and L_i for this to make sense, the ratio r_i is independent of such choice. The choice of a sign convention for computing linking number "lk" is tantamount to a choice of preferred orientation of S^3. We will use the convention under which the following has linking number $+1$.

In this setting the moves of the rational surgery calculus revise a labelled link as follows:

(I) Replace L by a link L' of same (oriented) link type in S^3. Keep the same coefficients.

(II) Replace L by L' as shown:
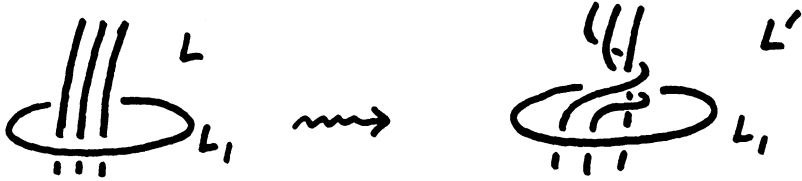
The new coefficients r_i' are defined:

$$\begin{cases} r_1' = \dfrac{1}{1/r_1 + \tau} & (\tau = \text{number of right-hand twists}), \\[2mm] r_i' = r_i + \tau[lk(L_i, L_1)]^2 & (i \neq 1). \end{cases}$$

(III) Introduce (or delete) a component of the link with coefficient ∞.

We point out that (V, J) is integral if and only if each r_i is an integer (finite). In this case r_i is exactly the "framing index" of Kirby [4].

EXAMPLE (continued). We will use the rational calculus to show that

the homology spheres resulting from surgery on the trefoil are Seifert fibre spaces. The following pictures all describe the same manifold.

Trivial insertion

Twist $(\tau = -1)$
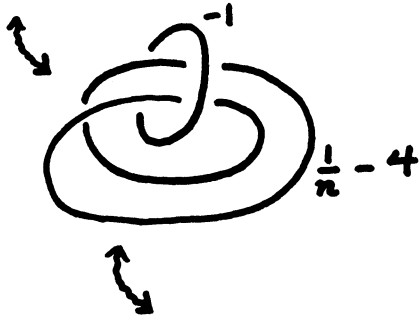
Homeomorphism and trivial insertion

Twist $(\tau = -2)$
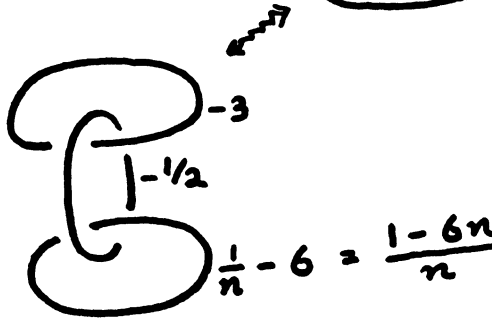
The $-\frac{1}{2}$ curve has complement fibred by circles, with the other curves as fibres. Using this one obtains a fibration of the surgery manifold with three singular fibres, of multiplicities 2, 3 and $|1 - 6n|$. (In case $n = 0$, the surgery manifold is S^3 and there are just two singular fibres.) According to Seifert [7], Satz 12, each homology sphere (except S^3) has a unique Seifert fibring. It follows that the manifolds constructed above are distinct for different n.

Proof of Lemma 3. Given (V, J), consider the associated labelled link $(L, r) = (L_1 \cup \cdots \cup L_n; r_1, \ldots, r_n)$. Using deletions, we may assume each r_i is finite. Write $r_i = p_i/q_i$ with $q_i > 0$. If each $q_i = 1$ we are done. Otherwise, suppose some component, L_1 say, has label $r_1 = p/q$, $q \geq 2$. Perform the following modifications on L:

Step 1. Unknot L_1. If one considers any "regular projection" of L_1, one may unknot L_1 by changing some crossings. This may be effected (following Hempel [3]) by insertions and twists thus:
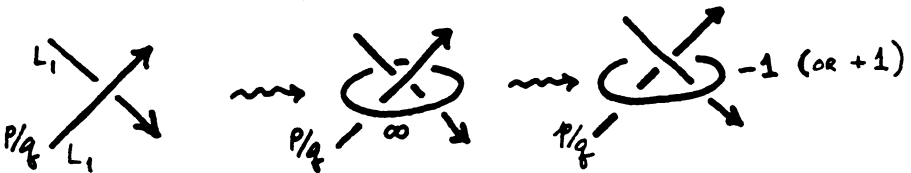
Step 2. By the following operation, we may revise p modulo q and assume $0 < p < q$.
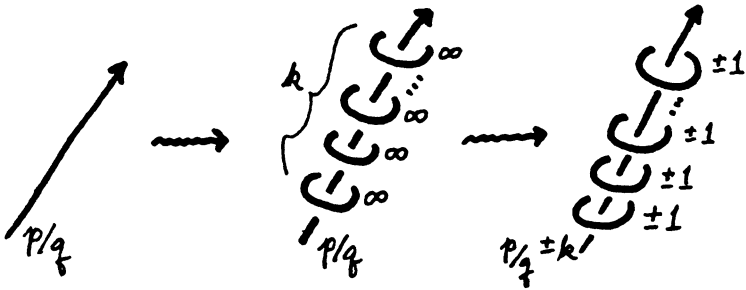
Step 3. Twist about L_1 with a left-hand twist. Then the coefficient $r_1 = p/q \, (0 < p < q)$ becomes

$$r_1' = \frac{1}{q/p - 1} = \frac{p}{q - p}.$$

Note that $0 < q - p < q$.

Thus for any component, we can reduce the denominator by insertions and twists. In the process, no new fractional labels have been introduced and the old fractional labels of components $L_2, \ldots, L_\mu$ are only changed by an integer. So inductively one may reduce all denominators to $+1$ and the lemma follows.

EXAMPLE (concluded). Here is a sequence which shows that the homology spheres constructed by $1/n$ surgery on the trefoil may be obtained by $+1$ surgery on some other knot (depending on n, of course).

(n full left-hand twists)
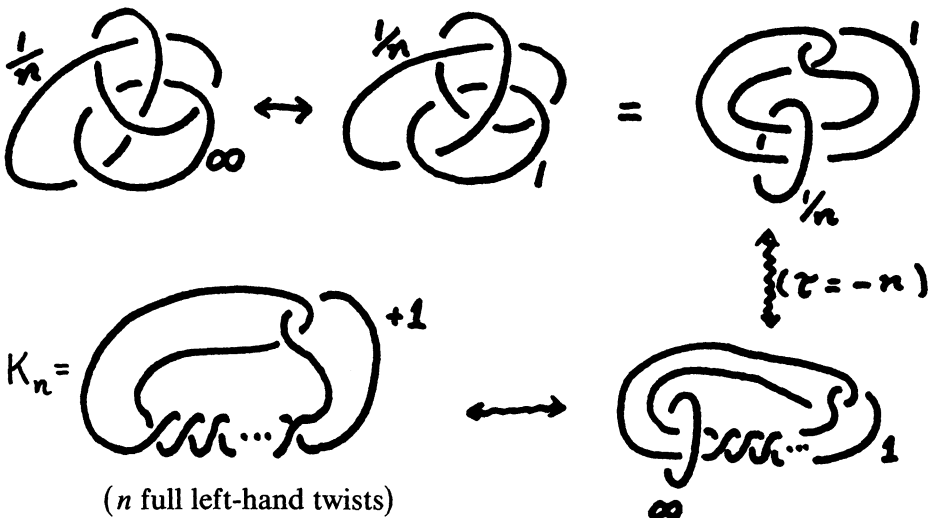
Note that K_1 is just the trefoil again! K_{-1} is the figure 8 knot.

An application.   Theorem 1 suggests possibilities for new tools for the study of 3-manifolds, coming from knot and link theory. Indeed any algebraic invariant of links (with coefficients) automatically becomes an invariant of oriented 3-manifolds, provided it is unchanged by the relation $\leftrightarrow$.

We will describe such an invariant. It extends a similar result of Fukuhara [9] for the Kirby calculus.

Let $M^3 = M(V, J)$ be the oriented 3-manifold with surgery data $V = V_1 \cup \cdots \cup V_n$, $J = J_1 \cup \cdots \cup J_n$. Consider the complementary fundamental group

$$G = \pi_1(S^3 - \mathring{V}),$$

and the normal subgroup

$$H = \{J_1, \ldots, J_n\}$$

normally generated by the J_i considered as loops in $S^3 - \mathring{V}$. Note that the class of J_i in G is well defined up to conjugacy. Next define

$$\overline{G} = G/[G, H], \qquad \overline{H} = H/[G, H],$$

where $[G, H]$ is the normal subgroup generated by commutators $ghg^{-1}h^{-1}$, $g \in G, h \in H$. Note that $[G, H] \lhd H$.

The group pair $(\overline{G}, \overline{H})$ may be called the *surgery group pair*, associated with the surgery data (V, J). We shall call group pairs (A, B) and $(A'B')$ *stably isomorphic* if for some integers m, n, there is a commutative diagram

$$A \times \mathbf{Z}^m \cong A' \times \mathbf{Z}^n$$
$$\text{(inclusion)} \times \text{id} \quad \uparrow \qquad \uparrow \quad \text{(inclusion)} \times \text{id}$$
$$B \times \mathbf{Z}^m \cong B' \times \mathbf{Z}^n.$$

THEOREM 2. *If (V, J) and (V', J') are surgery data in S^3 which produce orientation-preserving homeomorphic 3-manifolds, then their surgery group pairs are stably isomorphic.*

This means that the stable isomorphism class of any surgery description of an M^3 depends only on M. This is at least as sensitive an invariant as the fundamental group of M itself, noting the isomorphisms

$$\overline{G} \times \mathbf{Z}^m / \overline{H} \times \mathbf{Z}^m \cong \overline{G}/\overline{H} \cong G/H \cong \pi_1(M).$$

Proof of Theorem 2, and related results, will appear in a forthcoming paper.

Added in proof. See [10]. The stable isomorphism class of $(\overline{G}, \overline{H})$ depends only on $\pi_1(M)$.

Concluding remarks. Somewhat sharper versions of Theorem 1 may be obtained by closer inspection of the moves, I, II and III. We will say that the surgery data $(\hat{V}, \hat{J})$ is a *trivial extension* of (V, J) if $V \subset \hat{V}$ and $J \subset \hat{J}$ and for each component $\hat{V}_i$ of $\hat{V} - V$, the curve $\hat{J}_i = \hat{J} \cap \hat{V}_i$ is a meridian of $\hat{V}_i$. In other words $(\hat{V}, \hat{J})$ is the result of a finite number of trivial insertions on (V, J).

THEOREM 3. *If (V, J) and (V', J') are surgery data in S^3, the following are equivalent*:

(a) $M(V, J) \cong M(V', J')$ (*preserving orientation*).

(b) *There exist trivial extensions $(\hat{V}, \hat{J})$ and $(\hat{V'}, \hat{J'})$ of (V, J) and (V', J'), respectively, and an orientation preserving homeomorphism of pairs $(S^3\text{–int } \hat{V}, \hat{J}) \cong (S^3\text{–int } \hat{V'}, \hat{J'})$.*

(c) *There exist trivial extensions of (V, J) and (V', J') which are related by a finite sequence of twists* (II) *and a single homeomorphism* (I).

Proof. (c) $\Rightarrow$ (b) $\Rightarrow$ (a) is obvious. We argue (a) $\Rightarrow$ (c) as follows. If $M(V, J) \cong M(V', J')$, then $(V, J) \leftrightarrow (V', J')$ by Theorem 1. The sequence of moves I, II, III may be altered so that all moves of type I occur together, so may be lumped together into a single homeomorphism, placed anywhere we like in the sequence of moves from (V, J) to (V', J'). Also a twist on a particular component cannot occur before that component is inserted or after it is deleted. On the other hand, since different components are involved one may replace

$$(V^1, J^1) \overset{\text{twist}}{\to} (V^2, J^2) \overset{\text{trivial insertion}}{\to} (V^3, J^3)$$

by

$$(V^1, J^1) \overset{\text{trivial insertion}}{\to} (V^4, J^4) \overset{\text{twist}}{\to} (V^3, J^3).$$

So all the twists may be made to follow all the insertions. Likewise we can make them precede all the deletions, and we see that (a) $\Rightarrow$ (c).

Finally, we note that in the proof of Lemma 3 and in Fenn-Rourke's proof of Lemma 2, all the trivial insertions and deletions involve *unknotted* components. Call such an insertion (or deletion) *doubly trivial*.

THEOREM 4. *For surgery data, $(V, J) \leftrightarrow (V', J')$ if and only if they are related by a sequence of moves of types* I, II *and doubly trivial insertions and deletions.*

References

[1]   M. Dehn, *Über die Topologie des drei-dimensionalen Räumes*, Math. Ann., **75** (1910), 402–413.

[2]   R. A. Fenn and C. Rourke, *On Kirby's calculus of links*, Topology, **18** (1979), 1–15.

[3]   J. Hempel, *Construction of Orientable 3-Manifolds*, Topology of 3-manifolds (M. K. Fort, ed.), Prentice-Hall, N. J. 1962, 207–212.

[4]   R. Kirby, *A calculus for framed links in S^3*, Invent. Math., **45** (1978), 35–56.

[5]   W. B. R. Lickorish, *A representation of orientable combinatorial 3-manifolds*, Ann. Math., **76** (1962), 531–540.

[6]   D. Rolfsen, *Knots and Links*, Publish or Perish, Berkeley 1976.

[7]   H. Seifert, *Topologie dreidimensionaler gefaserter Räume*, Acta Math., **60** (1933), 147–238.

[8]   A. Wallace, *Modification and cobounding manifolds*, Canad. J. Math., **12** (1960), 503–528.

[9]   S. Fukuhara, *On framed link groups*, preprint.

[10] D. Rolfsen, *A stable group pair invariant of 3-dimensional manifolds*, Proc. Amer. Math. Soc. (to appear).

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