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**ENDOSCOPIC GROUPS AND BASE CHANGE C/R**

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# ENDOSCOPIC GROUPS AND BASE CHANGE $\mathbf{C}/\mathbf{R}$

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We consider a real reductive group  $G$  with complex points  $G(\mathbf{C})$ , Galois automorphism  $\sigma$ , and real points  $G(\mathbf{R}) = \{g \in G(\mathbf{C}) : \sigma(g) = g\}$ . In general, an irreducible admissible representation  $\Pi$  of  $G(\mathbf{C})$  equivalent to its Galois conjugate  $\Pi \circ \sigma$  need not be a lift from  $G(\mathbf{R})$ , even if  $G$  is quasi-split over  $\mathbf{R}$ . Following the results of  $L$ -indistinguishability we might expect this phenomenon to be related to the fact that  $\sigma$ -twisted conjugacy on  $G(\mathbf{C})$  need not be "stable", and therefore attempt to match the various "unstable" combinations of  $\sigma$ -twisted orbital integrals on  $G(\mathbf{C})$  with stable orbital integrals on certain groups  $H(\mathbf{R})$ . The principle of functoriality in the  $L$ -group would then suggest, with reservations in the nontempered case, a relation between the  $\sigma$ -twisted characters of representations of  $G(\mathbf{C})$  fixed up to equivalence by  $\sigma$  and the "dual lifts" to  $G(\mathbf{C})$  of stable characters on the groups  $H(\mathbf{R})$ .

In this paper we define the relevant groups  $H \dots$  they turn out to be the endoscopic groups from  $L$ -indistinguishability ... and prove a matching theorem for orbital integrals. As a preliminary to the proposed dual liftings of characters we also study the "factoring" of Galois-invariant Langlands parameters for  $G(\mathbf{C})$ .

**1. Introduction.** We begin with two simple examples. Let  $G(\mathbf{C}) = \mathbf{C}^\times$  and  $\sigma(z) = \bar{z}^{-1}$ ,  $z \in \mathbf{C}^\times$ , so that  $G(\mathbf{R}) = \{g \in G(\mathbf{C}) : \sigma(g) = g\}$  is the unit circle in  $\mathbf{C}^\times$ . A quasicharacter on  $\mathbf{C}^\times$  fixed by  $\sigma$ , i.e., trivial on the positive reals, need not be of the form  $z \rightarrow \chi(z\sigma(z)) = \chi(z/\bar{z})$ , with  $\chi$  a character on the unit circle. At the same time  $z \in \mathbf{C}^\times$  is stably  $\sigma$ -conjugate to  $-z$ , but not  $\sigma$ -conjugate (see [Sh6] for definitions). Let  $f \in C_c^\infty(\mathbf{C}^\times)$  and write  $f(r, \theta)$  for  $f(re^{i\theta})$ . Set  $H_1 = H_2 = G$ , so that  $H_1(\mathbf{R}) = S^1$ . Let

$$f_1(e^{i\theta}) = \frac{1}{2} \int_0^\infty (f(r, \theta/2) + f(r, \theta/2 + \pi)) dr/r$$

and

$$f_2(e^{i\theta}) = \frac{1}{2} e^{i\theta/2} \int_0^\infty (f(r, \theta/2) - f(r, \theta/2 + \pi)) dr/r$$

for  $-\pi < \theta < \pi$ . Then both  $f_1$  and  $f_2$  extend smoothly to  $S^1$ . If  $\chi$  is a character on  $S^1$  then  $f \rightarrow \int_{-\pi}^\pi \chi(e^{i\theta}) f_1(e^{i\theta}) d\theta$  is a distribution on  $\mathbf{C}^\times$  representing the usual lift of  $\chi$  to  $G(\mathbf{C})$ , i.e., representing the quasicharacter  $z \rightarrow \chi(z\sigma(z))$ . On the other hand,  $f \rightarrow \int_{-\pi}^\pi \chi(e^{i\theta}) f_2(e^{i\theta}) d\theta$  lifts  $\chi$  to the quasicharacter  $z = re^{i\theta} \rightarrow \chi(z\sigma(z))e^{i\theta}$ . We have therefore recovered the remaining Galois-invariant quasicharacters on  $\mathbf{C}^\times$ .

For a general group, however, there are difficulties more akin to those for  $L$ -indistinguishability. Consider  $G = \mathrm{SL}_2$ . Let

$$H(\mathbf{R}) = \left\{ r(\theta) = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \right\}.$$

Note that if  $\theta \not\equiv 0 \pmod{\pi}$  then  $r(\theta)$  and  $r(\theta + \pi)$  are stably  $\sigma$ -conjugate in  $G(\mathbf{C})$  but not  $\sigma$ -conjugate (see [Sh6, Lemma 2.5.2]). For  $f \in C_c^\infty(\mathrm{SL}_2(\mathbf{C}))$ , define

$$f_H(r(\theta)) = e^{i\theta/2}(e^{i\theta} - e^{-i\theta})(\Phi_f^\sigma(\theta/2) + \Phi_f^\sigma(\theta/2 + \pi)),$$

for  $-\pi < \theta < \pi$ , where

$$\Phi_f^\sigma(\theta) = \int_{G(\mathbf{C})/H(\mathbf{R})} f(\sigma(g)r(\theta)g^{-1}) \frac{dg}{d\theta},$$

$dg$  denoting a Haar measure on  $G(\mathbf{C}) = \mathrm{SL}_2(\mathbf{C})$ . It can be shown that  $f_H$  extends to a  $C^\infty$  function on  $H(\mathbf{R})$ . Then  $f \rightarrow \int_{H(\mathbf{R})} \chi f_H$  is a distribution on  $\mathrm{SL}_2(\mathbf{C})$  (see [Sh6, §5.4] for an explicit formula). L. Clozel has shown that this distribution is, up to a constant, the twisted character of a Galois-fixed equivalence class of representations of  $\mathrm{SL}_2(\mathbf{C})$ . It is easily verified that all such classes of (irreducible, admissible) representations of  $\mathrm{SL}_2(\mathbf{C})$  which are not lifts from  $\mathrm{SL}_2(\mathbf{R})$  are lifts in this way.

Returning to the general problem, we find it convenient to consider  $G(\mathbf{C})$  as the group of real points on a group  $\tilde{G}$ , and  $\sigma$  as the restriction to  $\tilde{G}(\mathbf{R})$  of an algebraic automorphism  $\alpha$  of  $\tilde{G}$  (cf. §2). Also, since  $(\tilde{G}, \alpha)$  is our starting point, rather than  $G$  itself, we may as well assume that  $G$  is quasi-split over  $\mathbf{R}$ .

In this paper we will be concerned with the matchings for  $\alpha$ -twisted orbital integrals on  $\tilde{G}(\mathbf{R})$ ; this includes the problem of determining what it is they should match. Theorem 7.1 is our main result, and §§2 to 6 are preparation for it. Also, as both a check on our definitions and a preliminary to the proposed dual liftings, we will consider the question of “factoring” Galois-invariant Langlands parameters for  $G(\mathbf{C})$  or, equivalently [L1]  $\alpha$ -invariant parameters for  $\tilde{G}(\mathbf{R})$ . Theorem 8.1 is the main result.

In [Sh6] we started a study of the matching problem for  $\alpha$ -twisted orbital integrals. We found that, despite various “technical” difficulties, the jump formulas for twisted orbital integrals on  $\tilde{G}(\mathbf{R})$  are closely related to those for ordinary orbital integrals on  $G(\mathbf{R})$ . Making convenient technical assumptions, we then put together a matching theorem involving the endoscopic groups from  $L$ -indistinguishability. In this paper we start afresh, making none of the technical assumptions of [Sh6]. We first define

the notion of *endoscopic group* for  $(\tilde{G}, \alpha)$ . This turns out to be the same as the notion of endoscopic group in  $L$ -indistinguishability [L3], [Sh4]. However, there is new information in the data for an endoscopic group  $H$  for  $(\tilde{G}, \alpha)$  and it is this information which allows us to formulate a matching theorem without the assumption (4.3.2) of [Sh6]. Moreover in relating the embeddings  ${}^L H \hookrightarrow {}^L \tilde{G}$  relevant to our present problem to the embeddings  ${}^L H \hookrightarrow {}^L G$  from  $L$ -indistinguishability we find a remarkable quasicharacter on  $\tilde{H}(\mathbf{R}) \simeq H(\mathbf{C})$  which allows us to dispense with the “cross-section for the norm” in [Sh6] (cf. Lemma 6.4).

As always, the twisted orbital integrals must be normalized. The normalization factors will be written in a form suitable for global applications [L3] and, more specifically, in a form to reflect the connection with  $L$ -indistinguishability for real groups. The proof of Theorem 7.1 itself relies heavily on the proof of the matching theorem for  $L$ -indistinguishability (see [Sh5] for an outline of the latter proof).

We will follow the notation of [Sh1]–[Sh7] as closely as possible, especially with respect to  $L$ -group data. However, we now write  $G(\mathbf{C})$  and  $G(\mathbf{R})$  in place of  $\mathbf{G}$  and  $G$ . The definitions in this paper may be presented in greater generality (cf. [Sh7]); in the general case there is no such intimate tie with  $L$ -indistinguishability.

**2. The groups  $G$ ,  $\tilde{G}$  and the automorphism  $\alpha$ .** Let  $G$  be a connected reductive linear algebraic group defined over  $\mathbf{R}$ . Assume that  $G$  is quasi-split over  $\mathbf{R}$ . In fixing the usual  $L$ -group data, we take  $G$  itself for  $G^*$ , a quasi-split inner form of  $G$ , and the identity map for  $\psi$ , an inner twist from  $G$  to  $G^*$ . Then  $B^*$  will be a Borel subgroup over  $\mathbf{R}$  in  $G$ , and  $T^*$  a maximal torus over  $\mathbf{R}$  in  $B^*$ . We form the dual  $({}^L G^0, {}^L B^0, {}^L T^0, \{X_r\})$  with  $r \in \Sigma({}^L B^0, {}^L T^0)$ , the set of simple roots of  ${}^L T^0$  in  ${}^L B^0$ . In fact it will be convenient to have fixed a root vector  $X_r$ , for any root  $r$  of  ${}^L T^0$  in  ${}^L G^0$ . We therefore fix a Chevalley basis and take for  $\{X_r, r \in \Sigma({}^L B^0, {}^L T^0)\}$  the vectors so provided. Then  ${}^L G = {}^L G^0 \rtimes W$ , with  $\sigma_G$  denoting the action of  $1 \times \sigma \in W$  on  ${}^L G^0$ . See [Sh 3, 4, or 5] for further explanation of the notation.

Let  $\tilde{G}$  be the group obtained from  $G$  by restriction of scalars from  $\mathbf{C}$  to  $\mathbf{R}$ . We realize  $\tilde{G}$  as  $G \times G$  with Galois automorphism  $\sigma_{\tilde{G}}: (x, y) \rightarrow (\sigma_G(y), \sigma_G(x))$ . Then  $\tilde{B}^* = B^* \times B^*$  will be the distinguished Borel subgroup defined over  $\mathbf{R}$  and  $\tilde{T}^* = T^* \times T^*$ . We realize the  $L$ -group  ${}^L \tilde{G}$  of  $\tilde{G}$  as follows. Set  ${}^L \tilde{G}^0 = {}^L G^0 \times {}^L G^0$ ,  ${}^L \tilde{B}^0 = {}^L B^0 \times {}^L B^0$ ,  ${}^L \tilde{T}^0 = {}^L T^0 \times {}^L T^0$ ,  $X_{(r, r')} = (X_r, X_{r'})$  for all roots  $r, r'$  of  ${}^L T^0$  in  ${}^L G^0$ , and define  $\sigma_{\tilde{G}}: {}^L \tilde{G}^0 \rightarrow {}^L \tilde{G}^0$  by  $\sigma_{\tilde{G}}(g, h) = (\sigma_G(h), \sigma_G(g))$ ,  $g, h \in {}^L G^0$ . Then  ${}^L \tilde{G} = {}^L \tilde{G}^0 \rtimes W$ , with  $\mathbf{C}^\times \times 1$  acting trivially and  $1 \times \sigma$  by  $\sigma_{\tilde{G}}$ .

Let  $\alpha: \tilde{G} \rightarrow \tilde{G}$  be the automorphism  $(x, y) \rightarrow (y, x)$ . We take the standard dual automorphism (cf. [Sh7]) of  $\alpha$ , and denote it by  $\alpha$  also. Thus:

$$\alpha((g, h) \times w) = (h, g) \times w, \quad g, h \in {}^L G^0, w \in W.$$

**3. Endoscopic groups for  $(\tilde{G}, \alpha)$ .** The following is a special case of the definitions in [Sh7]. Let  $s \in {}^L \tilde{G}^0$ . Then we set  $N(s) = s\alpha(s)$ ,  $\text{Cent}(N(s), {}^L \tilde{G}^0) = \{g \in {}^L \tilde{G}^0: g^{-1}N(s)g = N(s)\}$  and  $\text{Cent}_\alpha(s, {}^L \tilde{G}^0) = \{g \in {}^L \tilde{G}^0: g^{-1}s\alpha(g) = s\}$ . Call  $s$   $\alpha$ -semisimple if  $\text{Cent}_\alpha(s, {}^L \tilde{G}^0)$  is reductive. In §4 we will observe that  $s$  is  $\alpha$ -semisimple if and only if  $N(s)$  is semisimple (cf. Lemma 4.2). Let  $\tilde{Z}^W$  be the group of  $W$ -invariants in the center of  ${}^L \tilde{G}^0$ . Thus  $\tilde{Z}^W = {}^L \tilde{G}^0 \cap \text{Center}({}^L \tilde{G}) = \{(g, \sigma_G(g)) \times 1 \times 1: g \in \text{Center}({}^L G^0)\}$ . Also

$$\text{Cent}_\alpha(sz, {}^L \tilde{G}^0) = \text{Cent}_\alpha(s, {}^L \tilde{G}^0), \quad s \in {}^L \tilde{G}^0, z \in \tilde{Z}^W.$$

We will now use  $s$  to denote a coset of  $\tilde{Z}^W$  in  ${}^L \tilde{G}^0$  and  $\text{Cent}_\alpha(s, {}^L \tilde{G}^0)$  to denote  $\text{Cent}_\alpha(a, {}^L \tilde{G}^0)$  for  $a$  in the coset  $s$ . Following [Sh7], we consider tuples

$$(s, {}^L H_s^0, {}^L B_s^0, {}^L T_s^0, \{Y\}, \rho_s)$$

where

- (i)  $s \in {}^L \tilde{G}^0$  is a coset of  $\tilde{Z}^W$  consisting of  $\alpha$ -semisimple elements,
- (ii)  ${}^L H_s^0 = (\text{Cent}_\alpha(s, {}^L \tilde{G}^0))^0$ ,
- (iii)  ${}^L B_s^0$  is a Borel subgroup of  ${}^L H_s^0$ ,
- (iv)  ${}^L T_s^0 \subset {}^L B_s^0$  is a maximal torus in  ${}^L H_s^0$ ,
- (v)  $\{Y\}$  is a set of root vectors for the simple roots of  ${}^L T_s^0$  in  ${}^L B_s^0$ ,
- (vi)  $\rho_s: W \rightarrow \text{Aut}({}^L H_s^0, {}^L B_s^0, {}^L T_s^0, \{Y\})$  is a homomorphism which factors through  $\text{Gal}(\mathbf{C}/\mathbf{R})$  and is “realized in  $\text{Cent}_\alpha(s, {}^L \tilde{G})$ ”, i.e.  $\rho_s(w) = \text{ad } n(w)|_{{}^L H_s^0}$ ,  $w \in W$ , for some  $n(w) \in {}^L \tilde{G}^0 \times w$  such that  $n(w)^{-1}\alpha(n(w)) = a$  for each  $a$  in the coset  $s$ .

Let  ${}^L H_s = {}^L H_s^0 \rtimes W$ , the action of  $W$  on  ${}^L H_s^0$  being that defined by  $\rho_s$ . Often we will write  $\sigma_s$  for the automorphism  $\rho_s(1 \times \sigma)$ , and abbreviate  $(s, {}^L H_s^0, \dots, \rho_s)$  by  $(s, {}^L H_s)$ .

Two tuples

$$(s, {}^L H_s^0, {}^L B_s^0, {}^L T_s^0, \{Y\}, \rho_s) \quad \text{and} \quad (s', {}^L H_{s'}^0, {}^L B_{s'}^0, {}^L T_{s'}^0, \{Y'\}, \rho_{s'})$$

are *equivalent* if there exists  $g \in {}^L \tilde{G}^0$  such that  ${}^L H_{s'}^0 = g^{-1}{}^L H_s^0 g$ ,  ${}^L B_{s'}^0 = g^{-1}{}^L B_s^0 g$ ,  ${}^L T_{s'}^0 = g^{-1}{}^L T_s^0 g$ ,  $\{Y'\} = \{\text{Ad } g^{-1}(Y)\}$  and if  $n(w) \in \text{Cent}_\alpha(s, {}^L \tilde{G})$  realizes  $\rho_s(w)$  then  $g^{-1}n(w)g$  lies in  $\text{Cent}_\alpha(s', {}^L \tilde{G})$  and realizes  $\rho_{s'}(w)$ ,  $w \in W$ . The set of all equivalence classes will be denoted

$\mathfrak{S}(\tilde{G}, \alpha)$ . Using the results of the next section and Lemma 2.3.3 of [Sh4], we may show that  $\mathfrak{S}(\tilde{G}, \alpha)$  is a finite set. Since this fact will not be needed we omit the proof.

Finally, we call a quasi-split group  $H$  over  $\mathbf{R}$  an *endoscopic group* for  $(\tilde{G}, \alpha)$  if some  ${}^L H_s$  as above is an  $L$ -group for  $H$ .

**4. The relation between endoscopic groups for  $(\tilde{G}, \alpha)$  and endoscopic groups for  $G$ .** By the endoscopic groups for  $G$  we mean the groups “ $H$ ” of [Sh4], i.e. essentially the groups of [L1]. The set  $\mathfrak{S}(G)$ , or  $\mathfrak{S}(G, 1)$  in the more general notation of [Sh7], and the tuples used in its definition will be taken from [Sh4] (... there is a small difference in the definitions of [L3]).

We embed  ${}^L G$  “diagonally” in  ${}^L \tilde{G}$ , i.e. by the map  $g \times w \rightarrow (g, g) \times w$ ,  $g \in {}^L G^0$ ,  $w \in W$ , and will frequently *identify  ${}^L G$  with its image in  ${}^L \tilde{G}$* . As in [Sh4],  $Z^W$  will denote the set of  $W$ -invariants in the center of  ${}^L G^0$ .

By an  $\alpha$ -conjugacy class in  ${}^L \tilde{G}^0$ , we will mean a set  $\{g^{-1}a\alpha(g); g \in {}^L \tilde{G}^0\}$ , where  $a \in {}^L \tilde{G}^0$ .

LEMMA 4.1.

- (i) *Each  $\alpha$ -conjugacy class in  ${}^L \tilde{G}^0$  contains an element of the form  $(x, 1)$ ,  $x \in {}^L G^0$ .*
- (ii) *For  $x \in {}^L G^0$ ,  $\text{Cent}_\alpha((x, 1), {}^L \tilde{G}^0) = \text{Cent}(x, {}^L G^0)$ .*

Here, of course,  $\text{Cent}(x, {}^L G^0)$  has been identified with its image in  ${}^L \tilde{G}$  under the diagonal map.

*Proof.* Let  $a = (g_1, g_2) \in {}^L \tilde{G}^0$ ,  $g = (1, g_2)$ . Then  $g^{-1}a\alpha(g) = (1, g_2^{-1})(g_1, g_2)(g_2, 1) = (g_1 g_2, 1)$ , so that (i) is proved. (ii) is also a simple calculation.

LEMMA 4.2.  *$a \in {}^L \tilde{G}^0$  is  $\alpha$ -semisimple if and only if  $N(a) = a\alpha(a)$  is semisimple.*

*Proof.* Let  $a \in {}^L \tilde{G}^0$ . Choose  $g \in {}^L \tilde{G}^0$  such that  $g^{-1}a\alpha(g) = (x, 1)$ , for suitable  $x \in {}^L G^0$ . Then

$$\text{Cent}_\alpha(a, {}^L \tilde{G}^0) = g \text{Cent}_\alpha((x, 1), {}^L \tilde{G}^0) g^{-1} = g \text{Cent}(x, {}^L G^0) g^{-1}.$$

On the other hand,  $N(a) = g(x, x)g^{-1}$ , so that

$$\text{Cent}(N(a), {}^L \tilde{G}^0) = g(\text{Cent}(x, {}^L G^0) \times \text{Cent}(x, {}^L G^0))g^{-1}.$$

The lemma then follows from standard facts.

LEMMA 4.3. *Let  $s$  be a coset of  $\tilde{Z}^W$  in  ${}^L\tilde{G}^0$  consisting of  $\alpha$ -semisimple elements. Then there exists  $g \in {}^L\tilde{G}^0$  such that  $s' = g^{-1}s\alpha(g)$  has the property that  $\{\alpha(a): a \in s'\}$  is contained in  ${}^LG^0$ . Then  $\{\alpha(a): a \in s'\}$  is contained in a unique coset of  $Z^W$  in  ${}^LG^0$ . This coset, to be denoted  $N(s')$ , consists of semisimple elements.*

*Proof.* Let  $a \in s$ . Choose  $g \in {}^L\tilde{G}^0$  such that  $g^{-1}a\alpha(g) = (x, 1)$ , where  $x \in {}^LG^0$  is semisimple. Let  $s' = (x, 1)\tilde{Z}^W$ . Then if  $b \in s'$ ,  $b\alpha(b) = (x, x)(z\sigma_G(z), z\sigma_G(z))$ , for some  $z \in \text{Cent}({}^LG^0)$ . Thus, with our identifications,  $b\alpha(b) \in xZ^W$ , a coset of  $Z^W$  in  ${}^LG^0$  consisting of semisimple elements. The rest is clear.

LEMMA 4.4. *Each element of  $\mathfrak{S}(\tilde{G}, \alpha)$  has a representative  $(s, {}^LH_s)$  such that  $(N(s), {}^LH_s)$  is a representative for an element of  $\mathfrak{S}(G)$  i.e. such that  $\{\alpha(a): a \in s\}$  is contained in  ${}^LG^0$  (...so that  $N(s)$  is defined),  ${}^LH_s^0$  coincides with  $(\text{Cent}(N(s), {}^LG^0))^0$ , and  $\rho_s$  is “realized in  $\text{Cent}(N(s), {}^LG)$ .”*

*Proof.* We may take  $s = (x, 1)\tilde{Z}^W$ , some  $x \in {}^LT^0$ . Then  $N(s) = xZ^W$  and  $\text{Cent}_\alpha(s, {}^L\tilde{G}^0) = \text{Cent}(N(s), {}^LG^0)$ . We may also assume that  ${}^LT_s^0 = {}^LT^0$ ,  ${}^LB_s^0 = {}^LB^0 \cap {}^LH_s^0$  (... ${}^LT^0$  and  ${}^LB^0$  being identified with their images in  ${}^L\tilde{G}^0$ ) and that  $\{Y\} = \{X_r: r \in \Sigma({}^LB^0 \cap {}^LH_s^0, {}^LT^0)\}$ . Then  $\rho_s$  is a homomorphism of  $W$  into  $\text{Aut}({}^LH_s^0, {}^LB^0 \cap {}^LH_s^0, {}^LT^0, \{Y\})$ . Suppose that  $\rho_s(w) = \text{ad } n(w)|_{{}^LH_s^0}$ , where  $n(w) \in {}^L\tilde{G}^0 \times w$  satisfies  $n(w)^{-1}(x, 1)\alpha(n(w)) = (x, 1)$  (cf. (vi) in §3). Then  $n(w)^{-1}(x, x)n(w) = (x, x)$ . Also, if  $n(w) = (n_1(w), n_2(w)) \times w$  then calculation shows that for  $w \in \mathbf{C}^x \times 1$  we have  $n_1(w) = n_2(w)$  lies in the center of  ${}^LH_s^0$  and for  $w = 1 \times \sigma$  we have  $n_1(w) = xn_2(w)$ . Thus for all  $w \in W$ ,  $\rho_s(w) = \text{ad } m(w)|_{{}^LH_s^0}$  where  $m(w) = (n_1(w), n_1(w)) \times w \in {}^LG$ . Also,  $m(w)$  centralizes  $(x, x)$ . Thus  $\rho_s$  is “realized in  $\text{Cent}(N(s), {}^LG)$ ” and the lemma is proved.

LEMMA 4.5. *The correspondence in Lemma 4.4 induces a map*

$$\mathfrak{N}: \mathfrak{S}(\tilde{G}, \alpha) \rightarrow \mathfrak{S}(G).$$

*Proof.* We have to show that if  $(s, {}^LH_s)$  and  $(s', {}^LH_{s'})$  are as in Lemma 4.4, representing the same element of  $\mathfrak{S}(\tilde{G}, \alpha)$ , then the 5-tuples defining  ${}^LH_s$  and  ${}^LH_{s'}$  are conjugate under  ${}^LG^0$ . They are conjugate under  ${}^L\tilde{G}^0$ , by definition. It is easily checked that this conjugation may be replaced by one from  ${}^LG^0$ .

The map  $\mathfrak{U}$  need not be injective, as the example that  $G$  is a compact torus shows. However  $\mathfrak{U}$  does have finite fibers (which implies that  $\mathfrak{S}(\tilde{G}, \alpha)$  is finite, as asserted in the last section). Reversing the construction in the proof of Lemma 4.4 shows that  $\mathfrak{U}$  is surjective.

**5. Allowed embeddings of  ${}^L H_s$  in  ${}^L \tilde{G}$ .** Fix an element of  $\mathfrak{S}(\tilde{G}, \alpha)$ , with representative  $(s, {}^L H_s)$  chosen as in the proof of Lemma 4.4. In particular,  $s = (x, 1)\tilde{Z}^W$ ,  $x \in {}^L T^0$ , and  ${}^L H_s^0 = (\text{Cent}_\alpha(s, {}^L \tilde{G}^0))^0 = (\text{Cent}(N(s), {}^L G^0))^0$ . We may further assume that  ${}^L H_s^0$  is in standard position (cf. [Sh3, §2.2, Ex. 4.3.1]).

Suppose that  $\xi: {}^L H_s \hookrightarrow {}^L G$  is an admissible embedding, as in  $L$ -indistinguishability [L1], [Sh3]. Here we regard  ${}^L H_s^0$  as a subgroup of  ${}^L G^0$  yet to be embedded diagonally in  ${}^L \tilde{G}^0$ , and assume that  $\xi|_{{}^L H_s^0}$  is the inclusion map. The “diagonal” embedding of  ${}^L G$  in  ${}^L \tilde{G}$  then yields an embedding of  ${}^L H_s$  in  ${}^L \tilde{G}$ , again denoted  $\xi$ . Explicitly,  $\xi$  is of the form:

$$\begin{aligned}\xi(h \times 1 \times 1) &= (h, h) \times 1 \times 1, & h \in {}^L H_s^0, \\ \xi(1 \times z \times 1) &= (\xi_0(z), \xi_0(z)) \times z \times 1, & z \in \mathbf{C}^x,\end{aligned}$$

where  $\xi_0: \mathbf{C}^x \rightarrow \text{Cent}({}^L H_s^0)$  is a homomorphism satisfying  $\xi_0(\bar{z}) = \sigma_s(\xi_0(z))$ ,  $z \in \mathbf{C}^x$ , and

$$\xi(1 \times 1 \times \sigma) = (n_0, n_0) \times 1 \times \sigma,$$

where  $n_0 \in {}^L G^0$  normalizes  ${}^L T^0$ ,  $n_0 \sigma_G(n_0) = \xi_0(-1)$  and  $n_0 \times 1 \times \sigma \in {}^L G$  acts on  ${}^L H_s^0$  as  $\sigma_s = \rho_s(1 \times \sigma)$ . It follows immediately that  $\xi({}^L H_s) \subset \text{Cent}(N(s), {}^L G)$ . However, our present problem dictates (cf. §8) that we consider embeddings for which the image of  ${}^L H_s$  is contained in  $\text{Cent}_\alpha(s, {}^L \tilde{G})$ . That this is a quite different condition is indicated even by the example that  $G$  is a compact torus.

**DEFINITION 5.1.** Let  $(s, {}^L H_s)$  be a representative for an element of  $\mathfrak{S}(\tilde{G}, \alpha)$ . Then  $\tilde{\xi}: {}^L H_s \hookrightarrow {}^L \tilde{G}$  is an allowed embedding if:

- (i)  $\tilde{\xi}$  is an admissible homomorphism, i.e.  $\tilde{\xi}$  is a homomorphism such that  $\tilde{\xi}({}^L H_s^0 \times w) \subset {}^L \tilde{G}^0 \times w$ ,  $w \in W$ ,
- (ii) on  ${}^L H_s^0$ ,  $\tilde{\xi}$  is the inclusion mapping, and
- (iii)  $\tilde{\xi}({}^L H_s) \subset \text{Cent}_\alpha(s, {}^L \tilde{G})$ .

We return to our choice  $s = (x, 1)\tilde{Z}^W$ , etc. Once again it is more convenient to regard  ${}^L H_s^0$  as a subgroup of  ${}^L G^0$  yet to be embedded



diagonally in  ${}^L\tilde{G}^0$ . Then an allowed embedding  $\tilde{\xi}: {}^LH_s \hookrightarrow {}^L\tilde{G}$  is of the form:

$$\begin{aligned}\tilde{\xi}(h \times 1 \times 1) &= (h, h) \times 1 \times 1, & h \in {}^LH_s^0, \\ \tilde{\xi}(1 \times z \times 1) &= (\tilde{\xi}_0(z), \tilde{\xi}_0(z)) \times z \times 1, & z \in \mathbf{C}^\times,\end{aligned}$$

where  $\tilde{\xi}_0$  satisfies the same conditions as  $\xi_0$  earlier, and

$$\tilde{\xi}(1 \times 1 \times \sigma) = (xm_0, m_0) \times 1 \times \sigma$$

where  $m_0 \in {}^L\tilde{G}^0$  normalizes  ${}^LT^0$ ,  $xm_0\sigma_G(m_0) = \tilde{\xi}_0(-1)$ , and  $m_0 \times 1 \times \sigma \in {}^LG$  acts on  ${}^LH_s^0$  as  $\sigma_s$  (... then also  $xm_0 \times 1 \times \sigma$  acts on  ${}^LH_s^0$  as  $\sigma_s$ , as we have already used in the proof of Lemma 4.2).

Let  ${}^L\tilde{H}_s^0 = {}^LH_s^0 \times {}^LH_s^0$ . We of course regard  ${}^L\tilde{H}_s^0$  as a subgroup of  ${}^L\tilde{G}^0$ . Define an action of  $W$  on  ${}^L\tilde{H}_s^0$  by requiring  $\mathbf{C}^\times \times 1$  to act trivially and  $1 \times \sigma$  to act by the automorphism  $(h_1, h_2) \rightarrow (\sigma_s(h_2), \sigma_s(h_1))$ . If  ${}^LH_s$  is the  $L$ -group of  $H$  then  ${}^L\tilde{H}_s$  is the  $L$ -group of  $\tilde{H} = \text{Res}_{\mathbf{R}}^{\mathbf{C}} H$ .

**LEMMA 5.2.** *Let  $\tilde{\xi}$  be an allowed embedding of  ${}^LH_s$  in  ${}^L\tilde{G}$  and  $\xi$  be an admissible embedding of  ${}^LH_s$  in  ${}^LG \subset {}^L\tilde{G}$ . Then*

$$\tilde{\xi}(h \times w) = a(w)\xi(h \times w), \quad h \in {}^LH_s^0, w \in W,$$

where  $a(w)$  is a 1-cocycle of  $W$  in  $\text{Cent}({}^L\tilde{H}_s^0)$ .

*Proof.* This follows easily from our explicit description of  $\tilde{\xi}$  and  $\xi$ . The details are omitted.

Suppose that  $\tilde{\xi}, \tilde{\xi}'$  are both allowed embeddings of  ${}^LH_s$  in  ${}^L\tilde{G}$ . Then  $\tilde{\xi}'(w) = b(w)\tilde{\xi}(w)$ ,  $w \in W$ , where  $w \rightarrow b(w)$  is a 1-cocycle of  $W$  in the center of  ${}^LH_s^0$  embedded diagonally in  ${}^L\tilde{G}^0$ . We conclude then that the image of  ${}^LH_s$  under an allowed embedding is independent of the choice of embedding; we write thus simply “Image  ${}^LH_s$ .” Suppose next that  $(s, {}^LH_s)$  and  $(s', {}^LH_{s'})$  are equivalent in the sense of §3. Fix  $g \in {}^L\tilde{G}^0$  as in the definition. Suppose that  $\tilde{\xi}$  is an allowed embedding of  ${}^LH_s$  in  ${}^L\tilde{G}$ . Then  $ad\,g$  and  $\tilde{\xi}$  determine an allowed embedding of  ${}^LH_{s'}$  in  ${}^L\tilde{G}$ . We conclude then that there is an allowed embedding of  ${}^LH_s$  in  ${}^L\tilde{G}$  if and only if there is such an embedding of  ${}^LH_{s'}$ . Moreover, when embeddings exist we have  $g^{-1}(\text{Image } {}^LH_s)g = \text{Image } {}^LH_{s'}$ .

We defer a study of the existence of allowed embeddings. Recall, however, that if the center of  ${}^LG^0$  is connected then  ${}^LH_s$  embeds admissibly in  ${}^LG$  [L1]. The proof of this result can be used to show also that there is an allowed embedding of  ${}^LH_s$  in  ${}^L\tilde{G}$ .

**6. Ingredients for the matching theorem.** Fix an element of  $\mathfrak{S}(\tilde{G}, \alpha)$  with representative  $(s, {}^LH_s)$  satisfying  $s = (x, 1)\tilde{Z}^W$ , etc., as in the last section. We assume that  $\tilde{\xi}: {}^LH_s \hookrightarrow {}^L\tilde{G}$  is an allowed embedding. The main

purpose of this section is to attach to  $\tilde{\xi}$  normalizing factors to appear in the matching theorem of the next section. We will assume also that there is an admissible embedding of  ${}^L H_s$  in  ${}^L G$ , say  $\xi$ . The choice of  $\xi$  will not affect the normalization factors (cf. Lemma 6.2), but we write individual terms in the factors in a way that involves  $\xi$ , in order to make clear the relation with the factors from  $L$ -indistinguishability.

Let  $H$  be an endoscopic group for  $(\tilde{G}, \alpha)$  with  $L$ -group  ${}^L H_s$ . We fix a Borel subgroup  $B_H$  over  $\mathbf{R}$  containing the maximal torus  $T_H$  over  $\mathbf{R}$ , and assume that  $X^*(T_H) = X_*({}^L T^0) = X^*(T^*)$  and that  $\Sigma(B_H, T_H)$  is the dual of  $\Sigma({}^L B_s^0, {}^L T^0)$ . The group  $\tilde{H} = \text{Res}_{\mathbf{R}}^{\mathbf{C}} H$  will also play a role. We set  $\tilde{B}_H = B_H \times B_H$  and  $\tilde{T}_H = T_H \times T_H$ ;  ${}^L \tilde{H}_s$ , which appeared in the last section, is an  $L$ -group for  $\tilde{H}$ .

Since  $H$  is also an endoscopic group for  $G$  we may invoke many of the definitions from  $L$ -indistinguishability (cf. [L1], [Sh4]). Let  $T$  be a maximal torus over  $\mathbf{R}$  in  $G$ . A pseudodiagonalization (p.d.)  $\eta$  of  $T$  is a map from  $T$  to  $T^*$  of the form  $T \xrightarrow{\text{ad } x} T_0 \xrightarrow{\text{ad } m} T^*$ , where  $x \in \mathfrak{A}(T)$  [L1],  $T_0 = xTx^{-1}$  is standard (i.e. the maximal  $R$ -split torus in  $T_0$  lies in  $T^*$ ) and  $m$  belongs to the Levi group attached to  $T_0$ . Then  $\sigma_{(T, \eta)}$  denotes the transfer, by  $\eta$ , of the Galois action on  $T$  to  $T^*$ , and to  $X^*(T^*) = X_*({}^L T^0)$ ,  $X_*(T^*) = X^*({}^L T^0)$  and  ${}^L T^0 = X_*({}^L T^0) \otimes \mathbf{C}^x$ .

The set  $\mathfrak{T}_H(G) = \{(T, \eta): \sigma_{(T, \eta)} \in \Omega({}^L H_s^0, {}^L T^0) \sigma_s\}$ , where  $\Omega({}^L H_s^0, {}^L T^0)$  denotes the Weyl group of  $({}^L H_s^0, {}^L T^0)$ , is the starting point for the definitions of [Sh4, §2.4]. We will use it again. First, because  $G$  is quasi-split over  $\mathbf{R}$ , for each maximal torus  $T'$  over  $\mathbf{R}$  in  $H$  there exists  $h \in H(\mathbf{C})$  and  $(T, \eta) \in \mathfrak{T}_H(G)$  such that  $hT'h^{-1} = T_H$  and

$$X^*(T') \xrightarrow{\text{ad } h} X^*(T_H) = X^*(T^*) \xrightarrow{\eta^{-1}} X^*(T)$$

lifts to an isomorphism  $i(h, \eta): T' \rightarrow T$  over  $\mathbf{R}$ . We say that  $\gamma' \in H(\mathbf{R})$  *originates from*  $\gamma \in G(\mathbf{R})$  *via*  $(T, \eta)$  if  $\gamma'$  is the preimage of  $\gamma$  under some such map  $i(h, \eta)$ .

Recall that  $s = (x, 1)\tilde{Z}^W$ . Any element of this coset is of the form  $a = (xz, \sigma_G(z))$ , where  $z$  is in the center of  ${}^L G^0$ . But  $a\alpha(a) = (xz\sigma_G(z), xz\sigma_G(z))$ , an element of  ${}^L T^0 = \text{Hom}(X^*({}^L T^0), \mathbf{C}^x)$ . Also  $\sigma_s(x) = x$ . Thus  $\{a\alpha(a): a \in s\}$  defines a family of quasicharacters on  $X^*({}^L T^0)$ , each invariant under  $\sigma_{(T, \eta)}$ , for any  $(T, \eta) \in \mathfrak{T}_H(G)$ . Fix  $(T, \eta) \in \mathfrak{T}_H(G)$ . Then, on transfer to  $T$  via  $\eta$ , we get a family of quasicharacters on  $X_*(T)$ , each invariant under  $\sigma_T$ . On  $X_*(T_{\text{sc}})$ , the span of the coroots of  $T$  in  $G$ , these quasicharacters all coincide and so we have defined a single quasicharacter of the type used in  $L$ -indistinguishability (cf. [L1], also [Sh4, §2.4]). Moreover on  $\{\lambda^\vee \in X_*(T): \sigma_T \lambda^\vee = -\lambda^\vee\}$ , the quasicharacters

coincide again. We therefore obtain a single character on

$$\{\lambda^\vee \in X_*(T): \sigma_T \lambda^\vee = -\lambda^\vee\} / \{\mu^\vee - \sigma_T \mu^\vee : \mu^\vee \in X_*(T)\}$$

and thence by Tate-Nakayama duality, a character on  $H^1(T) = H^1(\text{Gal}(\mathbf{C}/\mathbf{R}), T(\mathbf{C}))$ . Unless otherwise indicated,  $\kappa$  will denote both the quasicharacter on  $X_*(T_{\text{sc}})$  and the character on  $H^1(T)$  attached to  $s$  and the pair  $(T, \eta) \in \mathfrak{T}_H(G)$ .

With  $G$  embedded diagonally in  $\tilde{G}$ , we have  $\tilde{T} = \text{Res}_{\mathbf{R}}^{\mathbf{C}} T$  naturally embedded in  $\tilde{G}$  as  $\text{Cent}(T, \tilde{G}) = T \times T$ , for any maximal torus  $T$  over  $\mathbf{R}$  in  $G$ . The norm from  $\tilde{T}$  to  $T$  is obtained from the map  $\tilde{T}(\mathbf{R}) \rightarrow T(\mathbf{R})$  defined by  $\delta = (t, \sigma_G(t)) \rightarrow \delta\alpha(\delta) = (t\sigma_G(t), t\sigma_G(t))$ . As in [Sh6] we regard the norm from  $\tilde{G}$  to  $G$  (...or from  $\tilde{T}$  to  $T$ ) as an (injective) map from the set of stable regular  $\alpha$ -semisimple twisted conjugacy classes in  $\tilde{G}(\mathbf{R})$  (...or in  $\tilde{T}(\mathbf{R})$ ) to the set of stable regular semisimple conjugacy classes in  $G(\mathbf{R})$  (...or to  $T(\mathbf{R})$ ). by Lemma 2.4.3(ii) of [Sh6] this norm from  $\tilde{G}$  to  $G$  can be recovered from the norms from  $\tilde{T}$  to  $T$ , as  $T$  ranges over the maximal tori over  $\mathbf{R}$  in  $G$ .

Note that if  $\eta: T \rightarrow T^*$  is a p.d., then so is  $\eta \times \eta: \tilde{T} \rightarrow \tilde{T}^*$ . Thus we can use  $\eta$  to transfer data from  $\tilde{T}$  to  $\tilde{T}^*$  or from  $\tilde{T}^*$  to  $\tilde{T}$ .

We come then to the normalizing factors. The admissible embedding  $\xi: {}^L H_s \hookrightarrow {}^L G$  has been fixed, and  ${}^L H_s$  chosen to satisfy the conditions of [Sh3, Sh4]. We may therefore write  $\xi = \xi(\mu^*, \lambda^*)$ , for suitable  $\mu^*, \lambda^* \in X_*({}^L T^0) \otimes \mathbf{C}$ , and define the attached correction (quasi) characters  $\Lambda_{(T, \eta)}$  on  $T(\mathbf{R})$ , for  $(T, \eta) \in \mathfrak{T}_H(G)$ . Although the notation does not reflect it,  $\Lambda_{(T, \eta)}$  depends on the choice of  $\xi$ .

Since  $\tilde{\xi}: {}^L H_s \hookrightarrow {}^L \tilde{G}$  has also been fixed, we have the 1-cocycle  $a(w)$  of  $W$  in  $\text{Center}({}^L \tilde{H}_s^0)$  from Lemma 5.2. A procedure in [L2] attaches to  $a(w)$  a quasicharacter on  $\tilde{H}(\mathbf{R})$ . This quasicharacter determines a pair  $(\tilde{\mu}_0, \tilde{\lambda}_0)$  of elements from  $X^*(\tilde{T}_H) \otimes \mathbf{C} = X_*({}^L \tilde{T}^0) \otimes \mathbf{C}$ . We may also recover  $(\tilde{\mu}_0, \tilde{\lambda}_0)$  directly from the 1-cocycle  $a(w)$ . Thus define  $\tilde{\mu}_0, \tilde{\lambda}_0$  by

$$\lambda^\vee(a(z \times 1)) = z^{\langle \tilde{\mu}_0, \lambda^\vee \rangle} \bar{z}^{\langle \sigma_s \tilde{\mu}_0, \lambda^\vee \rangle}, \quad z \in \mathbf{C}^\times,$$

$$\lambda^\vee(a(1 \times \sigma)) = e^{2\pi i \langle \tilde{\lambda}_0, \lambda^\vee \rangle}$$

for  $\lambda^\vee \in X^*({}^L \tilde{T}^0)$ . Then  $\tilde{\mu}_0$  is uniquely determined and  $\tilde{\lambda}_0$  is uniquely determined modulo

$$X_*({}^L \tilde{T}^0) + \{\tilde{\lambda} - \tilde{\sigma}_s \tilde{\lambda} : \tilde{\lambda} \in X_*({}^L \tilde{T}^0) \otimes \mathbf{C}\}.$$

Also

$$\tilde{\mu}_0 - \tilde{\sigma}_s \tilde{\mu}_0 \in X_*({}^L \tilde{T}^0), \quad 1/2(\tilde{\mu}_0 - \tilde{\sigma}_s \tilde{\mu}_0) \equiv \tilde{\lambda}_0 + \tilde{\sigma}_s \tilde{\lambda}_0 \pmod{X_*({}^L \tilde{T}^0)},$$

and

$$\langle \tilde{\mu}_0, \lambda^\vee \rangle = 0, \quad \langle \tilde{\lambda}_0, \lambda^\vee \rangle \in \mathbf{Z}$$

whenever  $\lambda^\vee$  lies in the span of the roots of  ${}^L\tilde{T}^0$  in  ${}^L\tilde{H}_s^0$  (cf. §9.1 of [Sh3]). Here we have used  $\tilde{\sigma}_s$  to denote the action of  $1 \times 1 \times \sigma \in {}^L\tilde{H}_s$ .

Let  $(T, \eta) \in \mathfrak{T}_H(G)$ . Then on transferring  $(\tilde{\mu}_0, \tilde{\lambda}_0)$  to  $\tilde{T}$  using  $\eta$  we obtain the data also denoted  $(\tilde{\mu}_0, \tilde{\lambda}_0)$  for a quasicharacter on  $\tilde{T}(\mathbf{R})$  (cf. [Sh3, §4.1]). This quasicharacter will be denoted  $a_{(T, \eta)}$ .

LEMMA 6.1.

$a_{(T, \eta)}$  is  $\alpha$ -invariant.

*Proof.* We describe  $a_{(T, \eta)}$  explicitly. Let  $\delta = (t, \sigma_T(t)) \in \tilde{T}(\mathbf{R})$ . Write  $t$  as  $\exp X$ ,  $X \in \text{Lie}(T(\mathbf{C})) = X_*(T) \otimes \mathbf{C}$ . Then  $\sigma_T(t) = \exp \sigma_T(\bar{X})$ , where if  $X = \sum_{i=1}^n \lambda_i^\vee \otimes z_i$  then  $\sigma_T(\bar{X}) = \sum_{i=1}^n \sigma_T(\lambda_i^\vee) \otimes \bar{z}_i$ . Because  $a(\mathbf{C}^\times \times 1)$  lies in the diagonal subgroup of  $\text{Center}({}^L\tilde{H}_s^0)$ , as is evident from the form of the embeddings  $\xi$  and  $\tilde{\xi}$  (cf. last section), we must have  $\tilde{\mu}_0$  lying in the diagonal subspace of  $X_*({}^L\tilde{T}^0) \otimes \mathbf{C} = (X_*({}^LT^0) \otimes \mathbf{C}) \times (X_*({}^LT^0) \otimes \mathbf{C})$ . Thus we write  $\tilde{\mu}_0$  as  $(\mu_0, \mu_0)$ ,  $\mu_0 \in X_*({}^LT^0) \otimes \mathbf{C}$ . As usual, we transfer  $\mu_0$  to  $X^*(T) \otimes \mathbf{C}$  via  $\eta$  without change in notation. Then

$$a_{(T, \eta)}(\delta) = e^{\mu_0(X + \sigma_T(\bar{X}))}.$$

Since  $\alpha(\delta) = (\exp \sigma_T(\bar{X}), \exp X)$  it is now clear that  $a_{(T, \eta)}(\alpha(\delta)) = a_{(T, \eta)}(\delta)$ , and the lemma is proved.

Note that  $a_{(T, \eta)}$  is uniquely determined by the class of  $a(w)$  in  $H^1(W, \text{Center}({}^L\tilde{H}_s^0))$ , but is affected by a change in  $\xi$  or  $\tilde{\xi}$ . The dependence on  $\tilde{\xi}$  of our normalization factors is to be expected; the dependence on  $\xi$  is not.

LEMMA 6.2. Fix  $(T, \eta) \in \mathfrak{T}_H(G)$  and  $\delta \in \tilde{T}(\mathbf{R})$ . Then  $\alpha_{(T, \eta)}(\delta) \Lambda_{(T, \eta)}(\delta \alpha(\delta))$  depends on  $\tilde{\xi}$  alone.

*Proof.* The embedding  $\xi$  may be replaced only by  $h \times w \rightarrow a_0(w)\xi(h \times w)$ , where  $a_0(w)$  is a 1-cocycle of  $W$  in the center of  ${}^LH_s^0$  embedded diagonally in the center of  ${}^L\tilde{H}_s^0$ . Then  $a(w)$  is replaced by  $a_0(w)^{-1}a(w)$ . The cocycle  $a_0(w)$  defines first a quasicharacter  $\chi$  on  $H(\mathbf{R})$  and second a quasicharacter  $\tilde{\chi}$  on  $\tilde{H}(\mathbf{R})$ . As before, we use  $\eta$  to transfer data and define quasicharacters  $\chi_{(T, \eta)}$  on  $T(\mathbf{R})$  and  $\tilde{\chi}_{(T, \eta)}$  on  $\tilde{T}(\mathbf{R})$ . Since  $\Lambda_{(T, \eta)}$  is replaced by  $\chi_{(T, \eta)}\Lambda_{(T, \eta)}$  and  $a_{(T, \eta)}$  by  $\tilde{\chi}_{(T, \eta)}^{-1}a_{(T, \eta)}$ , we have only to show that  $\tilde{\chi}_{(T, \eta)}(\delta) = \chi_{(T, \eta)}(\delta \alpha(\delta))$ . Define parameters  $\mu_1, \lambda_1 \in X_*({}^LT^0) \otimes \mathbf{C}$  for  $\chi$  as usual; use the same symbols for their transfer to  $X^*(T) \otimes \mathbf{C}$

via  $\eta$ . For  $\tilde{\chi}$  we can use parameters  $\tilde{\mu}_1 = (\mu_1, \mu_1)$ ,  $\tilde{\lambda}_1 = (\lambda_1, \lambda_1)$  in  $X_*(^L\tilde{T}^0) \otimes \mathbb{C}$  (...or  $X^*(\tilde{T}) \otimes \mathbb{C}$ , after transfer). Since  $\tilde{\chi}$  is clearly  $\alpha$ -invariant (see the last proof), we may take  $\delta = (\exp X, \exp X)$ ,  $X \in \text{Lie}(T(\mathbb{R}))$ . Then  $\tilde{\chi}(\delta) = e^{\langle 2\mu_1, X \rangle}$  and  $\chi(\delta\alpha(\delta)) = \chi(\delta^2) = e^{\langle \mu_1, 2X \rangle}$ , so that the lemma is proved.

The next lemma is simple but very useful (cf. proof of Lemma 6.4). Each element of  $H^1(T)$  can be represented by a cocycle  $\sigma \rightarrow \exp i\pi\lambda^\vee$ , where  $\lambda^\vee \in X_*(T)$  and  $\sigma_T\lambda^\vee = -\lambda^\vee$ . We will use  $\exp i\pi\lambda^\vee$  to denote this cocycle and its class in  $H^1(T)$ ; of course,  $\exp i\pi\lambda^\vee$  also denotes an element of  $T(\mathbb{R}) \subset \tilde{T}(\mathbb{R})$ . Recall that to  $(T, \eta) \in \mathfrak{G}_H(G)$  and our fundamental datum  $s = (x, 1)\tilde{Z}^W$  we have attached a character  $\kappa$  on  $H^1(T)$ .

LEMMA 6.3.

$$a_{(T, \eta)}(\exp i\pi\lambda^\vee) = \kappa(\exp i\pi\lambda^\vee)$$

for all  $\lambda^\vee \in X_*(T)$  such that  $\sigma_T\lambda^\vee = -\lambda^\vee$ .

Note that the left side alone appears to depend on the choice of  $\xi$  and  $\tilde{\xi}$ . However a quasicharacter  $\tilde{\chi}$  as in the last proof annihilates  $\exp i\pi\lambda^\vee$ , if  $\lambda^\vee \in X_*(T)$  and  $\sigma_T\lambda^\vee = -\lambda^\vee$ . Indeed we then have  $i\pi\lambda^\vee \in \text{Lie}(T(\mathbb{R}))$ , so that  $\tilde{\chi}(\exp i\pi\lambda^\vee) = e^{2\pi i \langle \mu_1, \lambda^\vee \rangle} = 1$ , since  $\frac{1}{2}(\mu_1 - \sigma_T\mu_1) \equiv (\lambda_1 + \sigma_T\lambda_1) \pmod{X^*(T)}$  implies that  $\langle \frac{1}{2}(\mu_1 - \sigma_T\mu_1), \lambda^\vee \rangle = \langle \mu_1, \lambda^\vee \rangle$  lies in  $\mathbb{Z}$ . It then follows that neither side of the formula depends on  $\xi$  or  $\tilde{\xi}$ .

*Proof of Lemma 6.3.* First we evaluate the right side. The cocycle  $\sigma \rightarrow \exp i\pi\lambda^\vee$  corresponds under the Tate-Nakayama isomorphism to the coset of  $\lambda^\vee$  in

$$H^{-1}(X_*(T)) \\ = \left\{ \mu^\vee \in X_*(T) : \sigma_T\mu^\vee = -\mu^\vee \right\} / \left\{ \nu^\vee - \sigma_T\nu^\vee : \nu^\vee \in X_*(T) \right\}.$$

Thus  $\kappa(\exp i\pi\lambda^\vee) = \lambda^\vee(x)$ , where  $s = (x, 1)\tilde{Z}^W$  was used to define  $\kappa$ . Note that we have transferred  $\lambda^\vee$  to  $^LT^0$  via  $\eta$ .

For the left side, we write  $a(z \times 1) = (a_0(z), a_0(z))$ ,  $z \in \mathbb{C}^\times$ , and  $a(1 \times \sigma) = (xb_0, b_0)$ , where  $a_0(z)$ ,  $b_0$  lie in the center of  $^LH_s^0$ . Since  $i\pi\lambda^\vee \in \text{Lie}(T(\mathbb{R}))$ , we have  $a_{(T, \eta)}(\exp i\pi\lambda^\vee) = e^{2\pi i \langle \mu_0, \lambda^\vee \rangle} = \lambda^\vee(a_0(-1))$ , where again we have transferred  $\lambda^\vee$  to  $^LT^0$  without change in notation (cf. proof of Lemma 6.1). On the other hand,  $a(1 \times \sigma)\tilde{\sigma}_s(a(1 \times \sigma)) = a(-1)$  implies that  $a_0(-1) = xb_0\sigma_s(b_0) = xb_0\sigma_{(T, \eta)}(b_0)$ . Since  $\sigma_{(T, \eta)}\lambda^\vee = -\lambda^\vee$ , we have that  $\lambda^\vee(a_0(-1)) = \lambda^\vee(x)$ , and the lemma is proved.

We continue with  $(T, \eta) \in \mathfrak{T}_H(G)$  and associated character  $\kappa$  on  $H^1(T)$ . Fix a set  $\{u = \exp i\pi\lambda^\vee : \lambda^\vee \in X_*(T), \sigma_T\lambda^\vee = -\lambda^\vee\}$  such that the cocycles  $\sigma \rightarrow \exp i\pi\lambda^\vee$  form a complete set of (noncohomologous) representatives for the elements of  $H^1(T)$ .

For  $f \in C_c^\infty(\tilde{G}(\mathbf{R}))$ , and Haar measures  $dt$  on  $T(\mathbf{R})$ ,  $d\tilde{g}$  on  $\tilde{G}(\mathbf{R})$  form (cf. [Sh6]):

$$\Phi_f^{(T, \alpha, \kappa)}(\delta, dt, d\tilde{g}) = \sum_u \kappa(u) \int_{\tilde{G}(\mathbf{R})/T(\mathbf{R})} f(\alpha(\tilde{g})u\delta\tilde{g}^{-1}) \frac{d\tilde{g}}{dt},$$

for  $\delta \in \tilde{T}(\mathbf{R})$  such that  $\delta\alpha(\delta)$  is regular. Note that for all  $\delta \in \tilde{T}(\mathbf{R})$ ,  $\delta\alpha(\delta)$  lies in  $T(\mathbf{R})^0$ , the identity component of  $T(\mathbf{R})$ .

LEMMA 6.4.

$$\gamma \rightarrow a_{(T, \eta)}(\delta) \Phi_f^{(T, \alpha, \kappa)}(\delta, dt, d\tilde{g}),$$

if  $\delta\alpha(\delta) = \gamma$ ,  $\gamma \in T(\mathbf{R})_{\text{reg}}^0 = T(\mathbf{R})^0 \cap G_{\text{reg}}$ , is a well-defined function on  $T(\mathbf{R})_{\text{reg}}^0$ .

*Proof.* By Lemma 6.3,

$$a_{(T, \eta)}(\delta) \Phi_f^{(T, \alpha, \kappa)}(\delta, dt, d\tilde{g}) = \sum_u a_{(T, \eta)}(u\delta) \int_{\tilde{G}(\mathbf{R})/T(\mathbf{R})} f(\alpha(\tilde{g})u\delta\tilde{g}^{-1}) \frac{d\tilde{g}}{dt}$$

which we will write as  $\Phi(\delta)$ . If  $\delta\alpha(\delta) = \delta'\alpha(\delta')$  then  $\delta' = v\delta$ , where  $v\alpha(v) = 1$ ,  $v \in \tilde{T}(\mathbf{R})$ . Then it is easily seen that  $v = t^{-1}\alpha(t)u$  for some  $t \in \tilde{T}(\mathbf{R})$  and  $u$  as in the summation. Since  $a_{(T, \eta)}$  is  $\alpha$ -invariant we then have  $\Phi(\delta') = \Phi(v\delta) = \Phi(u\delta)$  which clearly coincides with  $\Phi(\delta)$ . Thus the lemma is proved.

Finally, suppose that  $(T, \eta) \in \mathfrak{T}_H(G)$  and that  $i(h, \eta): T' \rightarrow T$  is defined over  $\mathbf{R}$ . Then the Haar measure  $dt$  on  $T(\mathbf{R})$  is transported via  $i(h, \eta)$  to a Haar measure  $dt'$  on  $T'(\mathbf{R})$ ;  $dt'$  is independent of the choice of  $h$ . Also, we say that  $\gamma' \in T'(\mathbf{R})_{\text{reg}}$  is *not a norm* if it is not in the image of the norm map from  $\tilde{T}' = \text{Res}_{\mathbf{R}}^{\mathbf{C}} T'$  to  $T'$ , i.e.  $\gamma'$  does not lie in the identity component of  $T'(\mathbf{R})$ . Then if  $\gamma'$  originates from  $\gamma \in T(\mathbf{R})_{\text{reg}}$  via  $(T, \eta)$ ,  $\gamma$  is not in the image of the norm from  $\tilde{T}$  to  $T$  (and conversely...).

We have not assumed that  $\xi$  or  $\tilde{\xi}$  is of “unitary type” [Sh3]. It is easily checked that there is a quasicharacter  $\chi$  on  $H(\mathbf{R})$  such that  $|\chi(\gamma')\Lambda_{(T, \eta)}(\gamma)a_{(T, \eta)}(\delta)| = 1$  if  $\gamma'$  originates from  $\gamma = \delta\alpha(\delta)$  via  $(T, \eta)$ . We then define  $\mathcal{C}_{\tilde{\xi}}(H(\mathbf{R}))$  to be the set of functions  $f$  on  $H(\mathbf{R})$  such that  $f\chi$  belongs to  $\mathcal{C}(H(\mathbf{R}))$ , the Schwartz space of  $H(\mathbf{R})$ . As the notation indicates, this space does not depend on the choice of  $\chi$ . For  $f \in \mathcal{C}_{\tilde{\xi}}(H(\mathbf{R}))$  the

stable orbital integrals  $\Phi_f^{(T',1)}(\gamma', dt', dh)$ ,  $\gamma' \in T'(\mathbf{R}) \cap H_{\text{reg}}$  (cf. [Sh4] etc.) are well-defined.

It remains now to recall the factor  $\Delta_{(T,\eta)}$  from  $L$ -indistinguishability. Thus

$$\Delta_{(T,\eta)} = (-1)^{q(G,H)} \epsilon(T, \eta) \Lambda_{(T,\eta)}' \Delta_{(T,\eta)},$$

where  $q(G, H)$  is an integer,  $(-1)^{q(G,H)}$  being inserted only for convenience,  $\epsilon(T, \eta) = \pm 1$  is defined implicitly,  $\Lambda_{(T,\eta)}$  is as earlier in this section and  $\Delta_{(T,\eta)}$  is a discriminant function (see [Sh4, §3] for further details).

## 7. The matching theorem.

**THEOREM 7.1.** *Let  $H$  be an endoscopic group for  $(\tilde{G}, \alpha)$ , with  $L$ -group  ${}^L H_s$  chosen as earlier. Suppose that  $\tilde{\xi}: {}^L H_s \hookrightarrow {}^L \tilde{G}$  is an allowed embedding and that  $\xi: {}^L H_s \hookrightarrow {}^L G$  is admissible (for  $L$ -indistinguishability). Then for each  $f \in C_c^\infty(\tilde{G}(\mathbf{R}))$  there exists  $f_H \in \mathcal{C}_{\tilde{\xi}}(H(\mathbf{R}))$  such that:*

$$\Phi_{f_H}^{(T',1)}(\gamma', dt', dh) = \begin{cases} \Delta_{(T,\eta)}(\gamma) a_{(T,\eta)}(\delta) \Phi_f^{(T,\alpha,\kappa)}(\delta, dt, d\tilde{g}), \\ \text{if } \gamma' \text{ originates from } \gamma = \delta\alpha(\delta) \\ \text{via } (T, \eta) \in \mathfrak{T}_H(G), \\ 0 \quad \text{if } \gamma \text{ is not a norm.} \end{cases}$$

Here it is assumed that  $\gamma'$  originates from *regular* elements in  $G(\mathbf{R})$ . Then  $\gamma'$  is regular in  $H(\mathbf{R})$  [Sh2];  $T'$  is the maximal torus containing  $\gamma'$ . Recall that  $\Delta_{(T,\eta)}$  depends on  $\xi$  alone, that  $a_{(T,\eta)}$  depends on both  $\xi$  and  $\tilde{\xi}$ , and that  $\Delta_{(T,\eta)}(\gamma) a_{(T,\eta)}(\delta)$  depends on  $\tilde{\xi}$  alone... as long as  $(T, \eta)$  and  $\delta$  are fixed.

**REMARK.** We have used  $C_c^\infty(\tilde{G}(\mathbf{R}))$  instead of the more natural  $\mathcal{C}(\tilde{G}(\mathbf{R}))$  since the necessary analysis of “twisted  $F_f$ ” (cf. [Sh6]), for  $f$  a Schwartz function, has not been carried out. Work of L. Clozel now in progress should settle this matter and allow us to replace  $C_c^\infty(\tilde{G}(\mathbf{R}))$  by  $\mathcal{C}(\tilde{G}(\mathbf{R}))$ .

*Proof of the theorem.* Let  $\gamma' \in H(\mathbf{R})$ . Suppose that  $\gamma'$  originates from  $\gamma \in G_{\text{reg}}$  via  $(T, \eta)$  and from  $\bar{\gamma}$  via  $(\bar{T}, \bar{\eta})$ . Choose  $\delta$  so that  $\delta\alpha(\delta) = \gamma$ . Write  $\bar{\gamma}$  as  $y\gamma y^{-1}$  and  $\bar{\eta}$  as  $\omega_H \circ \eta \circ \text{ad } y^{-1}$ , where  $\omega_H \in \Omega(H, T_H) \subset \Omega(G, T^*)$  and  $y \in \mathfrak{X}(T)$  (cf. [Sh4, §3]). Then for  $\bar{\delta}$  such that  $\bar{\delta}\alpha(\bar{\delta}) = \bar{\gamma}$  we may take  $y\delta y^{-1}$ , where  $y \in G$  has been identified with its image in  $\tilde{G}$  under the diagonal embedding. With this choice of  $\bar{\delta}$  we have  $a_{(\bar{T}, \bar{\eta})}(\bar{\delta}) = a_{(T,\eta)}(\delta)$ . The relation between  $\Delta_{(\bar{T}, \bar{\eta})}(\bar{\gamma})$  and  $\Delta_{(T,\eta)}(\gamma)$  is described in [Sh4, §3].

For fixed  $(T, \eta) \in \mathfrak{T}_H(G)$  the function

$$\gamma' \rightarrow a_{(T, \eta)}(\delta) \Delta_{(T, \eta)}(\gamma) \Phi_f^{(T, \alpha, \kappa)}(\delta, dt, d\tilde{g}),$$

if  $\gamma'$  originates from  $\gamma = \delta\alpha(\delta)$  via  $(T, \eta)$ , is well-defined and invariant under  $\mathfrak{U}(T')$ . To prove this we invoke [Sh4, Propositions 2.4.5 and 3.1.2] and [Sh6, Lemma 4.3.2]. These results show that we have only to check that  $a_{(T, \eta)}(\delta^\omega) = a_{(T, \eta)}(\delta)$  for  $\omega$  an element of the Weyl group  $\Omega(G, T)$  of  $(G, T)$  which commutes with the Galois action on  $T$  and “comes from  $H$ ” (i.e.  $\omega \in \Omega_0(G, T) \cap \Omega^{(\kappa)}(G, T)$  as in [Sh4, Proposition 2.4.5]). But this invariance of  $a_{(T, \eta)}$  follows easily from the fact that  $\langle \tilde{\mu}_0, \lambda^\vee \rangle = 0$  for  $\lambda^\vee$  in the span of the roots of  ${}^L\tilde{T}^0$  in  ${}^L\tilde{H}_s^0$  (see the proof of Lemma 6.1).

Suppose now that we fix a “framework of Cartan subgroups [Sh3], [Sh4, §3.2]. Thus we have specified certain pairs  $(T_n, \eta_n) \in \mathfrak{T}_H(G)$  and embeddings  $i_n = i(h_n, \eta_n): T'_n \rightarrow T_n$  over  $\mathbf{R}$ ; the set  $\{T'_n(\mathbf{R})\}$  provides a complete family of representatives, without redundancy, for the conjugacy classes of Cartan subgroups of  $H(\mathbf{R})$ . Given  $\gamma' \in T'_n(\mathbf{R})$ , set  $\gamma = i_n(\gamma')$ , and choose any  $\delta$  such that  $\delta\alpha(\delta) = \gamma$ . Call  $\gamma'$   $G$ -regular if  $\gamma$  is regular. Then for each  $n$  we may consider the function on the  $G$ -regular elements of  $T'_n(\mathbf{R})$  given by

$$\Phi_n(\gamma', dt', dh) = \begin{cases} \varepsilon_n \hat{\Delta}_{(T_n, \eta_n)}(\gamma) a_{(T_n, \eta_n)}(\delta) \Phi_f^{(T_n, \alpha, \kappa_n)}(\gamma, dt, d\tilde{g}) \\ \text{if } \gamma' \in T'_n(\mathbf{R})^0, \\ 0 \quad \text{if } \gamma' \notin T'_n(\mathbf{R})^0, \end{cases}$$

where  $\varepsilon_n = \pm 1$  (to be chosen),  $\hat{\Delta}_{(T, \eta)} = \varepsilon(T_n, \eta_n) \Delta_{(T_n, \eta_n)}$  (i.e.  $\hat{\Delta}_{(T, \eta)}$  is  $\Delta_{(T, \eta)}$  with the  $\varepsilon(T, \eta)$  removed), and  $\kappa_n$  is the “ $\kappa$ ” associated to  $(T_n, \eta_n)$ . Note that  $\{\kappa_n |_{X_*((T_n)_{\text{sc}})}\}$  is exactly the set  $\{\kappa_n\}$  from [Sh2, §7] and [Sh3, §2].

Suppose that we are able to show that there exists  $f_H \in \mathcal{C}_{\xi}^{\sim}(H(\mathbf{R}))$  such that

$$(*) \quad \Phi_{f_H}^{(T'_n, 1)}(\gamma', dt', dh) = \begin{cases} \Phi_n(\gamma', dt', dh) & \text{if } \gamma' \in T'_n(\mathbf{R})^0, \\ 0 & \text{if } \gamma' \notin T'_n(\mathbf{R})^0, \end{cases}$$

for all  $G$ -regular  $\gamma'$  in  $T'_n(\mathbf{R})$  and for all  $n$  provided  $\varepsilon_m \varepsilon_n = \varepsilon(m, n)$  whenever  $T'_m(\mathbf{R})$  and  $T'_n(\mathbf{R})$  are adjacent Cartan subgroups. Here  $\varepsilon(m, n)$  is as defined in [Sh4, §3.5] (cf. [Sh2]). Then we shall take  $\varepsilon_n = \varepsilon(T_n, \eta_n)$ , so that by the results of  $L$ -indistinguishability (exp. [Sh4, §3.5]) there does exist  $f_H$  satisfying (\*). It is then routine to verify that  $f_H$  satisfies the statement of our theorem (see the first paragraph of this proof; similar arguments for  $L$ -indistinguishability are given in [Sh4, §3]).



Returning to the condition on the existence of  $f_H$ , we have only to show that our family  $\{\Phi_n(, , )\}$  behaves like the family  $\{\Phi_n\}$  of [Sh2, §9] (cf. [Sh4, §3.2]). The invariance and growth requirements being satisfied (clearly), only the “jump conditions” remain. Thus we need the jump formulas for the functions  $\Psi_{(T,\eta)}$ :

$$\gamma \rightarrow \begin{cases} a_{(T,\eta)}(\delta) \hat{\Delta}_{(T,\eta)}(\gamma) \Phi^{(T,\alpha,\kappa)}(\delta, dt, d\tilde{g}) & \text{if } \gamma \in T(\mathbf{R})_{\text{reg}}^0, \\ 0 & \text{if } \gamma \in T(\mathbf{R})_{\text{reg}} - T(\mathbf{R})^0. \end{cases}$$

These are contained essentially in the analysis of §§4 and 5 of [Sh6]. To be more precise, we seek analogues of Lemmas 5.2.2 and 5.2.5 of [Sh6], when “ $\Delta_T \Phi^\tau$ ” is replaced by the function above (with the necessary adjustment in the choice of positive system for the imaginary roots of  $T$  used to define the factor  $\hat{\Delta}_{(T,\eta)}$ ). The proof of the analogue of Lemma 5.2.2 is straightforward; because of notational complications we omit further details. Note that the “ $\kappa$ -signature” [Sh2] which appears depends only on  $\kappa|_{X_*(T_{\text{sc}})}$ , i.e. the jump is indeed like that from  $L$ -indistinguishability. The analogue of Lemma 5.2.5 will be stronger than the original statement, because we no longer need the assumption “ $\kappa(\alpha^\vee) = 1$  if (5.2.3) holds.” We now have the exact analogue of [Sh2, Proposition 9.1] from  $L$ -indistinguishability. Indeed, let  $\gamma_0$  be a semiregular element in  $T(\mathbf{R})$  such that  $\lambda(\gamma_0) = 1$ , where  $\lambda$  is an imaginary root such that  $\kappa(\lambda^\vee) = -1$ . We wish to show that  $\Psi_{(T,\eta)}$  is smooth on some neighborhood of  $\gamma_0$ . We may assume that  $\gamma_0 \in T(\mathbf{R})^0$ . Fix  $\delta_0 \in T(\mathbf{R})^0$  such that  $\delta_0^2 = \gamma_0$ . For  $\gamma$  close to  $\gamma_0$  choose  $\delta$  close to  $\delta_0$  such that  $\delta^2 = \gamma$ . It will be sufficient to show that  $\delta \rightarrow \Psi_{(T,\eta)}(\delta^2)$  is smooth near  $\delta_0$ . This follows immediately from Lemma 4.3.3 of [Sh6]. Note that this type of argument could not be used in the proof of Lemma 5.2.5 of [Sh6] because the “cross-section for the norm” was not smooth near  $\gamma_0$ .

We now complete the proof of Theorem 7.1 by the arguments already indicated.

**8. The dual lifting.** Again we fix an element of  $\mathfrak{S}(\tilde{G}, \alpha)$  and choose a convenient representative  $(s, {}^L H_s)$  for this element, as in §5. Let  $H_s$  be the corresponding endoscopic group. Since  $H_s$  is, by definition, quasi-split over  $\mathbf{R}$ , the set  $\Phi(H_s)$  [L2] consists of all equivalence classes of admissible homomorphisms  $\phi: W \rightarrow {}^L H_s$ . Suppose that  $\tilde{\xi}: {}^L H_s \hookrightarrow {}^L \tilde{G}$  is an allowed embedding. Then  $\tilde{\xi}$  induces a map, also to be denoted  $\tilde{\xi}$ , from  $\Phi(H_s)$  to  $\Phi(\tilde{G})$ ; the image of the class of  $\phi: W \rightarrow {}^L H_s$  is the class of  $\tilde{\phi} = \tilde{\xi} \circ \phi: W \rightarrow {}^L \tilde{G}$ . It is easily checked that the image of  $\Phi(H_s)$  in  $\Phi(\tilde{G})$  is independent of the choice for  $\tilde{\xi}$ . By the remarks at the end of §5 it is also independent of the choice for  $(s, {}^L H_s)$ .

On the other hand, the automorphism  $\alpha$  of  $\tilde{G}$  has a standard dual [Sh7], again denoted  $\alpha$ :

$$\alpha((g, h) \times w) = (h, g) \times w, \quad h, g \in {}^L G^0, w \in W.$$

If  $\phi: W \rightarrow {}^L \tilde{G}$  is admissible then so is  $\alpha \circ \phi: W \rightarrow {}^L \tilde{G}$ . We write  $\{\phi\}$  for the class of  $\phi$  and  $\{\phi\}^\alpha$  for the class of  $\alpha \circ \phi$ . Then  $\Phi(\tilde{G})^\alpha = \{\{\phi\} \in \Phi(\tilde{G}): \{\phi\}^\alpha = \{\phi\}\}$ .

For *each* element of  $\mathfrak{S}(\tilde{G}, \alpha)$  we fix a representative  $(s, {}^L H_s)$  as before, and assume that *each*  ${}^L H_s$  has an allowed embedding  $\tilde{\xi}$  in  ${}^L \tilde{G}$ . Also, we will use  $\bigcup_{H_s}$  to denote a union over the corresponding endoscopic groups.

THEOREM 8.1.

$$\Phi(\tilde{G})^\alpha = \bigcup_{H_s} \tilde{\xi}(\Phi(H_s)).$$

*Proof.* Let  $\phi: W \rightarrow {}^L H_s$  be admissible. Set  $\tilde{\phi} = \tilde{\xi} \circ \phi$ . We may assume that  $\phi(\mathbf{C}^x \times 1) \subset {}^L T^0 \times \mathbf{C}^x \times 1$ . Then clearly  $\tilde{\phi}$  and  $\alpha \circ \tilde{\phi}$  coincide on  $\mathbf{C}^x \times 1$ . We write  $\phi(1 \times \sigma)$  as  $n_H \times 1 \times \sigma \in {}^L H_s$ , and  $\tilde{\xi}(1 \times 1 \times \sigma)$  as  $(xm_0, m_0) \times 1 \times \sigma$  (cf. §5). Then  $\tilde{\phi}(1 \times \sigma) = (xn_H m_0, n_H m_0) \times 1 \times \sigma$  and

$$\begin{aligned} (\alpha \circ \tilde{\phi})(1 \times \sigma) &= (n_H m_0, xn_H m_0) \times 1 \times \sigma \\ &= (x^{-1}, x) \tilde{\phi}(1 \times \sigma) = g^{-1} \tilde{\phi}(1 \times \sigma) g, \end{aligned}$$

where  $g = (x, 1)$ . Then clearly  $\alpha \circ \tilde{\phi} = \text{ad } g^{-1} \circ \tilde{\phi}$ , and so  $\tilde{\xi}(\Phi(H)) \subset \Phi(\tilde{G})^\alpha$ .

Suppose now that  $\tilde{\phi}: W \rightarrow {}^L \tilde{G}$  is admissible and that  $\{\tilde{\phi}\}^\alpha = \{\tilde{\phi}\}$ . Then it is sufficient to show that  $\tilde{\phi}$  factors through some  ${}^L H_s$  (not necessarily among our fixed representatives) embedded (via an allowed embedding) in  ${}^L \tilde{G}$ .

Let  $S_\phi^\alpha = \{a \in {}^L \tilde{G}^0: a\tilde{\phi}(w)a^{-1} = (\alpha \circ \tilde{\phi})(w), w \in W\}$ . Then  $S_\phi^\alpha$  is nonempty. If  $a_0$  lies in  $S_\phi^\alpha$  then so does  $a_0 z$ , for  $z \in \tilde{Z}^W$ . In fact, then  $S_\phi^\alpha = a_0 S_\phi$ , where  $S_\phi$  is the centralizer of  $\tilde{\phi}(W)$  in  ${}^L \tilde{G}^0$ ... recall that the results of [Sh4], with a little extra argument for the case  $\tilde{\phi}$  unbounded, show that  $S_\phi = S_\phi^0 \tilde{Z}^W$ ,  $S_\phi^0$  denoting the identity component in  $S_\phi$ . Choose  $s = a_0 \tilde{Z}^W$  contained in  $S_\phi^\alpha$ . Assume that  $s$  consists of  $\alpha$ -semisimple elements (... we will prove below that such an  $s$  exists). Then set  ${}^L H_s^0 = (\text{Cent}_\alpha(s, {}^L \tilde{G}^0))^0$ , and select  ${}^L B_s^0$ ,  ${}^L T_s^0$  and  $\{Y\}$  as in §3. To define a suitable action of  $W$  on  ${}^L H_s^0$  we have just to give a homomorphism of

$\text{Gal}(\mathbf{C}/\mathbf{R})$  into  $\text{Aut}({}^L H_s^0, {}^L B_s^0, {}^L T_s^0, \{Y\})$  such that  $\sigma_s$ , the image of  $\sigma$ , is “realized in  $\text{Cent}_\alpha(s, {}^L \tilde{G}) = \text{Cent}_\alpha(a_0, {}^L \tilde{G})$ ”. But

$$\tilde{\phi}(1 \times \sigma)^{-1} a_0 \alpha(\tilde{\phi}(1 \times \sigma)) = a_0.$$

Thus  $\tilde{\phi}(1 \times \sigma)$  normalizes  ${}^L H_s^0$ . We may write  $\text{ad } \tilde{\phi}(1 \times \sigma)|_{{}^L H_s^0}$  as  $\omega \sigma_s$ , where  $\omega$  is an inner automorphism of  ${}^L H_s^0$  and  $\sigma_s \in \text{Aut}({}^L H_s^0, {}^L B_s^0, {}^L T_s^0, \{Y\})$ . Note that  $\sigma_s^2 = 1$  and is “realized in  $\text{Cent}_\alpha(s, {}^L \tilde{G})$ ”. Using the associated  $W$ -action we form  ${}^L H_s$  and so obtain a representative  $(s, {}^L H_s)$  for an element of  $\mathfrak{S}(\tilde{G}, \alpha)$ . We claim that  $\tilde{\phi}$  factors through  ${}^L H_s$ . Thus, suppose that  $\tilde{\xi}: {}^L H_s \hookrightarrow {}^L \tilde{G}$  is an allowed embedding. Then for each  $w \in W$ ,  $\tilde{\phi}(w)$  lies in  $\text{Cent}_\alpha(s, {}^L \tilde{G})$  and acts on  ${}^L H_s^0 = (\text{Cent}_\alpha(s, {}^L \tilde{G}^0))^0$  as an element  $n(w) \times w$  of the image of  ${}^L H_s$  in  ${}^L \tilde{G}$ . By definition,  $n(w) \times w \in \text{Cent}_\alpha(s, {}^L \tilde{G})$ . Thus  $\tilde{\phi}(w) = a(w)(n(w) \times w)$ , where  $a(w) \in \text{Cent}_\alpha(s, {}^L \tilde{G}^0)$  centralizes  ${}^L H_s^0$ . But then  $a(w)$  lies in the center of  ${}^L H_s^0$ . Hence  $\tilde{\phi}$  factors through  ${}^L H_s$ .

It remains now to show that  $S_\phi^\alpha$  contains an  $\alpha$ -semisimple element. If we replace  $\tilde{\phi}$  by  $\text{ad } g \circ \tilde{\phi}$ ,  $g \in {}^L \tilde{G}^0$ , then we must replace  $S_\phi^\alpha$  by  $\alpha(g) S_\phi^\alpha g^{-1}$ . Therefore we may assume that  $S_\phi^\alpha$  contains an element  $(x^{-1}, 1)$ ,  $x \in {}^L G^0$  (cf. Lemma 4.1). Then we write  $\tilde{\phi}(w)$  as  $(\phi_1(w), \phi_2(w)) \times w$  and obtain from  $(x^{-1}, 1)\tilde{\phi}(w)(x, 1) = \alpha(\tilde{\phi}(w))$ ,  $w \in W$ , that  $\phi_1(z \times 1) = \phi_2(z \times 1)$ ,  $z \in \mathbf{C}^x$ , and  $\phi_1(1 \times \sigma) = x \phi_2(1 \times \sigma)$ ; also,  $x$  lies in the centralizer  $S_0$  of the image of the homomorphism  $\hat{\phi}_1: w \rightarrow \phi_1(w) \times w$  of  $W$  into  ${}^L G$ . Write  $x = x_u x_s$ , where  $x_u \in S_0$  is unipotent and  $x_s \in S_0$  is semisimple. Then with the same  $\phi_1$  and with  $x_s$  in place of  $x$  we can use the formulas above for  $\tilde{\phi}$  to define  $\tilde{\phi}_0: W \rightarrow {}^L \tilde{G}$  such that  $S_{\tilde{\phi}_0}^\alpha$  contains  $(x_s^{-1}, 1)$ . But  $\tilde{\phi}_0$  is easily seen to be equivalent to  $\tilde{\phi}$  because  $x_u^{-1}$ , being unipotent and fixed by  $\hat{\phi}_1(W)$ , can be written as  $v(\hat{\phi}_1(1 \times \sigma)v)^{-1}$ ,  $v \in \text{Cent}(\phi_1(\mathbf{C}^x \times 1), {}^L G^0)$ . Since  $(x_s^{-1}, 1)$  is  $\alpha$ -semisimple our proof of Theorem 8.1 is complete.

According to Langlands’ functoriality principle this factoring of the  $\alpha$ -fixed parameters  $\{\tilde{\phi}\}$  should be reflected in character theory. Let  $\tilde{\phi} \in \Phi(\tilde{G})$  be  $\alpha$ -fixed (we now drop the  $\{ \}$  from the notation for parameters). Then the  $L$ -packet  $\Pi_{\tilde{\phi}}$  consists of a single infinitesimal equivalence class of irreducible admissible representations fixed by the automorphism  $\alpha: \tilde{G}(\mathbf{R}) \rightarrow \tilde{G}(\mathbf{R})$  (… this is easily checked, see also [C1]). Thus the twisted character  $\chi_\phi^\alpha$  of  $\Pi_{\tilde{\phi}}$  is well-defined up to sign (see [C1] for a detailed discussion, especially concerning the question of signs). Assume that  $\tilde{\phi}$  is bounded, i.e. if  $\tilde{\phi}(w) = \tilde{\phi}_0(w) \times w$ ,  $w \in W$ , then  $\tilde{\phi}_0(W)$  is bounded. Then  $\chi_\phi^\alpha$  is tempered [C1, Theorem 5.12]. On the other hand, suppose that  $\tilde{\phi}$  is the lift of  $\phi \in \Phi(H)$ , in the sense afforded by Theorem

8.1. Then  $\phi$  is essentially bounded, so that the  $L$ -packet  $\Pi_\phi$  consists of essentially tempered (equivalence classes of) representations. Thus  $\chi_\phi = \sum_{\pi \in \Pi_\phi} \chi_\pi$ ,  $\chi_\pi$  denoting the ordinary character of  $\pi$ , is a stable essentially tempered distribution on  $H(\mathbf{R})$  [Sh1, Lemma 5.2].

Theorem 7.1 provides a correspondence  $(f, f_H)$  between  $C_c^\infty(G(\mathbf{R}))$  and  $\mathcal{C}_{\tilde{\xi}}(H(\mathbf{R}))$ . As mentioned already, an adequate analysis of the “twisted  $F_f$  transform” would provide a correspondence between  $\mathcal{C}(\tilde{G}(\mathbf{R}))$  and  $\mathcal{C}_{\tilde{\xi}}(H(\mathbf{R}))$ ; it would also give a dual lifting of stable tempered distributions on  $H(\mathbf{R})$  to twisted-invariant tempered distributions on  $\tilde{G}(\mathbf{R})$ , with eigendistributions mapping to eigendistributions (see [Sh4, §4] for the analogous arguments in the case of  $L$ -indistinguishability). Nevertheless, with the correspondence of Theorem 7.1 we can define  $(\text{Lift } \chi_\phi)(f) = \chi_\phi(f_H)$ ,  $f \in C_c^\infty(\tilde{G}(\mathbf{R}))$ . Writing  $\chi_\phi(f_H)$  as  $\int_{H(\mathbf{R})} f_H(h) \chi_\phi(h) dh$ , and applying the Weyl Integration Formula, the matching theorem and the twisted analogue of the Weyl Integration Formula, we find that  $\text{Lift } \chi_\phi$  is a twisted-invariant distribution on  $\tilde{G}(\mathbf{R})$  represented by a function explicitly computed in terms of  $\chi_\phi$ . Moreover, this function transforms under the center of the universal enveloping algebra of  $\tilde{G}(\mathbf{C})$  according to the infinitesimal character of  $\chi_\phi^\alpha$ . We may therefore ask if  $\text{Lift } \chi_\phi$  coincides with  $\chi_\phi^\alpha$  up to a constant (depending only on  $G$  and  $H$ , once the sign for  $\chi_\phi^\alpha$  has been suitably fixed). According to [C1] with some minor additional arguments, this is true if  $H = G$ ; recall that we are assuming that  $\tilde{\phi}$  is bounded, so that  $\phi$  is an essentially bounded parameter. Work of L. Clozel now in progress should provide the answer to our question for the case  $H \neq G$ .

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