Pacific Journal of Mathematics

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Vol. 110, No. 2 October 1984

PACIFIC JOURNAL OF MATHEMATICS
Vol. 110, No. 2, 1984

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Suppose M is a closed orientable 3-manifold. Then H. Hilden et al. proved that M is a 3-fold branched covering of S^3 branched over a fibered knot. In this paper we prove that, if M is irreducible and atoroidal, then M is either a 3-fold branched covering of S^3 branched over a simple, fibered knot, or a 2-fold branched covering of a closed orientable 3-manifold whose Heegaard genus is at most one.

Hilden [4], Hirsch [5] and Montesinos [11] proved independently that a closed, connected and orientable 3-manifold M is a 3-fold irregular branched covering of S^3 branched over a knot K. Further, it is known that K may be chosen to be a fibered knot. We do not know a reference for this refinement, which we need for our main theorem, so we give in §1 a sketch of the proof, shown to us by Hilden. Our main result is:

THEOREM. *Let M be a closed, connected and orientable 3-manifold. Suppose M is atoroidal and irreducible. Then at least one of the following holds.*

(i) *M is a 3-fold (cyclic or irregular) branched covering of S^3 branched over a simple, fibered knot.*

(ii) *There exist a closed, connected and orientable 3-manifold N whose Heegaard genus is at most one and a simple link L in N such that M is a 2-fold branched covering of N branched over L.*

Here M *atoroidal* means M contains no embedded incompressible torus. As is well known, classifying closed orientable 3-manifolds essentially reduces to the case of atoroidal irreducible 3-manifolds, by the Unique Prime Decomposition Theorem [9] and the Torus Decomposition Theorem [6], [7].

Recently Thurston announced that, if an atoroidal and irreducible 3-manifold M is a regular (in particular cyclic) branched covering of a closed, orientable 3-manifold, then M has a *geometric structure* (i.e. M admits a complete riemannian metric in which any two points have isometric neighborhoods). By this result and our Theorem, if M is a closed, orientable 3-manifold which is atoroidal and irreducible, then M

has a geometric structure or is a 3-fold irregular branched covering of S^3 branched over a simple, fibered knot.

By similar methods (see [**15**] for details) one can prove:

Suppose that Σ *is a homotopy* 3-*sphere, not necessarily irreducible. Then* Σ *is a* 3-*fold irregular branched covering of* S^3 *branched over a simple, fibered knot* K.

If the branch set K is a torus knot, then Σ is a graph manifold. By Montesinos [**10**, p. 249, Lemma 1], Σ is homeomorphic to S^3. Hence any homotopy 3-sphere is homeomorphic to S^3 or a 3-fold irregular branched covering of S^3 branched over a fibered, hyperbolic knot.

We would like to thank Professors Mitsuyoshi Kato, Hiroshi Noguchi and the referee for helpful comments and suggestions. We also would like to thank Professor Hugh M. Hilden for informing us of his useful results.

1. Preliminaries. In this paper we work in the piecewise linear category and every 3-manifold is orientable.

Let L be a link in S^3 and $\omega\colon \pi_1(S^3 - L) \to \Theta_3$ a transitive representation, where Θ_3 is the symmetric permutation group of 3-symbols. We say that ω is *simple* if it represents each meridian by a transposition in Θ_3. Then we denote by $M(L, \omega)$ the 3-fold irregular branched covering of S^3 (branched over L) which is determined by ω. We consider a regular projection of L. Let B be a 3-ball in S^3 as shown in Figure 1(a). In Figure 1(a), α, β and γ are three different transpositions in Θ_3 such that $\omega(x_\alpha) = \alpha$, $\omega(x_\beta) = \beta$, $\omega(x_\gamma) = \gamma$, where x_α (resp. x_β, x_γ) is the Wirtinger generator associated to an overpass x'_α (resp. x'_β, x'_γ).

We change the pair (L, ω) to (L', ω') as shown in Figure 1(b). By Montesinos [**11**], $M(L, \omega)$ is homeomorphic to $M(L', \omega')$. We say that (L', ω') is obtained by doing a *double-Montesinos move on* (L, ω) *in* B. We note that the number of components of L is equal to that of L'.

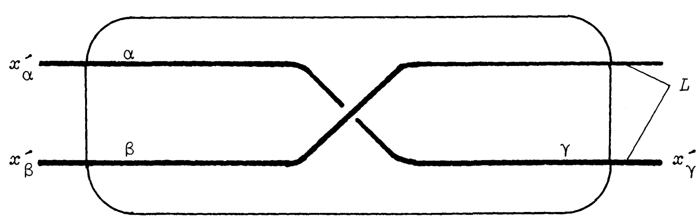

FIGURE 1(a)

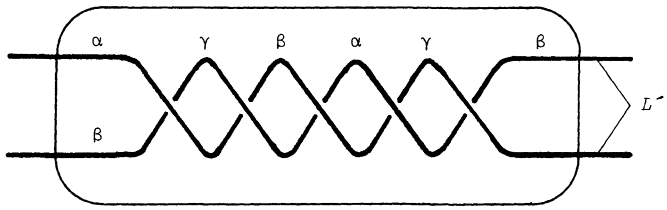

FIGURE 1(b)

Now we give a sketch of the proof of the following theorem of Hilden.

THEOREM (*Hilden*). *Every closed, connected 3-manifold M is a 3-fold irregular branched covering of S^3 branched over a fibered knot.*

Sketch of proof. By Hilden [**4**], Hirsch [**5**] or Montesinos [**11**], M is a 3-fold irregular branched covering of S^3 branched over a knot K. Let ω be the representation associated to the branched covering. By Alexander's Theorem, K is represented by a closed braid (see [**1**, p. 42, Theorem 2.1]). By doing some double-Montesinos moves on (K, ω), we obtain a new pair (K', ω') such that K' is represented by a closed positive braid (i.e. each crossing of the representation is positive). Figure 2 indicates the result. By Stallings [**16**, Theorem 2], K' is a fibered knot. □

Let F be a 2-manifold embedded in a 3-manifold M. Then a 2-disk D embedded in M is called a *compressing disk for F in M* if $F \cap D = \partial D$ and ∂D is not contractible in F. If F has a compressing disk in M, then we say that F is *compressible in M*, otherwise *incompressible in M*.

Let X be a submanifold of a manifold Y. Then we denote by $N(X, Y)$ a regular neighborhood of X in Y.

Let K be a knot in S^3. Then $E(K) = S^3 - \operatorname{int} N(K, S^3)$ is called the *exterior of K in S^3*. We say that K is *simple* if $E(K)$ contains no incompressible torus which is not isotopic to $\partial E(K)$ in $E(K)$.

Let V be an unknotted solid torus in S^3 and K a knot in S^3 which is contained in V and such that ∂V is incompressible in $V - K$ and K is not isotopic in V to a core c of V. Let $f\colon V \to S^3$ be an embedding such that $f(c)$ is knotted in S^3 and $f(l)$ is homologous to zero in $S^3 - \operatorname{int} f(V)$, where l is a meridian of the solid torus $S^3 - \operatorname{int} V$. We set $T = f(\partial V)$. Then T is an incompressible torus in $E(f(K))$ which is not isotopic to $\partial E(f(K))$. We say that $f(c)$ is the *companion of $f(K)$ for T*, $f(K)$ is the *satellite of $f(c)$ for T* and K is the *preimage of $f(K)$ for T*. By Myers [**12**, Proposition 9.11], if $f(K)$ is a fibered knot, then $f(c)$ and K are also fibered knots and $g(f(c))$, $g(K) < g(f(K))$, where $g(K)$ denotes the genus of K.

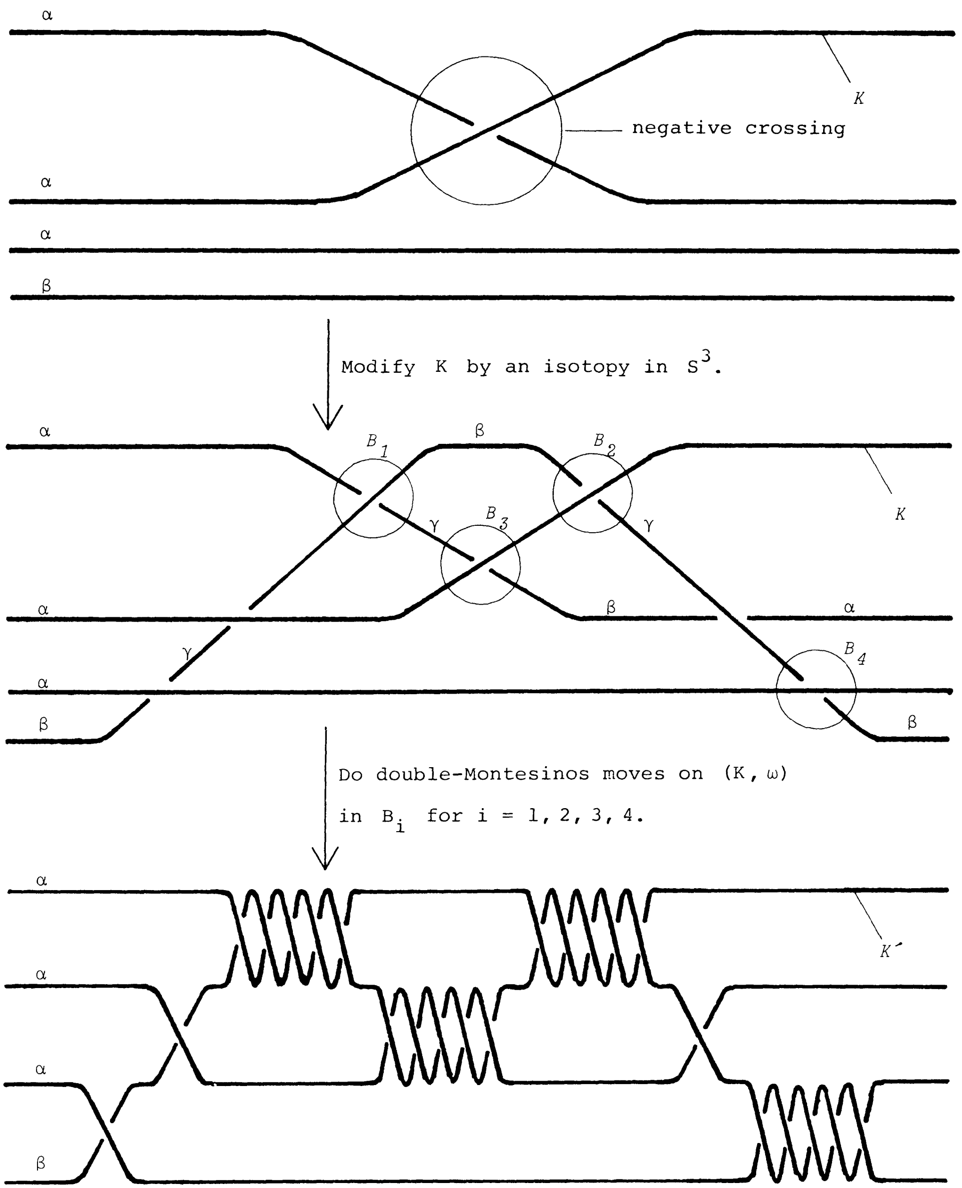

FIGURE 2

Let T be a torus in an atoroidal, irreducible 3-manifold M and D a compressing disk for T in M. Let f: $D \times I \to M$ be an embedding such that $f(D \times \{\frac{1}{2}\}) = D$ and $f(D \times I) \cap T = f(\partial D \times I)$, where $I = [0, 1]$. We say that $S = (T - \text{int}(T \cap f(D \times I))) \cup f(D \times \{0\}) \cup f(D \times \{1\})$ is a 2-sphere obtained by doing *surgery on T along D*. Obviously $S \cap D = \varnothing$.

Since M is irreducible, S bounds a 3-ball B in M. If $B \cap D = \emptyset$, then T bounds a solid torus $B \cup f(D \times I)$ in M with a meridian disk D. If $B \supset D$, then T bounds a compact 3-manifold $N = \overline{(B - f(D \times I))}$ in M such that $(N, \partial D)$ is homeomorphic to $(E(K), m)$, where K is a knot in S^3 and $m \subset \partial E(K)$ is a meridian of a solid torus $N(K, S^3) = S^3 - \operatorname{int} E(K)$. Then we say that $(N, \partial D)$ is a *knot space-meridian pair*. Let l be a simple loop in ∂N which meets ∂D transversely at a single point (hence l is not contractible in ∂N) and is homologous to zero in N. Then we say that l is a *longitude* of $(N, \partial D)$.

Let A, B be two manifolds. Then we denote by $A \cong B$ that A is homeomorphic to B.

We prove the following three lemmas.

LEMMA 1. *Let M be a connected, closed 3-manifold which is irreducible and atoroidal. Let p: $M \to S^3$ be a 3-fold irregular branched covering branched over a knot K. If K is a composite knot, then M is a 2-fold branched covering of S^3 branched over a prime factor K_0 of K.*

REMARK. By Gordon and Litherland [**3**, Theorem 2], K_0 is simple. By Myers [**12**, Proposition 9.11], if K is fibered, then K_0 is also fibered.

Proof. Since K is composite, there exists a 2-sphere S embedded in S^3 which bounds two 3-balls B_1, B_2 in S^3 such that $B_1 \cap B_2 = S$ and $\alpha_i = B_i \cap K$ is a knotted arc in B_i for $i = 1, 2$. Since the representation associated to p is simple and S meets K transversely at two points, $p^{-1}(S)$ consists of two 2-spheres S_1, S_2 such that $p|S_1$: $S_1 \to S$ is a homeomorphism and $p|S_2$: $S_2 \to S$ is a 2-fold branched covering branched over $K \cap S$. Since M is irreducible, either $p^{-1}(B_1)$ or $p^{-1}(B_2)$ is disconnected. We may assume that $p^{-1}(B_1)$ consists of two components N_1 and N_2 such that $\partial N_i = S_i$ for $i = 1, 2$. Then $p|N_2$: $N_2 \to B_1$ is a 2-fold branched covering branched over α_1. If N_2 is a 3-ball, then α_1 is unknotted in B_1 by the Branched Covering Theorem [**13**], a contradiction. Thus $M - \operatorname{int} N_2$ is a 3-ball. We may extend $p|S_2$: $S_2 \to S$ to a 2-fold branched covering q: $\tilde{C} \to C$ branched over an unknotted arc α in C, where $\tilde{C}$, C are 3-balls. Then $p|N_2 \cup q$: $N_2 \cup_{S_2} \tilde{C} \to B_1 \cup_S C$ is a 2-fold branched covering branched over a knot $K_0 = \alpha_1 \cup \alpha$ in $B_1 \cup_S C \cong S^3$. Obviously we have $N_2 \cup_{S_2} \tilde{C} \cong M$. By the above remark, K_0 is simple. Hence, in particular, K_0 is a prime factor of K. This completes the proof. □

LEMMA 2. *Let T_1, T_2 be tori and $p\colon T_1 \to T_2$ a covering. Suppose that l is a simple loop in T_1 which is not contractible in T_1. Then l is isotopic to a simple loop l_1 in T_1 such that $p(l_1)$ is a simple loop in T_2 and $p \mid l_1\colon l_1 \to p(l_1)$ is a covering.* (*We say that l_1 is in good position with respect to p.*)

Proof. We suppose that every loop is oriented. Let α, β be generators of $\pi_1(T_2) \approx Z \times Z$. Then we suppose that a map $p \mid l\colon l \to T_2$ represents $n(p\alpha + q\beta)$ in $\pi_1(T_2)$, where $n, p, q \in Z$, $n \neq 0$ and $(p, q) = 1$. Let l_2 be a simple loop in T_2 which represents $p\alpha + q\beta$ in $\pi_1(T_2)$. Let $\pi\colon S^1 \to l_2$ be an n-fold cyclic covering and $i\colon l_2 \to T_2$ an inclusion. Since $p \mid l$ is homotopic to $i \circ \pi$, $i \circ \pi$ has a lift $\tilde{\pi}$ with respect to p. Then it is easy to show that $l_1 = \tilde{\pi}(S^1)$ satisfies the conclusions of Lemma 2. □

LEMMA 3. *Let M_0 be a compact, connected 3-manifold whose boundary consists of n tori $T_1, \ldots, T_n$ $(n \geq 1)$, and let M_k $(k = 1, \ldots, n)$ be a compact, connected 3-manifold such that ∂M_k is an incompressible torus in M_k. If $M = M_0 \cup_{T_1 = \partial M_1} M_1 \cdots \cup_{T_n = \partial M_n} M_n$ is atoroidal, then each T_k is compressible in M_0.*

Proof. If $n = 1$, the proof is trivial. We suppose $n > 1$. Then it suffices to prove that T_1 is compressible in M_0. We set $P = M_0 \cup_{T_1 = \partial M_1} M_1$ and $Q = M - \operatorname{int} M_1$. Then $M = P \cup_{T_2 = \partial M_2} M_2 \cdots \cup_{T_n = \partial M_n} M_n$. By induction on n, for $k > 1$, T_k is compressible in P.

We suppose that T_1 is incompressible in M_0. Since $T_1 = \partial M_1$ is incompressible in M_1, it also is in P. Since T_k $(k > 1)$ is compressible in P, $(j \circ i_k)_*\colon \pi_1(T_k) \to \pi_1(P)$ is not injective, where $i_k\colon T_k \subset M_0$ and $j\colon M_0 \subset P$. Since $j_*\colon \pi_1(M_0) \to \pi_1(P) \approx \pi_1(M_0) *_{\pi_1(T_1)} \pi_1(M_1)$ is injective, $(i_k)_*$ is not injective. Hence there exists a compressing disk D_k for T_k in M_0. By using an elementary innermost disk technique, we may assume $D_k \cap D_l = \emptyset$ for $2 \leq k < l \leq n$. Let S_k $(k = 2, \ldots, n)$ be a 2-sphere in M_0 obtained by doing surgery on T_k along D_k such that $S_k \cap S_l = \emptyset$ for $k \neq l$. Then S_k bounds a compact 3-manifold N_k in Q such that $N_k \supset M_k \cup D_k$. Since T_1 is compressible in Q,

$$j'_* \circ i'_*\colon \pi_1(T_1) \to \pi_1(Q - \operatorname{int}(N_2 \cup \cdots \cup N_k)) \to \pi_1(Q)$$
$$\approx \pi_1(Q - \operatorname{int}(N_2 \cup \cdots \cup N_k)) * \pi_1(N_2) * \cdots * \pi_1(N_k)$$

is not injective, where $i'\colon T_1 \subset Q - \operatorname{int}(N_2 \cup \cdots \cup N_k)$ and $j'\colon Q - \operatorname{int}(N_2 \cup \cdots \cup N_k) \subset Q$. Since j'_* is injective, i'_* is not injective. Hence T_1 is compressible in $Q - \operatorname{int}(N_2 \cup \cdots \cup N_k) \subset M_0$, a contradiction. Thus T_1 must be compressible in M_0. This completes the proof. □

2. Proof of Theorem. Let M be a closed, connected 3-manifold which is atoroidal and irreducible, and let $p: M \to S^3$ be a 3-fold irregular branched covering branched over a fibered knot K.

We suppose K is not simple, that is, int $E(K)$ contains an incompressible torus T which is not isotopic to $\partial E(K)$. Then $p^{-1}(T)$ consists of one, two, or three tori in M.

Let X be a compact orientable 2-manifold which is properly embedded in a compact 3-manifold Y. We denote by Y_X the compact 3-manifold obtained by splitting Y along X.

We use a weighted graph to study the configuration of $p^{-1}(T)$ in M.

To each component of $M_{p^{-1}(T)}$, we associate a vertex v with weight i and denote the component by $M(v)$. The weight i indicates that $p\,|\,M(v)$: $M(v) \to p(M(v))$ is an i-fold branched or unbranched covering. Let V be a solid torus in S^3 bounded by T. Obviously V contains K. We color a vertex v black if $p(M(v)) = V$, otherwise white.

To each component of $p^{-1}(T)$, we associate an edge e with weight i and direction, and denote the component by $T(e)$. The weight i indicates that $p\,|\,T(e)$: $T(e) \to T$ is an i-fold covering. We say that v is a vertex of e if $\partial M(v)$ contains $T(e)$. An edge e is directed, $v_1 \underset{e}{\to} v_2$, means $T(e)$ is compressible in the component of $M_{T(e)}$ which contains $M(v_2)$ (we note that M is atoroidal). An edge may have two directions. The two ends of an edge have opposite colors.

Thus we obtain the weighted graph Γ associated to $(M, p^{-1}(T))$.

The *valency* of a vertex v is the number of all edges of Γ with v as a common vertex.

LEMMA 4. *The graph* Γ *associated to* $(M, p^{-1}(T))$ *satisfies the following properties.*

(i) *Let* v_0 *be a white vertex of* Γ *with valency* 1 *and* e_0 *the unique edge with* v_0 *as a vertex. Then* e_0 *is directed only away from* v_0.

(ii) *Let* v_1 *be a black vertex of* Γ *with weight* 1 (*hence the valency of* v_1 *is* 1) *and* e_1 *the unique edge with* v_1 *as a vertex. Then* e_1 *is directed only toward* v_1.

(iii) *The total sum of the weights of all edges with* v *as a common vertex is equal to the weight of* v.

(iv) Γ *is a tree.*

(v) *The number of all black vertices of* Γ *is at most two. The number of white vertices is at most three.*

It follows that Γ is one of the five graphs Γ_i in Figure 3. (Lemma 4 does not determine the directing of the edge e in Γ_2 nor of e_1 in Γ_4.)

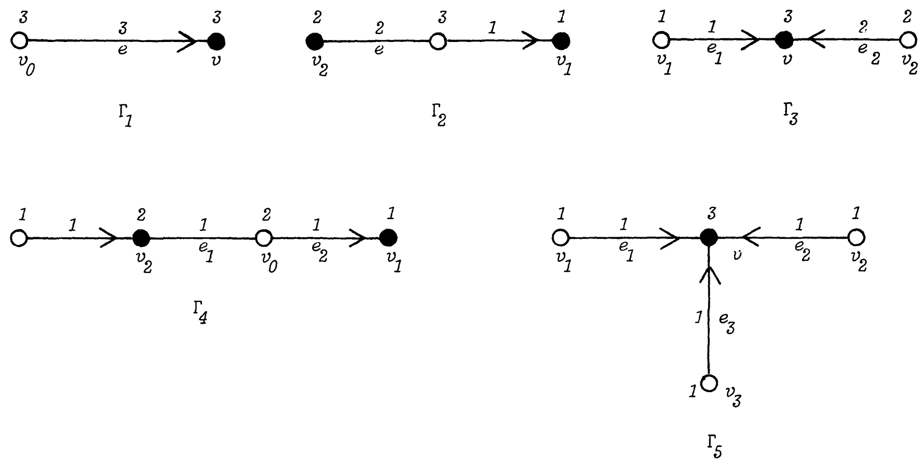

FIGURE 3

Proof of Lemma 4. (i) If $T(e)$ is compressible in $M(v_0)$, then T is compressible in $S^3 - \text{int}\, V$, a contradiction.

(ii) Since $p\,|\,M(v_1)$: $M(v_1) \to V$ is a homeomorphism, $M(v_1)$ is a solid torus. Hence $T(e_1) = \partial M(v_1)$ is compressible in $M(v_1)$.

(iii) If $p\,|\,M(v)$: $M(v) \to V$ (or $S^3 - \text{int}\, V$) is i-fold, then $p\,|\,\partial M(v)$: $\partial M(v) \to T$ is also i-fold. This gives (iii).

(iv) Let e be an edge of Γ. Since $T(e)$ bounds a compact 3-manifold N in M such that $\partial N = T(e)$ (see §1), $T(e)$ separates M into two components. Therefore Γ is a tree.

(v) If Γ has three black vertices v_1, v_2, v_3, then every $p\,|\,M(v_i)$: $M(v_i) \to V$ is 1-fold. Hence $p\,|\,M(v_i)$ is a homeomorphism. This contradicts that the branch set K of p is contained in V. □

Proof of Theorem. By Lemma 1 we may assume the branch set K is a prime, fibered knot. We prove the theorem by induction on $g(K)$. Let Γ be the graph associated to $(M, p^{-1}(T))$. By Lemma 2, we may assume that every non-contractible simple loop in $p^{-1}(T)$ is in good position with respect to $p\,|\,p^{-1}(T)$: $p^{-1}(T) \to T$.

Case 1. $\Gamma = \Gamma_1$.

Let D be a compressing disk for $T(e)$ in $M(v)$. We set $\partial D = \mu$. Then $m = p(\mu)$ is a meridian of V. It is easy to show that $p^{-1}(m)$ is either connected (i.e. $p^{-1}(m) = \mu$) or has three components μ_1 $(= \mu)$, μ_2, μ_3. If the latter case holds, we may extend $p|T(e)$: $T(e) \to T$ to a 3-fold unbranched covering q: $V_1 \to V$, where V_1 is a solid torus with meridians μ_1, μ_2, μ_3. Then $q \cup p|M(v_0)$: $V_1 \cup_{T(e)} M(v_0) \to S^3$ is a 3-fold unbranched covering. This contradicts that S^3 has no non-trivial covering. Hence we have $p^{-1}(m) = \mu$. Then we may extend $p|T(e)$: $T(e) \to T$ to a 3-fold cyclic branched covering r: $V_2 \to V$ branched over a core c of V, where V_2 is a solid torus with a meridian μ. Then $r \cup p|M(v_0)$: $V_2 \cup_{T(e)} M(v_0) \to S^3$ is a 3-fold cyclic branched covering branched over c. Since c in S^3 is the companion of K for T, c is fibered and $g(c) < g(K)$. If $(M(v_0), \mu)$ is a knot space-meridian pair, then $V_2 \cup_{T(e)} M(v_0) \cong S^3$. By the Branched Covering Theorem, c (hence V) is unknotted in S^3. Therefore $T = \partial V$ is compressible in a solid torus $S^3 - \text{int}\, V$, a contradiction. Hence $M(v)$ is a solid torus with a meridian μ. Therefore we have $V_2 \cup_{T(e)} M(v_0) \cong M$. By [3, Theorem 2], c is simple. Thus $r \cup p|M(v_0)$ satisfies the conclusion of (i).

Case 2. $\Gamma = \Gamma_2$ and $\partial M(v_2)$ is compressible in $M(v_2)$.

Let D be a compressing disk for $T(e)$ in $M(v_2)$. We set $\mu = \partial D$. Then $m = p(\mu)$ is a meridian of V. If $p^{-1}(m) \cap T(e)$ consists of two components μ, μ', we may extend $p|T(e)$: $T(e) \to T$ to a 2-fold unbranched covering q: $V_1 \to V$, where V_1 is a solid torus with meridians μ, μ'. Then

$$q \cup (p|(M - \text{int}\, M(v_2))): V_1 \cup_{T(e)} (M - \text{int}\, M(v_2)) \to S^3$$

is a 3-fold unbranched covering, a contradiction. Therefore we have $p^{-1}(m) \cap T(e) = \mu$. Since $p|M(v_2)$: $M(v_2) \to V$ is a 2-fold (cyclic) branched covering, by the Equivariant Dehn's Lemma [**8**, Theorem 5], we may assume $g \cdot D = D$ for all $g \in G$, where G ($\cong Z_2$) is the group of the branched covering. By the argument of Gordon and Litherland [**3**], $p(D)$ is a meridian disk of V and $p(D) \cap K$ is a single point. By Schubert [**14**, §14, Satz 1], K is a composite knot. This contradicts our assumption. Thus Case 2 cannot occur.

Case 3. $\Gamma = \Gamma_2$ and $\partial M(v_2)$ is incompressible in $M(v_2)$.

We set $M_0 = M - \text{int}\, M(v_2)$. Let D be a compressing disk for $T(e)$ in M_0. By a remark in §1, either M_0 is a solid torus with a meridian disk D, or $(M(v_2), \partial D)$ is a knot space-meridian pair. We set $\partial D = \mu$.

(3.1) We suppose M_0 is a solid torus. If $p|\mu$: $\mu \to p(\mu)$ is a 2-fold covering (resp. a homeomorphism), then we may extend $p|T(e)$: $T(e) \to T$ to q: $M_0 \to V_1$ which is a 2-fold branched covering branched over a core c of V_1 (resp. a 2-fold unbranched covering), where V_1 is a solid torus with a meridian $p(\mu)$. Then

$$p|M(v_2) \cup q\colon M = M(v_2) \cup_{T(e)} M_0 \to V \cup_T V_1$$

is a 2-fold branched covering branched over a link $K \cup c$ (resp. a knot K). We set $N = V \cup_T V_1$. Thus $p|M(v_2) \cup q$ satisfies the conclusions of (ii).

(3.2) We suppose that $(M(v_2), \mu)$ is a knot space-meridian pair. By the argument of (3.1), we may extend $p|T(e)\colon T(e) \to T$ to a 2-fold branched or unbranched covering $r\colon V_2 \to V_3$, where V_2, V_3 are solid tori with meridians μ, $p(\mu)$ respectively. Then

$$p|M(v_2) \cup r\colon M(v_2) \cup_{T(e)} V_2 \to V \cup_T V_3$$

is a 2-fold branched covering. Since $(M(v_2), \mu)$ is a knot space-meridian pair, $M(v_2) \cup_{T(e)} V_2$ is homeomorphic to S^3. Hence we have $\pi_1(V \cup_T V_3) = 1$, so $V \cup_T V_3$ is homeomorphic to S^3. By Fox [**2**, pp. 165–166], the branch set of $p|M(v_2) \cup r$ is connected. Therefore $r\colon V_2 \to V_3$ must be an unbranched covering, so $p|\mu\colon \mu \to p(\mu)$ is a homeomorphism. Let λ be a longitude of $(M(v_2), \mu)$. Since $l = p(\lambda)$ is homologous to zero in V, l is a meridian of V. Since $V \cup_T V_3 \cong S^3$, we may assume $l \cap p(\mu)$ consists of a single point. Since $p|\mu\colon \mu \to p(\mu)$ is a homeomorphism, $p^{-1}(l) \cap \mu$ consists of a single point. Hence $p^{-1}(l) \cap T(e)$ is connected, i.e. $p^{-1}(l) \cap T(e) = \lambda$. Therefore we may extend $p|T(e)\colon T(e) \to T$ to a 2-fold branched covering $s\colon V_4 \to V$ branched over a core c of V, where V_4 is a solid torus with a meridian λ. Then $s \cup p|M_0\colon V_4 \cup_{T(e)} M_0 \to S^3$ is a 3-fold irregular branched covering branched over c. Since c in S^3 is the companion of K for T, c is a fibered knot and $g(c) < g(K)$. We set $N = N(D, M_0)$. Since $\lambda \cap \mu$ consists of a single point, $B_1 = V_4 \cup_{T(e) \cap N} N$ is a 3-ball in $V_4 \cup_{T(e)} M_0$. Since $(M(v_2), \mu)$ is a knot space-meridian pair, $B_2 = M(v_2) \cup_{T(e) \cap N} N$ is a 3-ball in M. Since

$$V_4 \cup_{T(e)} M_0 - \operatorname{int} B_1 \cong \overline{(M_0 - N)} \cong M - \operatorname{int} B_2,$$

we have $V_4 \cup_{T(e)} M_0 \cong M$. Hence the result follows by induction.

Case 4. $\Gamma = \Gamma_3$.

By Lemma 3, $T(e_2)$ is compressible in $M(v)$. Let D_2 be a compressing disk for $T(e_2)$ in $M(v)$. We set $\mu_2 = \partial D_2$ and $m_2 = p(\mu_2)$. Since $p(D_2) \subset V$, m_2 is a meridian m of V. If $p^{-1}(m) \cap T(e_2)$ consists of two components μ_2, μ_2', then we may extend $p|T(e_2)\colon T(e_2) \to T$ to a 2-fold unbranched covering $q\colon V_1 \to V$, where V_1 is a solid torus with meridians μ_2, μ_2'. Then

$$q \cup p|M(v_2)\colon V_1 \cup_{T(e_2)} M(v_2) \to S^3$$

is a 2-fold unbranched covering, a contradiction. Hence we have $p^{-1}(m) \cap T(e_2) = \mu_2$. Then we may extend $p|T(e_2)$: $T(e_2) \to T$ to a 2-fold branched covering r: $V_2 \to V$ branched over a core c of V, where V_2 is a solid torus with a meridian μ_2. Then

$$r \cup p|M(v_2): V_2 \cup_{T(e_2)} M(v_2) \to S^3$$

is a 2-fold branched covering branched over c. If $(M(v_2), \mu_2)$ is a knot space-meridian pair, then $V_2 \cup_{T(e_1)} M(v_2) \cong S^3$. This gives a contradiction as in Case 1. Hence M_1 is a solid torus with a meridian μ_2. Therefore we have $V_2 \cup_{T(e_2)} M(v_2) \cong M$. Thus $r \cup p|M(v_2)$ satisfies the conclusion of (ii).

Case 5. $\Gamma = \Gamma_4$.

We may extend a homeomorphism $p|T(e_1)$: $T(e_1) \to T$ to a homeomorphism q: $V_1 \to V$, where V_1 is a solid torus bounded by $T(e_1)$. Then

$$q \cup p|\left(M(v_0) \cup_{T(e_2)} M(v_1)\right): V_1 \cup_{T(e_1)} \left(M(v_0) \cup_{T(e_2)} M(v_1)\right) \to S^3$$

is an unbranched 2-fold covering, a contradiction. Thus Case 5 cannot occur.

Case 6. $\Gamma = \Gamma_5$.,

Let D_i be a compressing disk for $T(e_i)$ in $M - \operatorname{int} M(v_i)$ for $i = 1, 2, 3$. By Lemma 3 we may assume $D_i \subset M(v)$ and $D_i \cap D_j = \varnothing$ for $i \neq j$. We set $\mu_i = \partial D_i$. Since $p(D_i) \subset V$, $m_i = p(\mu_i)$ is a meridian of V. We may assume $m_1 = m_2 = m_3$ $(= m)$. Since $p|M(v_i)$: $M(v_i) \to S^3 - \operatorname{int} V$ is a homeomorphism, $(M(v_i), \mu_i)$ is a knot space-meridian pair. Let λ_i be a longitude of $(M(v_i), \mu_i)$. We may assume $l = p(\lambda_1) = p(\lambda_2) = p(\lambda_3)$. Then l is a longitude of $(S^3 - \operatorname{int} V, m)$. We may extend a homeomorphism $p|T(e_i)$: $T(e_i) \to T$ to a homeomorphism q_i: $V_i \to \overline{V}$, where V_i (resp. $\overline{V}$) is a solid torus with a meridian λ_i (resp. l). Then

$$p|M(v) \cup \left(\bigcup_{i=1}^{3} q_i\right): M(v) \cup_{T(e_1)} V_1 \cup_{T(e_2)} V_2 \cup_{T(e_3)} V_3 \to V \cup_T \overline{V}$$

is a 3-fold irregular branched covering over K in $V \cup_T \overline{V}$ $(\cong S^3)$. As in Case 3 we have

$$M(v) \cup_{T(e_1)} V_1 \cup_{T(e_2)} V_2 \cup_{T(e_3)} V_3 \cong M.$$

Obviously K in $V \cup_T \overline{V}$ is the preimage of K (in $V \cup_T (S^3 - \operatorname{int} V)$) for T. Hence the result follows by induction. This completes the proof. □

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Received June 1, 1982 and in revised form November 22, 1982.

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The Pacific Journal of Mathematics ISSN 0030-8730 is published monthly by the Pacific Journal of Mathematics at P.O. Box 969, Carmel Valley, CA 93924. Application to mail at Second-class postage rates is pending at Carmel Valley, California, and additional mailing offices. Postmaster: Send address changes to Pacific Journal of Mathematics, P. O. Box 969, Carmel Valley, CA 93924.

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Pacific Journal of Mathematics
Vol. 110, No. 2 October, 1984