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Let $\mathcal{H}$ be a separable complex ∞-dimensional Hilbert space and let $\mathcal{F}$ be the Fock space of symmetric tensors over $\mathcal{H}$. We consider non-linear operators T from $\mathcal{H}$ to $\mathcal{F}$ defined on a dense subspace $\mathcal{D}$ in $\mathcal{H}$ with range in $\mathcal{F}$. A symmetry and reality condition is imposed on the operators T under consideration. They are generally unbounded and have different extensions $\tilde{T}$ defined on subspaces $\tilde{\mathcal{D}}$ in $\mathcal{H}$ containing $\mathcal{D}$. Generalizing a result of Arveson for bounded operators (alias functions from $\mathcal{H}$ to $\mathcal{F}$), we show that if T is affiliated with a maximal abelian von Neumann algebra in $B(\mathcal{H})$, then it follows that there is an extension $\tilde{T}$ of T which is unitarily equivalent to a (non-linear) multiplication operator.

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1. Introduction. Work on spectral theory of non-linear filters has been restricted so far to filters f which satisfy one of several possible boundedness conditions. In the expansion into power series $f \sim (f_n)_{n=0}^{\infty}$, when the coefficients f_n are sure functions, the finiteness condition $\Sigma \|f_n\|_{L^2}^2 < \infty$ is customarily assumed (cf. **[Wi]** and **[McK]**), and this is done for important mathematical reasons. Recently, Arveson **[Ar: 1, 2]** established a spectral theorem for non-linear "normal" processes. Here a different boundedness was assumed. The coefficients f_n were treated as operators on the space of symmetric n-tensors, $n = 1, 2, \ldots$, and the boundedness was expressed in terms of the resulting operator norms.

The various boundedness conditions, in the literature, seem to have been dictated by the mathematical framework, rather than the applications.

In this note we consider the spectral representations of non-linear processes where the boundedness condition is dropped.

The mathematical formulation is that of **[Ar: 2]**. We consider non-linear functions T from a given ∞-dimensional complex Hilbert space $\mathcal{H}$

into the symmetric Fock space $\mathcal{F}$ over $\mathcal{H}$. But our function T is only defined on a dense linear subspace $\mathcal{D}$ in $\mathcal{H}$. Generalizing Arveson, we drop the assumption that T be holomorphic (in the weak sense, i.e., for all $n = 1,\ldots,$ all $z_1,\ldots,z_n \in \mathcal{D}$, and $f \in \mathcal{F}$, the function

$$a_1,\ldots,a_n \to (T(a_1z_1 + \cdots + a_nz_n), f)_{\mathcal{F}} \quad [\text{inner product in } \mathcal{F}]$$

is entire analytic in the n complex variables $a_1,\ldots,a_n$).

Moreover, we do not place any boundedness condition on T; but instead, we restrict attention to *symmetric* functions. The symmetry condition is expressed in terms of coherent vectors in $\mathcal{F}$. These are vectors of the form

$$\Omega + z + 2^{-1/2}z \otimes z + \cdots + (n!)^{-1/2}\overbrace{z \otimes \cdots \otimes z}^{n \text{ fold}} + \cdots \tag{1}$$

where Ω is the normalized basis vector in $\mathcal{F}_0$ (ground state), and z is a fixed vector in $\mathcal{H}$. The vector in (1) is denoted by e^z, and we have

$$\|e^z\| = \exp\left(\tfrac{1}{2}\|z\|^2\right).$$

We say that T is *symmetric* if

$$(T(z), e^w) = (e^z, T(w)) \quad \text{for all } z, w \in \mathcal{D}.$$

2. Symmetric functions and holomorphy. Arveson considered *bounded* holomorphic functions T i.e.,

$$\sup_{\|z\| \le 1} \|T(z)\| < \infty.$$

Such a function is said to be *normal* if it is *affiliated* with a maximal abelian von Neumann algebra $\mathcal{A}$ in $B(\mathcal{H})$ (cf. [**Ar**: 2] for the definition of affiliation).

Multiplication operators are examples of such normal functions T. Let (X, μ) be a finite separable measure space, and let $t_n = t_n(x_1,\ldots,x_n)$ be a sequence of measurable symmetric functions (t_n on X^n) with a certain growth condition on $\|t_n\|_\infty$. Let $\mathcal{F}(X, \mu)$ be the symmetric Fock space over $L^2(X, \mu)$. Define

$$T_n(z)(x_1,\ldots,x_n) = t_n(x_1,\ldots,x_n)z(x_1)\cdots z(x_n) \tag{2}$$

for $z \in L^2(X, \mu)$. Then

$$T(z) = T_0(z) + T_1(z) + \cdots$$

is a multiplication operator.

THEOREM A (*Arveson* [**Ar**: 2]). *Let T: $\mathcal{H} \to \mathcal{F}$ be a bounded holomorphic normal function. Then T is unitarily equivalent to a multiplication operator based on some finite separable measure space; and conversely.*

It follows that every T, satisfying the conditions in Arveson's theorem, can be expressed in the form

$$T = F + iG$$

where F and G are commuting (see below) symmetric functions from $\mathcal{H}$ to $\mathcal{F}$. Simply, let F be a unitarily equivalent copy of the multiplication operator, given in (2) with $t_n(x_1,\ldots,x_n)$ replaced by $\operatorname{Re} t_n(x_1,\ldots,x_n)$, and similarly

$$G \approx \{\operatorname{Im} t_n(x_1,\ldots,x_n)\}.$$

A simple application of Segal's duality transform [**Se**: 1, §4] shows that every bounded symmetric function T: $\mathcal{H} \to \mathcal{F}$ extends to a symmetric (i.e., hermitian) *linear* operator $\tilde{T}$ in $\mathcal{F}$. However, even if T is bounded, the extension $\tilde{T}$ may be unbounded.

Suppose a function T from $\mathcal{H}$ to $\mathcal{F}$ can be expressed in the form $T(z) = F(z) + iG(z)$ where each of the functions F and G is symmetric. We then say F and G *commute* if the hermitian linear extensions $\tilde{F}$, resp. $\tilde{G}$, satisfy $(\tilde{F}f, \tilde{G}f) = (\tilde{G}f, \tilde{F}f)$ for all f in the common domain.

It follows that every T, satisfying the conditions in Theorem A, has a decomposition of the above form.

DEFINITION 1. A function T: $\mathcal{H} \to \mathcal{F}(\mathcal{H})$ is said to be strongly bounded if there is a bounded linear operator $\tilde{T}$: $\mathcal{F}(\mathcal{H}) \to \mathcal{F}(\mathcal{H})$ such that

$$\tilde{T}(e^z) = T(z) \quad \text{for all } z \in \mathcal{H}. \tag{3}$$

PROPOSITION 2. *Every strongly bounded function is bounded and holomorphic.*

Proof. Assume T is strongly bounded, and let $\tilde{T}$ be a bounded linear operator in $\mathcal{F}$ which satisfies (3). Then

$$\sup_z \|T(z)\| \leq \|\tilde{T}\| \exp(\tfrac{1}{2})$$

where the supremum is taken over z in $\mathcal{H}$, $\|z\| \leq 1$. Hence T is bounded.

For $f \in \mathfrak{F}$ we have $(T(z), f) = (\tilde{T}(e^z), f) = (e^z, \tilde{T}^*(f))$, and the entire analytic property is clear from the following:

LEMMA 3. *Let $\mathfrak{F}$ be the Fock space over a given complex Hilbert space $\mathfrak{K}$. Then for all $n = 1, 2, \ldots$, all $z_1, \ldots, z_n \in \mathfrak{K}$, and all $f \in \mathfrak{F}$, the function*

$$a_1, \ldots, a_n \to \left(\exp\left(\sum a_i z_i\right), f\right) \tag{4}$$

is entire analytic on $\mathbf{C}^n$.

Proof. There are three steps in the reasoning. Consider first $f = e^w$ for $w \in \mathfrak{K}$, second $f =$ some linear combination of coherent vectors e^w, and third $f \in \mathfrak{F}$, $\|f - f_n\|_{\mathfrak{F}} \to 0$, where f_n is a sequence in the algebraic linear span of the e^w's.

Consider now $f = e^w$, and $z = \sum a_i z_i$. Then

$$(e^z, f) = \sum_0^\infty (n!)^{-1}(z \otimes \cdots \otimes z, w \otimes \cdots \otimes w)_n = \sum_0^\infty (n!)^{-1}(z, w)^n$$

$$= \exp(z, w) = \exp(a_1(z_1, w) + \cdots + a_n(z_n, w)).$$

Hence, if f is chosen as in the first and second steps, the inner product (e^z, f) is entire analytic in the complex variables $a_1, \ldots, a_n$.

In the third step we note that if $\|f - f_n\|_{\mathfrak{F}} \to 0$, then

$$|(e^z, f_n) - (e^z, f)| \leq \exp\left(\frac{1}{2}\left\|\sum a_i z_i\right\|^2\right)\|f_n - f\| \to 0.$$

We have thus an approximation of the function $a_1, \ldots, a_n \to (e^z, f)$, given in (4), with entire analytic scalar functions, and the approximation is uniform on compact subsets of $\mathbf{C}^n$, i.e., the variables $a_1, \ldots, a_n$.

The lemma follows now from the Montel theorem.

At an important point in the proof of our Theorem 5 (§3), Lemma 3 yields a corresponding automatic holomorphic property of unbounded symmetric functions (T from $\mathfrak{K}$ to $\mathfrak{F}$) which are affiliated with a maximal abelian von Neumann algebra $\mathfrak{A} \subset \mathfrak{B}(\mathfrak{K})$.

EXAMPLE 4. Let $\mathfrak{K}$ be a complex Hilbert space with Fock space $\mathfrak{F}$. Let A be a linear operator in $\mathfrak{K}$ and let $\Omega(A)$ be the corresponding "quantized" operator in $\mathfrak{F}$, defined in [**Co**, Definition 3]:

$$\Omega(A) = \sum_{n=0}^{\infty} \sum_{i=1}^{n} A^{\delta(i,1)} \otimes \cdots \otimes A^{\delta(i,n)},$$

where $\delta(i, j) = 1$ if $i = j$, and 0 if $i \neq j$ (we define A^0 to be the identity operators I in $\mathfrak{K}$).

Then define $T_A(z) = \Omega(A)(e^z)$ for all z in the domain of A.

(i) The function T_I: $\mathcal{H} \to \mathcal{F}$ is symmetric, bounded, but *not* strongly bounded. Moreover, T_I *is* holomorphic.

(ii) Let the operator A in $\mathcal{H}$ be hermitian, i.e., $(Az, w) = (z, Aw)$ for all $z, w \in \mathcal{D}(A)$, the domain of A. Then T_A is symmetric and, moreover, T_A is bounded if and only if A is bounded.

Proof. By [**Co**, Theorem 2] the operator $\Omega(A)$ is never bounded in $\mathcal{F}$ when $A \neq 0$. It is unbounded in particular when $A = I$. Note that $\Omega(I) = N$ is the number operator; indeed

$$\Omega(I) = \sum_{n=0}^{\infty}{}^{\oplus} nI^{(n)} = N.$$

The function $z \to (e^z, f)$ is entire analytic on every finite-dimensional subspace $\mathcal{M} \subset \mathcal{H}$, for all fixed vectors f in $\mathcal{F}$, according to Lemma 3. Fixing $\mathcal{M}$ and an orthonormal basis $z_1, \ldots, z_n$, set $z(a) = \sum a_i z_i$ and $f = e^w$. We may then use the chain rule on the composed mapping from

$$a \to m = z(a)\colon \mathbf{C}^n \to \mathcal{M}$$

and

$$m \to (N(e^m), e^w)\colon \mathcal{M} \to \mathbf{C}.$$

Each of the two mappings is entire analytic so the chain rule applies. The analyticity of the second map is clear from the identity

$$(N(e^m), e^w) = \sum n!^{-1} n(m, w)^n = (m, w)\exp((m, w)).$$

The analyticity of $(T_I(z), f)$ for *all* $f \in \mathcal{F}$ now follows from the approximation argument at the end of the proof in Lemma 3. This concludes the proof of (i).

(ii) Suppose first that T_A is bounded. Then we have

$$\sup_z \|T_A(z)\| = M < \infty,$$

where the supremum is taken over all z in $\mathcal{H}$ such that $\|z\| \leq 1$, and e^z is in the domain of $\Omega(A)$. We have

$$\begin{aligned}(T_A(z), e^z) &= \sum (n!)^{-1} n(Az, z)\|z\|^{2(n-1)} \\ &= (Az, z)\exp(\|z\|^2).\end{aligned}$$

Hence:

$$|(Az, z)| = \exp\left(-\|z\|^2\right)|(T_A(z), e^z)|$$
$$\leq \exp\left(-\|z\|^2\right)\|T_A(z)\|\exp\left(\tfrac{1}{2}\|z\|^2\right).$$

Taking the supremum over $z \in \mathcal{D}(A)$, $\|z\| \leq 1$, we get

$$\sup_z |(Az, z)| \leq M.$$

Since A is hermitian, this implies boundedness of A, and $\|A\| \leq M$.

Assume, conversely, that A is bounded. Then

$$\|T_A(z)\|^2 = \sum (n!)^{-1} n \|Az\|^2 \|z\|^{2(n-1)}$$
$$+ \sum (n!)^{-1} n(n-1)(Az, z)^2 \|z\|^{2(n-2)}$$
$$= \left(\|Az\|^2 + (Az, z)^2\right)\exp\left(\|z\|^2\right).$$

It follows that T_A is bounded, and for the norm M we have

$$M \leq \left(2\|A\|^2 e^1\right)^{1/2}.$$

To check that the hermitian symmetry of A implies symmetry of non-linear function T_A, a more general observation is appropriate.

Observation 1. Let T be a function from $\mathcal{H}$ to $\mathcal{F}$ with domain $\mathcal{D}$, and suppose there is a hermitian symmetric operator $\tilde{T}$ in $\mathcal{F}$ such that $e^z \in \mathcal{D}(\tilde{T})$ and $\tilde{T}(e^z) = T(z)$ for all $z \in \mathcal{D}$. Then it follows that T is symmetric.

Proof. Obvious.

Note that the observation applies to $\tilde{T} = \Omega(A)$ for hermitian A, since then $\Omega(A)$ is hermitian in $\mathcal{F}$ by [**Co**, §1].

3. The extension theorem for unbounded symmetric functions. The verbatim parallel in the formulation of the spectral theorem for linear, and non-linear, operators (which is so striking in Arveson's theorem) is broken when the boundedness condition is dropped. Affiliation to a maximal abelian von Neumann algebra in $B(\mathcal{H})$ is not enough to insure a spectral representation (2) when T is unbounded symmetric, and *non-linear*, but is for unbounded *linear* operators, as is reflected in the interesting theorem of Stone [**St**].

THEOREM B (*Stone* [**St**]). *Let T be a densely defined linear operator in a complex Hilbert space $\mathcal{H}$ and assume T is affiliated with a maximal abelian von Neumann algebra $\mathcal{A}$ in $B(\mathcal{H})$.*

Then it follows that T is unitarily equivalent to a multiplication operator (in particular, T is essentially normal, i.e., the closure T^- is a normal operator).

Our main result is an analogy to Theorem B (just as Arveson's theorem generalizes the spectral theorem for bounded linear normal operators). But for unbounded T, additional structure is needed for establishing a spectral representation.

We need the presence of a *conjugation* J on $\mathcal{H}$, i.e., J conjugate-linear, and satisfying $J^2 = I$, and $(Jz, w) = (Jw, z)$, $z, w \in \mathcal{H}$.

We have

THEOREM C (*von Neumann* [**vN**; **RS**]). *Let S be a hermitian symmetric linear operator which commutes with a conjugation in a Hilbert space $\mathcal{H}$. Then it follows that S has selfadjoint extensions in $\mathcal{H}$.*

A given conjugation J on $\mathcal{H}$ clearly extends to a conjugation $\tilde{J}$ on Fock space $\mathcal{F}$ over $\mathcal{H}$ (by direct summing appropriate tensor powers of J).

We say that a function T from $\mathcal{H}$ to $\mathcal{F}$ *commutes with* J if the domain $\mathcal{D}$ of T is J-invariant and

$$T(Jz) = \tilde{J}T(z) \quad \text{for all } z \in \mathcal{D}.$$

THEOREM 5. *Let $\mathcal{H}$ be a separable complex Hilbert space (∞-dimensional), $\mathcal{F}$ the symmetric Fock space over $\mathcal{H}$, J a conjugation in $\mathcal{H}$, T a symmetric function from $\mathcal{H}$ to $\mathcal{F}$ with dense domain, and $\mathcal{A} \subset B(\mathcal{H})$ a maximal abelian von Neumann algebra. Assume*

(i) *$JAJ = A^*$ for all $A \in \mathcal{A}$.*

(ii) *T commutes with J.*

(iii) *T is affiliated with $\mathcal{A}$.*

Then it follows that T has a selfadjoint extension, unitarily equivalent to a multiplication operator, which is affiliated with $\mathcal{A}$. There is a spectral representation (2) *based on some finite separable measure space (X, μ) and on a sequence $\{t_n(x_1, \ldots, x_n)\}_{n=0}^{\infty}$ of measurable symmetric functions with values in the reals.*

Proof. We first note that T extends canonically to the linear subspace in $\mathcal{F}$ which is spanned by the (coherent) vectors $\{e^z : z \in \mathcal{D}\}$. We denote

this subspace by $E(\mathcal{D})$ and note that it is dense in the Fock space $\mathcal{F}$ over $\mathcal{H}$. Indeed, the density follows from a known ([**Se** 1, Theorem 3]) and easy algebraic argument coupled with the density of $\mathcal{D}$ in $\mathcal{H}$.

The following is a direct converse to Observation 1 in §2.

Observation 1′. There is a unique hermitian, symmetric, linear operator $\tilde{T}$ in $\mathcal{F}$ which is defined on $E(\mathcal{D})$ and satisfies

$$\tilde{T}(e^z) = T(z) \quad \text{for } z \in \mathcal{D}. \tag{5}$$

Proof. Consider a finite linear combination $f = \Sigma_k c_k e^{z_k}$, where $c_k \in \mathbf{C}$, and $z_k \in \mathcal{D}$ for $k = 1, 2, \ldots$. If $\tilde{T}$ is given as a linear operator satisfying (5), then necessarily

$$\tilde{T}(f) = \sum_k c_k T(z_k). \tag{6}$$

Hence, the uniqueness!

But it is possible to define a linear operator $\tilde{T}$ via the formula (6). For if $f = \Sigma c_k e^{z_k} = 0$, then

$$\begin{aligned}\left(\sum_k c_k T(z_k), e^w\right) &= \sum_k c_k (T(z_k), e^w) \\ &= \sum_k c_k (e^{z_k}, T(w)) = \left(\sum_k c_k e^{z_k}, T(w)\right) \\ &= (0, T(w)) = 0 \quad \text{for all } w \in \mathcal{D}.^{1}\end{aligned}$$

But the vectors in $\mathcal{F}$ of the form e^w span a dense subspace in $\mathcal{F}$, and it follows that $\tilde{T}$ is well defined through formula (6). If we consider $f = \Sigma c_j e^{z_j}$, and $g = \Sigma d_k e^{w_k}$ ($c_j, d_k \in \mathbf{C}$, $z_j, w_k \in \mathcal{D}$), then the argument used in showing that T is well defined as a linear operator with domain $E(\mathcal{D})$ yields the identity

$$(\tilde{T}(f), g) = (f, \tilde{T}(g)).$$

---

[1] Note that the extension argument works more generally for any function T from $\mathcal{H}$ to $\mathcal{F}$ (symmetric or not) which satisfies $(T(z), e^w) = (T(Jw), e^{J(z)})$ for all z, $w \in \mathcal{D}$. The resulting linear extension operator $\tilde{T}$ with domain $E(\mathcal{D}) \subset \mathcal{F}$ is formally normal, and T decomposes, $T = F + iG$, as a sum of commuting symmetric functions F, G, each commuting with J. If selfadjoint commuting extensions $\tilde{F}$ and $\tilde{G}$ can be found, for example by [**Ne**, Corollary 9.1], or [**St**] Theorem 21, alias Theorem B above, then the resulting joint spectral representation can be restricted to a spectral representation for T.

Indeed, the verification reduces to the identity

$$\big(T(z_j), e^{w_k}\big) = \big(e^{z_j}, T(w_k)\big)$$

in view of the sesquilinearity of the inner product $(\cdot, \cdot)$. Hence, the symmetry of T carries over to the extension $\tilde{T}$.

Observation 2. If U is a unitary operator in $\mathcal{H}$ satisfying

$$T(Uz) = \Gamma(U)T(z) \tag{7}$$

for all $z \in \mathcal{D}$, then it follows that $\Gamma(U)$ commutes with $\tilde{T}$.

Proof. It is assumed that T is symmetric, and $\tilde{T}$ denotes the linear hermitian symmetric extension constructed in Observation 1′. For $f \in E(\mathcal{D}), f = \sum c_j e^{z_j}$ ($c_j \in \mathbf{C}$, $z_j \in \mathcal{D}$), we have

$$\Gamma(U)f = \sum c_j \Gamma(U)(e^{z_j}) = \sum c_j e^{U(z_j)}.$$

Since U commutes with T it follows that the domain $\mathcal{D}$ is invariant under U. Hence, $\Gamma(U)(f) \in E(\mathcal{D})$, and

$$\begin{aligned}\tilde{T}\Gamma(U)(f) &= \tilde{T}\Big(\sum c_j e^{U(z_j)}\Big) = \sum_j c_j T\big(U(z_j)\big) \\ &= \sum_j c_j \Gamma(U)\big(T(z_j)\big) = \Gamma(U)\tilde{T}(f).\end{aligned}$$

The commutativity of the linear operators $\Gamma(U)$ and $\tilde{T}$ follows.

Since commutativity for linear operators is preserved under closure, it follows that the closure of $\tilde{T}$ commutes with $\Gamma(U)$. We shall therefore, in the sequel, use the same symbol $\tilde{T}$ for $(\tilde{T})^-$.

We now have commutativity of $\tilde{T}$ and $\Gamma(U)$ for all unitaries U in the commutant of

$$\mathcal{A} \text{ in } B(\mathcal{H})\colon U \in \mathcal{A}' = \{X \in B(\mathcal{H})\colon XA = AX \text{ for all } A \in \mathcal{A}\}.$$

Since $\mathcal{A}$ is maximal abelian, $\mathcal{A} = \mathcal{A}'$.

Observation 3. There is a selfadjoint extension operator $\tilde{T}_1$ of $\tilde{T}$ which commutes with $\Gamma(U)$ for all unitaries U in $\mathcal{A}$.

Proof. By general theory the given conjugation J on $\mathcal{H}$ extends to a conjugation $\tilde{J}$ on $\mathcal{F}$ (by direct summing appropriate tensor powers of J (cf. [**Se**: 1, 2])). By assumption we have

$$T(Jz) = \tilde{J}T(z) \quad \text{for } z \in \mathcal{D}, \tag{8}$$

and

$$JAJ = A^* \quad \text{for } A \in \mathcal{Q}. \tag{9}$$

It follows from Observation 2 and (8) that $\tilde{J}$ commutes with $\tilde{T}$. The conjugation $\tilde{J}$ restricts to an isometry of the space $\mathcal{F}_+ = \{f_+ \in \mathcal{F}: \tilde{T}^* f_+ = if_+\}$ onto

$$\mathcal{F}_- = \{f_- \in \mathcal{F}: \tilde{T}^* f_- = -if_-\},$$

the von Neumann deficiency spaces for $\tilde{T}$. Consider the closed linear subspaces $\mathcal{F}_{\mathcal{Q}}(f_\pm)$ generated by $\{\Gamma(U)f_\pm : U \in \mathcal{Q} \text{ unitary}\}$, where $f_\pm$ denotes a pair of vectors in the respective spaces $\mathcal{F}_+$ and $\mathcal{F}_-$. We have

$$\tilde{J}\Gamma(U)f_+ = \Gamma(JUJ)\tilde{J}f_+ = \Gamma(U^*)\tilde{J}f_+ \in \mathcal{F}_{\mathcal{Q}}(\tilde{J}f_+), \tag{10}$$

and it follows that the vectors f_+ and $\tilde{J}f_+$ occur in pairs.

If $\mathcal{F}_+ = 0$, then $\tilde{T}$ is already selfadjoint and there is nothing to prove. If not, we may choose a family of normalized vectors $f_+ \in \mathcal{F}_+$ such that the spaces $\mathcal{F}_{\mathcal{Q}}(f_+)$ form a maximal orthogonal family in $\mathcal{F}_+$. It follows from (9) and (10) that the corresponding family $\mathcal{F}_{\mathcal{Q}}(\tilde{J}f_+)$ is maximally orthogonal in $\mathcal{F}_-$. For finite linear combinations

$$\varphi_+ = \sum\sum c_{jk}\Gamma(U_k)f_j^{(j)} \tag{11}$$

with $c_{jk} \in \mathbf{C}$, $U_j \in \mathcal{Q}'$, $j = 1, 2, \ldots$, we may define

$$\tilde{W}(\varphi_+) = \sum\sum c_{jk}\Gamma(U_k)\tilde{J}f_+^{(j)}. \tag{12}$$

We claim $\tilde{W}$ is well defined as a partial isometry of the orthogonal direct sum $\Sigma^{\oplus}\mathcal{F}_{\mathcal{Q}}(f_+)$ onto $\Sigma^{\oplus}\mathcal{F}_{\mathcal{Q}}(\tilde{J}f_+)$, and $\tilde{W}$ commutes with $\Gamma(U)$ for all unitaries U in $\mathcal{Q}$. Finally, we have $\Sigma^{\oplus}\mathcal{F}_{\mathcal{Q}}(f_+) = \mathcal{F}_+$ and $\Sigma^{\oplus}\mathcal{F}_{\mathcal{Q}}(\tilde{J}f_+) = \mathcal{F}_-$ by the maximality of the chosen family $\{f_+\}$. The reader is refered to [**Jo**] and [**Sl**] for details of the proof at this point.

The essential step in the above argument is the following identity:

$$\begin{aligned}
\left\| \sum\sum c_{jk}\,\Gamma(U_k)\tilde{J}f_+^{(j)} \right\|^2 &= \sum\sum\sum c_{jk}\bar{c}_{js}\left(\Gamma(U_s^*U_k)\tilde{J}f_+^{(j)}, \tilde{J}f_+^{(j)}\right) \\
&= \sum\sum\sum c_{jk}\bar{c}_{js}\overline{\left(\Gamma(JU_s^*U_kJ)f_+^{(j)}, f_+^{(j)}\right)} \\
&= \sum\sum\sum c_{jk}\bar{c}_{js}\overline{\left(\Gamma(U_k^*U_s)f_+^{(j)}, f_+^{(j)}\right)} \\
&= \sum\sum\sum c_{jk}\bar{c}_{js}\overline{\left(\Gamma(U_s)f_+^{(j)}, \Gamma(U_k)f_+^{(j)}\right)} \\
&= \left\| \sum\sum c_{jk}\Gamma(U_k)f_+^{(j)} \right\|^2.
\end{aligned}$$

Commutativity of U_k and $U_s \in \mathcal{Q}$ is used.

Returning to formulas (11) and (12), we see that

$$\|\tilde{W}(\varphi_+)\| = \|\varphi_+\|.$$

The asserted properties of $\tilde{W}$ follow quite easily from this.

Let $\tilde{T}_1$ be the selfadjoint operator in $\mathfrak{F}$ which extends $\tilde{T}$ and has Cayley transform

$$\tilde{W} \oplus (\tilde{T} - i)(\tilde{T} + i)^{-1} = \tilde{W}_1.$$

Then $\tilde{T}_1$ is the desired extension operator. It is selfadjoint since it has deficiency indices ([**RS**]) equal to $(0, 0)$. Indeed, it is the inverse Cayley transform of the isometry W with initial space, and final space, equal to $\mathfrak{F}$.

Moreover, the operator F: $\mathfrak{K} \to \mathfrak{F}$ defined by $F(z) = \tilde{W}_1(e^z)$ satisfies the assumptions in Arveson's theorem (Theorem A). Indeed, the boundedness and the analyticity are clear from Proposition 2 and Lemma 3. For $U \in \mathfrak{A}$ (unitary) and $z \in \mathfrak{D}$, we have

$$F(Uz) = \tilde{W}_1(e^{U(z)}) = \tilde{W}_1\Gamma(U)(e^z) = \Gamma(U)\tilde{W}_1(e^z) = \Gamma(U)F(z).$$

There is then, by Theorem A, a unitary isomorphism R of $\mathfrak{K}$ onto $L^2(X, \mu)$ (for some finite separable measure space (X, μ)) such that $G = \tilde{R}FR^*$ is a multiplication operator. Let $G(u) = G_0(u) + G_1(u) + \cdots$ be the decomposition of G in

$$\mathfrak{F}(X, \mu) := L^2(X^0, \mu^0) \oplus L^2(X, \mu) \oplus L_s^2(X^2, \mu^2) \oplus \cdots.$$

Then

$$G_n(u)(x_1, \ldots, x_n) = g_n(x_1, \ldots, x_n)u(x_1) \cdots u(x_n)$$

where the function g_n on X^n satisfies

$$|g_n(x_1, \ldots, x_n)| = 1 \quad \text{for a.e. } x_1, \ldots, x_n.$$

The function

$$t_n(x_1, \ldots, x_n) = i(1 + g_n)(1 - g_n)^{-1}(x_1, \ldots, x_n)$$

is defined, a.e., in X^n, $n = 1, 2, \ldots$. Indeed, $N(I - \tilde{W}_1) = N(I - \tilde{R}\tilde{W}_1\tilde{R}^*) = \{0\}$. From easy properties of multiplication operators, it follows that the set $\{(x_1, \ldots, x_n) \mid g_n(x_1, \ldots, x_n) = 1\}$ has μ^n-measure zero in X^n. As a result we note that the family of functions $\{t_n\}$ define a selfadjoint multiplication operator in $\mathfrak{F}(X, \mu)$ which extends the symmetric operator $\tilde{R}\tilde{T}\tilde{R}^*$. Therefore, the original non-linear operator T: $\mathfrak{D} \to \mathfrak{F}$ has an extension which is unitarily equivalent to a multiplication operator and the proof is completed.

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