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Let X be a semi-locally contractible metrizable space. We show that X is locally equi-connected (LEC) if and only if X has a local mixer introduced by van Mill and van de Vel $[MV_{1,2}]$.

Throughout this paper, all spaces are metrizable and maps are continuous. Let X be a space. We will use the same symbol ΔX to denote the diagonals of X^2 and X^3 , that is,

$$\Delta X = \{ (x, x) : x \in X \} \text{ or } = \{ (x, x, x) : x \in X \},\$$

and we will let

$$\Delta^* X = \{ (x, y, z) \in X^3 \colon x = y \text{ or } y = z \text{ or } z = x \}$$
$$= \bigcup_{x \in X} (X \times \{x\} \times \{x\} \cup \{x\} \times X \times \{x\} \cup \{x\} \times \{x\} \times X)$$

A local mixer for X is a map $\mu: U \to X$ of a neighborhood U of $\Delta^* X$ in X^3 to X which satisfies the following condition:

if $((x_n, y_n, z_n))_{n=1}^{\infty}$ is a sequence of points in X^3 such that the sequences $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ both converge to $a \in X$, then the sequences $(\mu(x_n, y_n, z_n))_{n=m}^{\infty}$, $(\mu(x_n, z_n, y_n))_{n=m}^{\infty}$ and $(\mu(z_n, x_n, y_n))_{n=m}^{\infty}$ converge to a for some m;

or, equivalently, (see [MV₂, Lemma 2.3]):

for each $x \in X$ and for each neighborhood V of x, there exists a neighborhood W of x such that

(#)
$$E(W) = (X \times W \times W) \cup (W \times X \times W)$$
$$\cup (W \times W \times X) \subset \mu^{-1}(V),$$
that is, $E(W) \subset U = \operatorname{dom} \mu$ and $\mu(E(W)) \subset V.$

When $U = X^3 = \text{dom } \mu$, we call μ a *mixer* for X. If X is compact, then (*) (or (#)) is equivalent to the condition that

$$\mu(x, x, y) = \mu(x, y, x) = \mu(y, x, x) = x$$

for all $x, y \in X$. The concept of a (local) mixer for a (compact) metric space was introduced by van Mill and van de Vel [$MV_{1,2}$].

We say that X is *locally equi-connected* (*LEC*) provided there is a map $\lambda: U \times [0, 1] \rightarrow X$, where U is a neighborhood of ΔX in X^2 , such that

$$\lambda(x, y, 0) = x, \quad \lambda(x, y, 1) = y \quad \text{for all } (x, y) \in U,$$

$$\lambda(x, x, t) = x \quad \text{for all } x \in X, t \in [0, 1];$$

the map λ is called a (*local*) equi-connecting function. When $U = X^2$, we say that X is equi-connected (EC). These concepts were introduced by Fox [F] (cf. [S], [H], [D] and [C]).

A space X is said to be *semi-locally contractible* if each point of X has a neighborhood which is contractible in X; a space X is said to be *semi-locally path-connected* if each point of X has a neighborhood whose any two points can be joined by a path in X. In $[\mathbf{MV}_{1,2}]$, van Mill and van de Vel showed that

(i) each semi-locally path-connected space admitting a (local) mixer is $(LC^{\infty}) C^{\infty}$;

(ii) each A(N)R has a (local) mixer; and

(iii) each contractible space admitting a mixer is EC and each semi-locally contractible *separable* space with a local mixer is LEC.

In this paper, we will show that each (L)EC space has a (local) mixer (Theorem I), and each semi-locally contractible space admitting a local mixer is LEC (Theorem II). These results generalize (ii) and (iii). From these results we obtain the following characterization of (local) equi-connectedness.

THEOREM. A metrizable space is (L)EC if and only if it is (semi-locally) contractible and has a (local) mixer.

First, we will prove the following:

THEOREM I. If X is LEC, then X has a local mixer. If X is EC, then X has a mixer.

Proof. Let U be a neighborhood of ΔX in X^2 and λ : $U \times [0, 1] \to X$ an equi-connecting function. For each $a \in X$, let U'_a be a neighborhood of a in X so that $U'^2_a \subset U$. There is an open neighborhood U''_a of a such that $U''_a \subset U'_a$ and $\lambda(U''^2_a \times [0, 1]) \subset U'_a$ [**D**, Lemma 2.3]. By [**BP**, Ch. II, Theorem 4.1], there exists a metric d on X compatible with the topology of X and such that $\{\{x \in X : d(x, a) \le 1\}: a \in X\}$ refines $\{U_a'': a \in X\}$. We define a metric d^* on X^3 by

$$d^*((x, y, z), (x', y', z')) = \max\{d(x, x'), d(y, y'), d(z, z')\}$$

Then d^* induces the product topology of X^3 .

For each $n \in \mathbb{N}$, define

$$V_0(n) = \{(x, y, z) \in X^3 : d^*((x, y, z), \Delta X) \le 1/n\}.$$

Observe that for each $(x, y, z) \in V_0(1)$ and for each $s, t \in [0, 1]$, $\lambda(\lambda(x, y, s), z, t)$ is well defined. Put

$$X_{1} = \bigcup_{a \in X} X \times \{a\} \times \{a\},$$

$$X_{2} = \bigcup_{a \in X} \{a\} \times X \times \{a\},$$

$$X_{3} = \bigcup_{a \in X} \{a\} \times \{a\} \times X,$$

and for each $n \in \mathbb{N}$ and for i = 1, 2, 3, define

$$V_i(n) = \{(x, y, z) \in X^3 : d^*((x, y, z), X_i) \le 1/4n\}.$$

Then we have a neighborhood

$$V = V_0(1) \cup V_1(1) \cup V_2(1) \cup V_3(1)$$

of $\Delta X = X_1 \cup X_2 \cup X_3$. For i = 1, 2, 3, put

$$Y_i = \bigcup_{n \in \mathbf{N}} (V_i(n) \setminus \operatorname{int} V_0(n)).$$

Since $V_i(n) \setminus \operatorname{int} V_0(n-1) \subset V_i(n-1) \setminus \operatorname{int} V_0(n-1)$ for each n > 1, it follows that

$$Y_i = \bigcup_{n>1} \left(V_i(n) \cap \left(V_0(n-1) \setminus \operatorname{int} V_0(n) \right) \right) \cup \left(V_i(1) \setminus \operatorname{int} V_0(1) \right),$$

that is, Y_i is a union of closed sets which is locally finite in $V \setminus \Delta X$. Thus Y_i is a closed neighborhood of $X_i \setminus \Delta X$ in $V \setminus \Delta X$. Moreover, Y_1 , Y_2 and Y_3 are mutually disjoint. Indeed, if not, then we can assume without loss of generality that there exists a point

$$(x, y, z) \in (V_1(n) \setminus \operatorname{int} V_0(n)) \cap (V_2(m) \setminus \operatorname{int} V_0(m))$$

for some $n \leq m$. Then

$$d^*((x, y, z), (b, a, a)) \le 1/4n$$

$$d^*((x, y, z), (a', b', a')) \le 1/4m \le 1/4n$$

for some $a, a', b, b' \in X$. Since

 $d(x, a) \le d(x, a') + d(a', z) + d(z, a) \le 3/4n < 1/n,$

we have $d^*((x, y, z), (a, a, a)) < 1/n$; hence $(x, y, z) \in int V_0(n)$. This is a contradiction.

Let $f, g: V \setminus \Delta X \to [0, 1]$ be Urysohn functions such that

$$f(Y_2 \cup Y_3) = 0$$
, $f(Y_1) = 1$; $g(Y_3) = 0$, $g(Y_1 \cup Y_2) = 1$.

We define $\mu: V \to X$ by

$$\mu(x, y, z) = \begin{cases} \lambda(\lambda(x, y, f(x, y, z)), z, g(x, y, z)) & \text{if } (x, y, z) \in V_0(1) \setminus \Delta X, \\ z & \text{if } (x, y, z) \in Y_1 \cup Y_2, \\ x & \text{if } (x, y, z) \in Y_3 \cup \Delta X. \end{cases}$$

Clearly, μ is well defined and continuous at each point of $V \setminus \Delta X$. We will show the condition (#), which implies μ is continuous at each point of ΔX and μ is a local mixer for X.

Let $a \in X$ and W be a neighborhood of a. By [D, Lemma 2.3] there exists a neighborhood W'' of a such that $W'' \subset \{x \in X : d(x, a) \le 1\}$ and

 $\lambda(\lambda(W''^2 \times [0,1]) \times W'' \times [0,1]) \subset W.$

Note that $W''^3 \subset V$ and $\mu(W''^3) \subset W$. Choose n > 1 so that

 $\{x \in X: d(x, a) \le 1/n + 1/4n\} \subset W''$

and put

 $W' = \{x \in X: d(x, a) \le 1/4n\}.$

Then it follows that

$$E(W') = (X \times W' \times W') \cup (W' \times X \times W') \cup (W' \times W' \times X)$$

$$\subset V_1(n) \cup V_2(n) \cup V_3(n) \subset V.$$

Observe that $E(W') \cap V_0(n) \subset W''^3$. It follows that $\mu(E(W') \setminus V_0(n)) \subset W$. Since $(E(W') \setminus V_0(n)) \cap V_i(n) \subset Y_i$ for i = 1, 2, 3, it is easily seen that $\mu(E(W') \setminus V_0(n)) \subset W$. Therefore $\mu(E(W')) \subset W$.

In the case U = X, using a metric d on X such that the diameter of X with respect to d is less than 1, we have a mixer μ for X because V = X. \Box

Next, improving the technique of van Mill and van de Vel $[MV_2, Theorem 3.1]$, we will prove the following without separability:

THEOREM II. Let X be a semi-locally contractible space. If X has a local mixer, then X is LEC.

Proof. Let μ be a local mixer for X. Using the semi-local contractibility of X and the A. H. Stone Theorem (e.g., see [**BP**, Ch. II Theorem 2.1]), X has a locally finite σ -discrete cover $\mathfrak{A} = \bigcup_{n \in \mathbb{N}} \mathfrak{A}_n$ by open sets contractible within X, where each \mathfrak{A}_n is discrete in X. For each $U \in \mathfrak{A}$, let $F_U: U \times [0, 1] \to X$ be a contraction of U onto some $x_U \in X$, that is,

$$F_U(x,0) = x$$
 and $F_U(x,1) = x_U$ for all $x \in U$.

Take a closed cover $\{A(U): U \in \mathfrak{A}\}$ and an open cover $\{B(U): U \in \mathfrak{A}\}$ of X so that

$$A(U) \subset B(U) \subset \operatorname{cl} B(U) \subset U$$

for each $U \in \mathcal{A}$. For each $U \in \mathcal{A}$, let $f_U: X \to [0, 1]$ be a Urysohn function such that

$$f_U(\operatorname{cl} B(U)) = 1$$
 and $f_U(X \setminus U) = 0$.

For each $n \in \mathbb{N}$, we define a homotopy $F^n: X \times [0, 1] \to X$ by

$$F^{n}(x,t) = \begin{cases} F_{U}(x, f_{U}(x) \cdot t) & \text{if } x \in U \in \mathcal{A}_{n}, \\ x & \text{otherwise.} \end{cases}$$

Since \mathfrak{A} is locally finite and each \mathfrak{A}_n is discrete, there exists an open cover \mathbb{V} of X each element of which meets at most finitely many elements of \mathfrak{A} and at most one element of \mathfrak{A}_n for each $n \in \mathbb{N}$. For each $V \in \mathbb{V}$, let k(V) be the number of $U \in \mathfrak{A}$ with $V \cap U \neq \emptyset$. And for each $0 \le i \le k(V)$, let $\mathfrak{A}(V, i)$ be an open cover of V such that

(1)
$$\mathfrak{W}(V, k(V)) \leq_{\mu} \mathfrak{W}(V, k(V) - 1) \leq_{\mu} \cdots \leq_{\mu} \mathfrak{W}(V, 1) \leq_{\mu} \mathfrak{W}(V, 0) = \{V\};$$

(2) if $W \in \mathfrak{V}(V, i)$, $U \in \mathfrak{A}$ and $W \cap A(U) \neq \emptyset$, then $W \subset B(U)$,

where $\mathfrak{W}' \leq_{\mu} \mathfrak{W}$ means that for each $W' \in \mathfrak{W}'$ there is some $W \in \mathfrak{W}$ such that

$$E(W') = (X \times W' \times W') \cup (W' \times X \times W')$$

 $\cup (W' \times W' \times X) \subset \mu^{-1}(W),$

where this relation is denoted by $W' \subset_{\mu} W$. Now we have an open neighborhood,

$$W^* = \bigcup \{ W^2 \colon W \in \mathfrak{V}(V, k(V)) \text{ for some } V \in \mathfrak{V} \},\$$

of ΔX in X^2 .

We will construct two maps

$$G, H: W^* \times [1, \infty) \to X$$

as follows:

$$G(x, y, 1) = x;$$
 $H(x, y, 1) = y;$

and for $t \in [n, n + 1]$, inductively,

$$G(x, y, t) = \mu(G(x, y, n), H(x, y, n), F^n(G(x, y, n), t - n));$$

$$H(x, y, t) = \mu(G(x, y, n), H(x, y, n), F^n(H(x, y, n), t - n)).$$

We will show that G and H are well defined and for each $W \in \mathfrak{V}(V, k(V))$, $V \in \mathfrak{V}$, there is some $n(W) \in \mathbb{N}$ such that

(3) if
$$x, y \in W$$
 and $t \ge n(W)$, then
 $G(x, y, t) = G(x, y, n(W)) = H(x, y, n(W)) = H(x, y, t).$

To this end, let $V \in \mathbb{V}$ and $W \in \mathfrak{W}(V, k(V))$. Set

$$\{U \in \mathfrak{A} \colon V \cap U \neq \emptyset\} = \{U_i \colon i = 1, 2, \dots, k(V)\},$$
$$U_i \in \mathfrak{A}_{n(i)} \quad \text{for } i = 1, 2, \dots, k(V),$$
$$n(1) < n(2) < \dots < n(k(V)).$$

By (1), take $W_i \in \mathfrak{M}(V, i)$, $i = 0, 1, \dots, k(V)$, so that

(4)
$$W = W_{k(V)} \subset_{\mu} W_{k(V)-1} \subset_{\mu} \cdots \subset_{\mu} W_1 \subset_{\mu} W_0 = V.$$

Since $W_{k(V)} \cap U = \emptyset$ for any $U \in \mathfrak{A}_n$, n < n(1), it follows that G and H are well defined on $W^2 \times [1, n(1)]$ and

G(x, y, t) = x, H(x, y, t) = y for each $(x, y, t) \in W^2 \times [1, n(1)]$. Suppose G and H are well defined on $W^2 \times [1, n(i)]$ and

(5)
$$G(W^2 \times [1, n(i)]) \cup H(W^2 \times [1, n(i)]) \subset W_{k(V)-i+1}.$$

From the definition and (4), it follows that G and H are well defined on $W^2 \times [n(i), n(i) + 1]$ and

$$G(W^2 \times [n(i), n(i) + 1]) \cup H(W^2 \times [n(i), n(i) + 1]) \subset W_{k(V)-i}.$$

Since $W_{k(V)-i} \cap U = \emptyset$ for any $U \in \mathfrak{A}_n$, n(i) < n < n(i+1), it follows that G and H are well defined on $W^2 \times [n(i) + 1, n(i+1)]$ and

$$G(x, y, t) = G(x, y, n(i) + 1), \quad H(x, y, t) = H(x, y, n(i) + 1)$$

for each $(x, y, t) \in W^2 \times [n(i) + 1, n(i + 1)],$

where we consider $n(k(V) + 1) = \infty$ and $[1, \infty] = [1, \infty)$. Thus, by induction, we conclude that G and H are well defined on $W^2 \times [1, \infty)$ and

satisfy condition (5) for all i = 1, 2, ..., k(V). Next, we will find $n(W) \in \mathbb{N}$. Since $\{A(U): U \in \mathfrak{A}\}$ covers X, there is a $U \in \mathfrak{A}$ such that $W \cap A(U) \neq \emptyset$, however $U = U_{i_0}$ for some $i_0 = 1, 2, ..., k(V)$ because $V \cap U \neq \emptyset$. Thus we have some $i_0 = 1, 2, ..., k(V)$ such that

$$W_{k(V)-i_0+1}\cap A(U_{i_0})\neq \emptyset.$$

From (2) and (5), it follows that

$$G(W^2 \times \{n(i_0)\}) \cup H(W^2 \times \{n(i_0)\}) \subset B(U_{i_0}) \subset U_{i_0} \in \mathfrak{A}_{n(i_0)}$$

Recall that $f_{U_{i_0}}(\text{cl } B(U_{i_0})) = 1$. This implies

$$F^{n(i_0)}(B(U_{i_0}) \times \{1\}) = F_{U_{i_0}}(B(U_{i_0}) \times \{1\}) = x_{U_{i_0}}.$$

Hence from the definition, we have

$$G(x, y, n(i_0) + 1) = H(x, y, n(i_0) + 1)$$

= $\mu(G(x, y, n(i_0)), H(x, y, n(i_0)), x_{U_{i_0}})$

for each x, $y \in W$. Put $n(W) = n(i_0) + 1$. Then (3) follows from the definition and the property of a (local) mixer.

Now we define $G', H': W^* \times [0, 1] \to X$ by

$$G'(x, y, t) = \begin{cases} G(x, y, 1/t) & \text{if } t \neq 0, \\ G(x, y, n(W)) & \text{if } t = 0 \text{ and } x, y \in W; \end{cases}$$
$$H'(x, y, t) = \begin{cases} H(x, y, 1/t) & \text{if } t \neq 0, \\ H(x, y, n(W)) & \text{if } t = 0 \text{ and } x, y \in W. \end{cases}$$

From (3) these are obviously continuous and

$$G' | W^* \times \{0\} = H' | W^* \times \{0\}.$$

Thus we have an equi-connecting function $\lambda: W^* \times [0, 1] \to X$ defined by

$$\lambda(x, y, t) = \begin{cases} G'(x, y, 1 - 2t) & \text{if } 0 \le t \le \frac{1}{2}, \\ H'(x, y, 2t - 1) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

In the following, we will consider after J. Dugundji [D] a condition that a space with a local mixer is an ANR. Let μ be a (local) mixer for a space X. For $A \subset X$ define $A^{(\mu,1)} = \mu(A^3)$ when $A^3 \subset \text{dom } \mu$, and inductively, define $A^{(\mu,n+1)} = \mu((A^{(\mu,n)})^3)$ when $(A^{(\mu,n)})^3 \subset \text{dom } \mu$. We define $A^{(\mu,\infty)} = \bigcup_{n \in \mathbb{N}} A^{(\mu,n)}$ if each $A^{(\mu,n)}$ is well defined. For $A \subset B$ ($\subset X$), we say that A is μ -stable in B provided $A^{(\mu,\infty)}$ is well defined and $A^{(\mu,\infty)} \subset B$. COROLLARY. Let X be semi-locally contractible. If X has a local mixer μ with the property:

(**) for each $x \in X$ and each neighborhood W of x, there is neighborhood V of x which is μ -stable in W,

then X is an ANR.

Proof. From Theorem II, X is LEC. By [**D**, Theorem 3.2], we may show that each open cover \mathfrak{W} of X has an open refinement \mathfrak{V} such that every partial realization $f: K^0 \to X$ in \mathfrak{V} of the 0-skeleton of any polytope K extends to a full realization of K in \mathfrak{W} . This follows from (**) and the following lemma:

LEMMA. Let X be semi-locally path-connected and have a local mixer μ . Assume that an open cover \mathfrak{W} of X has an open refinement \mathfrak{V} such that each $V \in \mathfrak{V}$ is μ -stable in some $W \in \mathfrak{W}$. Then every partial realization f: $K^0 \to X$ in \mathfrak{V} of the 0-skeleton of any polytope K extends to a full realization of K in \mathfrak{W} .

Proof. We define an extension of f over K by induction on the skeletons of K. Assume f has been extended to a map $f_n: K^n \to X$ so that

$$f_n(\sigma) \subset \bigcap \left\{ V^{(\mu,n)} \colon f(\sigma \cap K^0) \subset V \in \mathfrak{V} \right\}$$

for each closed simplex σ of K^n , where K^n denotes the *n*-skeleton of *K*. We denote the closed unit (n + 1)-ball and unit *n*-sphere in \mathbb{R}^{n+1} by \mathbb{B}^{n+1} and \mathbb{S}^n , respectively. Let τ be any (n + 1)-simplex and h_{τ} : $\mathbb{B}^{n+1} \to \tau$ a fixed homeomorphism. Note that

$$f_n h_{\tau}(\mathbf{S}^n) = f_n(\partial \tau) \subset \bigcap \left\{ V^{(\mu,n)} \colon f(\tau \cap K^0) \subset V \in \mathfrak{V} \right\}.$$

Using the technique of $[\mathbf{MV}_1$, Theorem 1.3], we have an extension g_{τ} : $\mathbf{B}^{n+1} \to X \text{ of } f_n h_{\tau} | \mathbf{S}^n$ such that

$$g_{\tau}(\mathbf{B}^{n+1}) \subset \bigcap \{ V^{(\mu,n+1)} \colon f(\tau \cap K^0) \subset V \in \mathbb{V} \}.$$

Define a map f_{n+1} : $K^{n+1} \to X$ by

$$f_{n+1} \mid \tau = g_{\tau} h_{\tau}^{-1} \colon \tau \to X$$

on each (n + 1)-simplex τ of K. Then f_{n+1} is an extension of f_n such that

$$f_{n+1}(\tau) \subset \bigcap \{ V^{(\mu,n+1)} : f(\tau \cap K^0) \subset V \in \mathfrak{V} \}$$

for each closed simplex τ of K^{n+1} . Thus we have an extension $\tilde{f}: K \to X$ defined by $\tilde{f} | K^n = f_n$ on each K^n . This extension \tilde{f} is obviously a full realization of K in \mathfrak{V} .

REMARK. Let $\mu: U \to X$ be a map of a neighborhood U of ΔX in X^3 to X. We will call μ a *local weak mixer* provided μ satisfies the following condition:

(w)
$$\mu(x, x, y) = \mu(x, y, x) = \mu(y, x, x) = x$$

if $(x, x, y), (x, y, x), (y, x, x) \in U.$

When $U = X^3$, we call μ a *weak mixer*. The properties of a local mixer used in the proof of Theorem II are (w) and:

(#)' for each $x \in X$ and for each neighborhood V of x in X, there exists a neighborhood W of x such that $W \times W \times X \subset \mu^{-1}(V)$,

and then dom μ is a neighborhood of $\Delta X \times X$ in X^3 rather than of $\Delta^* X$. And moreover, if we assume X is locally contractible then it suffices that dom μ is a neighborhood of ΔX in X^3 and (#)' can be replaced by:

each $x \in X$ has a neighborhood W_x in X such that for any

(#)" neighborhood V of x there is some neighborhood W of x with $W \times W \times W_x \subset \mu^{-1}(V)$.

If X is locally compact, then a (local) weak mixer satisfies (#)''. Thus we have

THEOREM. A locally compact metrizable space is LEC if and only if it is locally contractible and has a local weak mixer.

From this theorem, it follows that:

COROLLARY. Let X be locally compact and locally contractible. If X has a local weak mixer then it has a local mixer. And, moreover, if X is contractible then it has a mixer.

Supplement: In $[MV_2]$ it is a question whether every Banach space has a "natural" mixer. In Euclidean space let $\mu(x, y, z)$ be the inner center of the triangle with vertices x, y and z. Then μ is clearly the mixer. T.

Yagasaki gave a "natural" mixer for each *convex set X* in a *normed space* as follows:

$$\mu(x, y, z) = \begin{cases} \frac{1}{\|x - y\| + \|y - z\| + \|z - x\|} \\ \cdot \{\|y - z\| \cdot x + \|z - x\| \cdot y + \|x - y\| \cdot z\} & \text{if } (x, y, z) \notin \Delta X, \\ x \ (= y = z) & \text{if } (x, y, z) \in \Delta X, \end{cases}$$

where $\| \|$ denotes the norm. In fact, if $(x, y, z) \notin \Delta X$ and $\|x - a\|$, $\|y - a\| < \epsilon$, then

$$\begin{split} \|\mu(x, y, z) - a\| \\ &\leq \frac{1}{\|x - y\| + \|y - z\| + \|z - x\|} \\ &\times \{\|y - z\| \cdot \|x - a\| + \|z - x\| \cdot \|y - a\| + \|z - a\|\} \\ &< \frac{1}{\|y - z\| + \|z - x\|} \{\|y - z\| \cdot \epsilon + \|z - x\| \cdot \epsilon + \|x - y\| \cdot \|z - a\|\} \\ &= \epsilon + \frac{\|x - y\|}{\|y - z\| + \|z - x\|} \|z - a\|. \end{split}$$

If
$$||z - a|| < 2\varepsilon$$
, then

$$\frac{\|x-y\|}{\|y-z\|+\|z-x\|}\|z-a\| \le \|z-a\| < 2\varepsilon.$$

If
$$||z - a|| \ge 2\varepsilon$$
, then

$$\frac{||x - y||}{||y - z|| + ||z - x||} ||z - a|| < \frac{\varepsilon}{||z - a|| - \varepsilon} ||z - a||$$

$$= \frac{\varepsilon}{1 - \varepsilon/||z - a||} \le \frac{\varepsilon}{1 - 1/2} = 2\varepsilon,$$

because

$$||x - y|| \le ||x - a|| + ||y - a|| < 2\varepsilon,$$

$$||z - x|| \ge ||z - a|| - ||x - a|| > ||z - a|| - \varepsilon$$

and, similarly, $||y - z|| < ||z - a|| - \varepsilon$. Therefore $||\mu(x, y, z) - a|| < 3\varepsilon$. Since μ is symmetric, this implies μ is continuous at each point of ΔX (hence at any point of X^3) and μ satisfies (#) (equivalently, (*)). Therefore μ is a mixer for X.

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