Pacific Journal of Mathematics

SOME REMARKS ON THE CALCULATION OF STIEFEL-WHITNEY CLASSES AND A PAPER OF WU-YI HSIANG'S

McKenzie Y. K. Wang

Vol. 112, No. 2 February 1984

PACIFIC JOURNAL OF MATHEMATICS
Vol 112, No 2, 1984

SOME REMARKS ON THE CALCULATION OF STIEFEL-WHITNEY CLASSES AND A PAPER OF WU-YI HSIANG'S

MCKENZIE Y. K. WANG

In this paper we employ the techniques introduced by Wu-Yi Hsiang in [4] to perform Stiefel-Whitney class calculations for the possibilities of connected principal isotropy type listed in Theorems 1–3 of [4]. We show that some of the possibilities listed there do not occur if we assume in addition that sufficiently many Stiefel-Whitney classes of the G-manifold vanish. We therefore obtain a slightly shorter list of possibilities of connected principal isotropy type for compact connected Lie group actions on parallelizable manifolds. Stiefel manifolds which are not spheres, for example, fall under this category. We also give an example of how our results may be used to study actions on Stiefel manifolds. As this paper is actually a supplement to [4], we refer the reader to it for notation and general philosophy.

1. Preliminaries on Stiefel-Whitney classes and 2-weights. In this section we set the stage for our calculations, which will be described in the next section. Basically, the statements here are parallel to those in §§1 and 2 of [4].

DEFINITION. Let $\psi\colon H \to G$ be a homomorphism of compact Lie groups. Suppose P and Q are, respectively, maximal 2-tori of H and G, and $\psi(P) \subseteq Q$. Let $\{y_i\}$ be a basis for $\mathrm{Hom}(Q, Z/2) \approx H^1(BQ, Z/2)$ and $\psi^*\colon H^1(BQ, Z/2) \to H^1(BP, Z/2)$. Then $\{x_i = \psi^* y_i\}$ are called the 2-weights of ψ relative to (P, Q). In particular, if $\psi = \mathrm{Ad}\, H$, $G = O(\mathfrak{h})$, then the 2-weights are called the 2-roots of H relative to the maximal 2-tori (P, Q).

We list below the non-zero 2-roots of some classical groups.

G	Q	2-roots	multiplicity
U(n)	diag($\varepsilon_1,\ldots,\varepsilon_n$), $\varepsilon_i = \pm 1$	$y_i - y_j$, $i<j$	2
SU(n)	diag($\varepsilon_1,\ldots,\varepsilon_n$), $\varepsilon_i = \pm 1$, $\sum_1^n \varepsilon_i = 0$	"	2
O(n)	diag($\varepsilon_1,\ldots,\varepsilon_n$), $\varepsilon_i = \pm 1$	"	1
SO(n)	diag($\varepsilon_1,\ldots,\varepsilon_n$), $\varepsilon_i = \pm 1$, $\sum_1^n \varepsilon_i = 0$	"	1
Sp(n)	diag($\varepsilon_1,\ldots,\varepsilon_n$), $\varepsilon_i = \pm 1$	"	4

We shall also need to know the maximal 2-tori in the exceptional Lie groups G_2 and F_4. We refer the reader to the papers [**1**, **2**] for details. However, to summarize, let us mention that if the Cayley numbers are given by $R + Re_1 + Re_2 + Re_3 + \cdots + Re_7$, then G_2 is the group Aut(Cay). Suppose S_i: Cay $\to$ Cay are given by $S_i(e_{i+1}) = e_{i+1}$, $S_i(e_{i+5}) = e_{i+5}$, $S_i(e_{i+6}) = e_{i+6}$ and $S_i(e_j) = -e_j$ if $j \neq i+1, i+5, i+6$. Then $Q = \{1, S_1, \ldots, S_7\}$ are the automorphisms of the Cayley numbers which form a maximal 2-torus of G_2. We let x_i in Hom(Q, $Z/2$) be given by $x_i(S_j) = \delta_{ij}$. The 2-roots of G_2 with respect to Q are $x_i + x_j$, $i < j$, with multiplicity 2, $x_1 + x_2 + x_3$ with multiplicity 2, together with x_i with multiplicity 2.

In the case of F_4, note that Spin(9) $\subseteq F_4$ and every maximal 2-torus in F_4 is conjugate to one in Spin(9). Consequently, we may take the maximal 2-tori in Spin(9) to be maximal 2-tori in F_4. Now $G_2 \subseteq$ Spin(7) $\subseteq$ Spin(8) $\subseteq$ Spin(9), and center(Spin(7)) $\simeq Z/2$, center(Spin(8)) $\simeq (Z/2)^2$. Therefore, we may parametrize a maximal 2-torus in Spin(9) by x_1, x_2, x_3, y, z, where the x_i's come from G_2, y from center(Spin(7)) and z from center(Spin(8)). Then the 2-roots of F_4 consist of x_i, $x_i + x_j$ with $i < j$, $x_1 + x_2 + x_3$, all with multiplicity 4, together with y, $y + x_i$, $y + x_i + x_j$, $y + x_1 + x_2 + x_3$, z, $z + x_i$, $z + x_i + x_j$, $z + x_1 + x_2 + x_3$, $y + z$, $y + z + x_i$, $y + z + x_i + y_j$, $y + z + x_1 + x_2 + x_3$ with multiplicity 1 and $i < j$.

PROPOSITION. *Let ξ be a principal H-bundle, where H is compact Lie. Let Q be a maximal 2-torus of H and Φ: $H \to$* SO(n) (*respectively* O(n), SU(n), U(n), Sp(n)) *a representation of H. Assume $\Phi(Q) \subseteq Q'$, the standard maximal 2-torus of* SO(n) (*respectively* O(n), SU(n), U(n), Sp(n)). *Let π: $E(\xi)/Q \to B(\xi)$ and $\{w_i\}$ be the 2-weights of Φ with respect to (Q, Q') counted with multiplicity. If η is the extension of ξ by the representation Φ, and i: $E(\xi)/Q \to BQ$ is the classifying map of the principal Q-bundle $Q \to E(\xi) \to E(\xi)/Q$, then*

$$i^* \prod_{w_i \in \Omega_2(\Phi)} \left(1 + w_i^d\right) = \pi^*\left(w_* \eta\right),$$

where $d =$ 1, 2, 4, *and $w_*\eta$ denotes the total Stiefel-Whitney class, the total Chern class reduced* mod 2, *or the total quaternion Pontrjagin class reduced* mod 2. *$\Omega_2(\Phi)$ denotes the system of* 2-*weights of the representation Φ.*

Proof. This is basically Theorem 11.3 in [**1**]. We have a commutative diagram

$$\begin{tikzcd}
E(\xi) \arrow[rr] \arrow[ddr] & & E(\xi)/Q \arrow[rr, "\pi"] \arrow[dr] \arrow[ddd, "i"] & & B(\xi) \arrow[ddd, "i'"] & \\
& & & E(\eta)/Q' \arrow[ur, "\pi'"] \arrow[ddd, "j"] & & \\
& E(\eta) \arrow[urr] & & & & \\
& & BQ \arrow[rr, "\alpha"] \arrow[dr, "\Phi"'] & & BH \arrow[dr, "\Phi"] & \\
& & & BQ' \arrow[rr, "\alpha'"] & & BG
\end{tikzcd}$$

We consider first the cases of $\mathrm{SO}(n)$ and $\mathrm{O}(n)$. For this recall that $H^*(\mathrm{BO}(n);\ Z/2) \to H^*(BQ';\ Z/2) \simeq Z/2[y_1,\ldots,y_n]$ identifies $H^*(\mathrm{BO}(n);\ Z/2)$ with $Z/2[w_1,\ldots,w_n] \subseteq Z/2[y_1,\ldots,y_n]$. The w_i's are elementary symmetric polynomials in the y_i's which form a basis for $H^1(BQ'; Z/2)$. Now

$$\pi^*\big(w_*\eta\big) = \pi^* i'^* \Phi^* w_* = i^* \alpha^* \Phi^* w_* = i^* \Phi^* \alpha'^* w_*,$$

where w_* denotes the total universal Stiefel-Whitney class. But $\alpha'^* w_* = \prod_{i=1}^{n}(1 + y_i)$, so

$$\pi^*\big(w_*\eta\big) = i^*\Phi^*\left(\prod_{i=1}^{n}(1 + y_i)\right) = i^*\prod_{1}^{n}(1 + \Phi^* y_i)$$
$$= i^* \prod_{w_i \in \Omega_2(\Phi)} (1 + w_i).$$

For the case of $\mathrm{SO}(n)$, the only difference is that $w_1 = 0$.

Next we consider the parallel cases of $\mathrm{SU}(n)$, $\mathrm{U}(n)$ and $\mathrm{Sp}(n)$. As an example, we discuss the case of $\mathrm{U}(n)$ in detail. The only observation we have to make is that

$$H^*(\mathrm{BU}(n); Z/2) = Z/2[c_1,\ldots,c_n] \subseteq Z/2[y_1,\ldots,y_n] \simeq H^*(BQ'; Z/2),$$

where c_i is the ith elementary symmetric polynomial in the y_i^2's. Therefore, if c_* is the universal total Chern class reduced mod 2, $c_* = \prod(1 + y_i^2)$. This accounts for the power of 2 in the proposition. For $\mathrm{SU}(n)$, $c_1 = 0$. The case of $\mathrm{Sp}(n)$ is similar. □

Let H be a compact connected Lie group and Ψ: $H \to \mathrm{SO}(m) = G$ (respectively $\mathrm{SU}(m)$, $\mathrm{Sp}(m)$) almost faithful representations of H. We should think of $\Psi(H)$ as the connected component of some isotropy group of a classical group action on a compact manifold M which has some vanishing Stiefel-Whitney classes. Let $x \in M$ and $\Psi(H) = G_x^0$.

Then we have $G/\Psi(H) \xrightarrow{p} G/G_x \xrightarrow{h} G(x) \xhookrightarrow{j} M$. Employing reduction 2 in §1 of [4], we have

$$\tau(G/\Psi(H)) = p^!\tau(G/G_x) = p^!h^!\tau(G(x))$$

in $\widetilde{\mathrm{KO}}(G/\Psi(H))$. It is well known that $\tau(G/\Psi(H))$ is the extension of the principal $\Psi(H)$ bundle $\Psi(H) \to G \to G/\Psi(H)$ by the isotropy representation. If x is on a principal orbit, then $\nu(G(x))$, the normal bundle to the orbit through x, is trivial. Since Stiefel-Whitney classes obey a product formula and are stable characteristic classes, vanishing conditions on the Stiefel-Whitney classes of M pull back to vanishing conditions on those of $\tau(G/\Psi(H))$.

Let us denote the principal $\Psi(H)$ bundle above by $\alpha(\Psi, H)$. Then

$$\tau(G/\Psi(H)) = \alpha(\Psi, H)(\mathrm{Ad}\, G \,|\, \Psi(H) - \mathrm{Ad}\,\Psi(H)),$$

so the fundamental observation of Hsiang (reduction 4 in §1 of [4]) allows us to transfer the vanishing conditions on $w_i(G/\Psi(H))$ to $w_i(\alpha(\Psi, H)(\mathrm{Ad}\,\Psi(H)))$. In Proposition 1, we can set $\xi = \alpha(\Psi, H)$, $\Phi = \mathrm{Ad}\,\Psi(H)$, and $\{\alpha_i\}$ = the 2-roots of $\mathrm{Ad}\,\Psi(H)$ counted according to multiplicity. Then provided that $\Phi(\Psi(Q)) \subseteq Q'$, we have

$$\pi^*\left(w_*\eta\right) = i^*\left(\prod_{\alpha_i \in \Delta_2(\Phi)} (1 + \alpha_i)\right).$$

As in [4] we must determine $\mathrm{Ker}\, i^*$. The arguments in [4] go through if we replace T by $\Psi(Q)$ and if we use $Z/2$ coefficients. We must, however, justify the existence of the Serre spectral sequence for the fibration π_G: $E_G \times_{\Psi(Q)} G \to B\Psi Q$, since the base is not simply-connected. However, we simply have to note that we have here a principal G-bundle with G connected, and it is well-known that this is a case where we have simple coefficients.

To compute $\mathrm{Ker}\, i^*$, let us consider $G = \mathrm{SU}(m)$ or $\mathrm{Sp}(m)$ first. These groups have no torsion. Let c_* be the total universal Chern class reduced mod 2 in $H^*(\mathrm{BSU}(m); Z/2)$, and q_* the total universal quaternionic Pontrjagin class reduced mod 2 in $H^*(\mathrm{BSp}(m); Z/2)$. Then the argument in [4] shows that $\mathrm{Ker}\, i^* = \mathrm{Ker}\, \pi_G^*$ = the ideal in $H^*(B\Psi Q; Z/2)$ generated by the images of all differentials of the Serre spectral sequence of π_G. If λ: $B\Psi Q \to BG$ is the classifying map of the principal G-bundle π_G, $\mathrm{Ker}\, i^*$ = the ideal in $H^*(B\Psi Q; Z/2)$ generated by $\{\lambda^* c_i\}$ or, in the case of $\mathrm{Sp}(m)$, by $\{\lambda^* q_i\}$. We apply Proposition 1 again with $\xi = [\Psi Q \to E_G \to B\Psi Q]$, $H = \Psi(Q)$, Φ = the inclusion of $\Psi(Q)$ into G, $\eta = \pi_G$, $i = \mathrm{id}$ to obtain $\lambda^* c_* = \prod_{w_i \in \Omega_2(\Phi)}(1 + w_i^2)$ or $\lambda^* q_* = \prod_{w_i \in \Omega_2(\Phi)}(1 + w_i^4)$. So $\mathrm{Ker}\, i^*$

is the ideal $((W\Phi)_2^2,\ldots,(W\Phi)_m^2)$ for $G = \mathrm{SU}(m)$ and $((W\Phi)_1^4,\ldots,(W\Phi)_m^4)$ for $G = \mathrm{Sp}(m)$, where $(W\Phi)_j$ is the jth elementary symmetric polynomial in the w_i's.

$$w_i \in \Omega_2(\Phi).$$

If $G = \mathrm{SO}(m)$, then G has 2-torsion. However, the generators of $H^*(\mathrm{SO}(m);\ Z/2) \simeq \Delta(x_1,\ldots,x_{m-1})$ are universally transgressive. So the problems of transgression encountered in Pontrjagin classes actually do not occur here. Thus exactly the same argument as in the previous cases yields $\operatorname{Ker} i^* =$ the ideal $((W\Phi)_2,\ldots,(W\Phi)_m)$.

REMARK. We emphasize here that the weights w_i in the above discussion lie in $H^1(B\Psi Q;\ Z/2)$ and the equation

$$\pi^*\left(W_*\eta\right) = i^* \prod_{\alpha_i \in \Delta_2(\Phi)} \left(1 + \alpha_i^d\right)$$

belongs to $H^*(E_{\alpha(\Psi,H)}/\Psi Q;\ Z/2)$. Order two elements in Q may very well go to 0 under Ψ. This will be a point for us to be careful about in our calculations. However, it turns out that there are always enough points in our 2-tori so that we need not worry about elements going to 0.

2. Symmetric products of 2-roots of simple Lie groups. In this section we shall compute certain symmetric products of the 2-roots of some of the simple Lie groups. If K is a compact connected Lie group, we let WK_i denote the ith symmetric product of the 2-roots of K, SK_i denote the ith symmetric sum of the 2-roots of K. If $y_1,\ldots,y_n$ are variables, then σ_i will denote the ith symmetric polynomial in the y_j's. s_i will denote the ith symmetric sum in the y_j's.

We begin with the observation that $W\mathrm{SO}(n)_* = \prod_{i<j}(1 + y_i + y_j)$, $W\mathrm{SU}(n)_* = \prod_{i<j}(1 + y_i + y_j)^2$, and $W\mathrm{Sp}(n)_* = \prod_{i<j}(1 + y_i + y_j)^4$. Formally we must have $W\mathrm{SU}(n)_{2i} = W\mathrm{SO}(n)_i^2$. For the case of $\mathrm{Sp}(n)$, if we use the sub-2-torus of the standard 2-torus, consisting of $\operatorname{diag}(\varepsilon_1,\ldots,\varepsilon_n)$, $\varepsilon_i = \pm 1$, $\sum \varepsilon_i = 0$, then we have $\tilde{W}\mathrm{Sp}(n)_{4i} = W\mathrm{SO}(n)_i^4$. We note the proof of Proposition 1 does not use the maximality of Q. Neither does the discussion of the vanishing conditions and $\operatorname{Ker} i^*$. All this means that while symmetric products of 2-roots of $\mathrm{SU}(n)$ and $\mathrm{Sp}(n)$ may be hard to compute directly, they can be obtained from the results of $\mathrm{SO}(n)$. We therefore begin by computing $W\mathrm{SO}(n)_i$.

Let $k(n)$ be $2n + 1$ if we are considering $\mathrm{SO}(2n + 1) = B_n$, and $k(n) = 2n$ if we are considering $\mathrm{SO}(2n) = D_n$. The basic technique of calculation is the use of Newton's formulas, followed by reduction mod 2.

PROPOSITION. *Let WH_i denote the ith symmetric product of the 2-roots of the Lie group H. Then we have*:

(a) $H = \mathrm{SU}(n) = A_{n-1}$,

$WH_{\mathrm{odd}} = 0$,

$WH_2 = 0$,

$WH_4 = n\sigma_2^2$,

$WH_6 = n\sigma_3^2$,

$$WH_8 = \begin{cases} ([n/2]+1)\sigma_2^4 + \sigma_4^2 & \text{if } n \text{ is odd}, \\ ([n/2]+1)\sigma_2^4 & \text{if } n \text{ is even}, \end{cases}$$

$$WH_{10} = \begin{cases} \sigma_5 & \text{if } n \text{ is odd}, \\ 0 & \text{if } n \text{ is even}, \end{cases}$$

$$WH_{12} = \begin{cases} \sigma_6^2 + ([n/2]+1)\sigma_2^6 + ([n/2]+1)\sigma_3^4 & \text{if } n \text{ is odd}, \\ ([n/2]+1)\sigma_3^4 & \text{if } n \text{ is even}; \end{cases}$$

(b) $H = \mathrm{SO}(2n+1)$,

$WH_1 = 0, \quad WH_2 = \sigma_2, \quad WH_3 = \sigma_3$,

$WH_4 = \sigma_4 + (n+1)\sigma_2^2$,

$WH_5 = \sigma_5$,

$WH_6 = \sigma_6 + (n+1)\sigma_2^3 + (n+1)\sigma_3^2$;

(c) $H = \mathrm{Sp}(n) = C_n$,

$WH_{\mathrm{odd}} = 0 = WH_2$,

$WH_4 = (n+1)\sigma_1^4$,

$\tilde{W}H_4 = 0$,

$\tilde{W}H_8 = n\sigma_2^4$,

$\tilde{W}H_{12} = n\sigma_3^4$,

$$\tilde{W}H_{16} = \begin{cases} \sigma_4^4 + ([n/2]+1)\sigma_2^8 & \text{if } n \text{ is odd}, \\ ([n/2]+1)\sigma_2^8 & \text{if } n \text{ is even}; \end{cases}$$

(d) $H = \mathrm{SO}(2n)$,

$WH_1 = WH_2 = WH_3 = 0$,

$WH_4 = (n+1)\sigma_2^2$,

$WH_5 = 0$,

$WH_6 = (n+1)\sigma_3^2$,

$WH_8 = n\sigma_4^2 + \sigma_2^4 + \left(\left(\tfrac{11}{6}\right)n^2 + \left(\tfrac{2}{3}\right)n^4 + \left(\tfrac{5}{2}\right)n\right)\sigma_2^4 + \sigma_3\sigma_5$,

(e) $$(WG_2)_* = 1 + \left(\sigma_1^8 + \sigma_1^2\sigma_3^2 + \sigma_2^4\right) + \left(\sigma_1^6\sigma_3^2 + \sigma_3^4 + \sigma_1^2\sigma_2^2\sigma_3^2 + \sigma_1^4\sigma_2^4\right) + \left(\sigma_3^4\sigma_1^2 + \sigma_1^4\sigma_2^2\sigma_3^2\right);$$

(f) *Let*

$$A = \sigma_1^4 + \sigma_1\sigma_3 + \sigma_2^2, \quad B = \sigma_1^3\sigma_3 + \sigma_3^2 + \sigma_1\sigma_2\sigma_3 + \sigma_1^2\sigma_2^2,$$
$$C = \sigma_3^2\sigma_1 + \sigma_1^2\sigma_2\sigma_3,$$

and

$$W(u) = 1 + u + u^2 + u^3 + (u^4 + A) + (u^5 + Au) + (u^6 + Au^2 + B) + (u^7 + Au^3 + C).$$

Then $(WF_4)_* = W(0)^4W(y)W(z)W(y+z)$. *In particular*, $(WF_4)_* \neq 1$ *since for* $y = 1$, $z = x_1 = x_2 = x_3 = 0$, $(WF_4)_* = 0 = 1 + (WF_4)_1 + \cdots$.

Proof. Among (a)–(d), the essential calculations occur with the $\mathrm{SO}(n)$'s by a previous remark. Let $k(n)$ be $2n+1$ or $2n$ as described. It is clear that $WH_1 = 0$.

We have $WH_2 = (\frac{-1}{2})SH_2 = (\frac{-1}{2})k(n)s_2 = k(n)\sigma_2 = \sigma_2$ for $k(n) = 2n+1$ and 0 for $k(n) = 2n$. $SH_3 = (k(n)-4)s_3 = (k(n)-4)3\sigma_3$. So $WH_3 = (k(n)-4)\sigma_3 = k(n)\sigma_3$. Similarly, $SH_4 = k(n)s_4 + (\frac{1}{2})(\frac{4}{2})s_2^2 = k(n)(-4\sigma_4 + 2\sigma_2^2) + 3(4\sigma_2^2) = -4k(n)\sigma_4 + 2k(n)\sigma_2^2 + 12\sigma_2^2$. $WH_4 = -(\frac{1}{4})(-4k(n)\sigma_4 + 2k(n)\sigma_2^2 + 12\sigma_2^2) + (\frac{1}{8})k(n)^2 4\sigma_2^2 \equiv \sigma_4 + (n+1)\sigma_2^2$ if $k(n) = 2n+1$ and $(n+1)\sigma_2^2$ if $k(n) = 2n$. Other calculations are similar, though the demand for perseverance increases.

We obtain the formulas for (a) by either direct computation or by the fact that $W\mathrm{SU}(n)_{2i} = W\mathrm{SO}(n)_i^2$. The formulas in (c), (e), (f) are obtained by direct calculation or by the relationship $\tilde{W}\mathrm{SP}(n)_{4i} = W\mathrm{SO}(n)_i^4$, where $\tilde{W}$ are the symmetric products calculated using the corank 1 sub-2-torus of the maximal 2-torus chosen. □

3. Connected principal isotropy types and Stiefel-Whitney classes. Let G denote $\mathrm{SO}(m)$, $\mathrm{SU}(m)$, or $\mathrm{Sp}(m)$ and let G act smoothly on a manifold M whose first three Pontrjagin classes and all Stiefel-Whitney classes vanish. As can be seen, we do not really need the full force of the latter vanishing condition. For applications, we have in mind all stably parallelizable manifolds whose first three Pontrjagin classes vanish. For example, Stiefel manifolds and sufficiently high-dimensional homotopy spheres have this property.

At this point we refer the reader to Theorems 1–3 in [**4**]. Basically, we show that certain cases in these theorems cannot occur if Stiefel-Whitney classes vanish. As described at the end of §1, if $\Psi\colon H \to G$ is an almost faithful representation of H and $\Phi = \mathrm{Ad}(\Psi(H))$ takes $\Psi(Q)$ to a subtorus of Q', then we have the following vanishing condition:

$$(WH)_i = 0 \mod\left((W\Phi)_1^d, \ldots, (W\Phi)_{[i/d]}^d\right), \qquad d = 1, 2, 4,$$

where WH_i is the ith symmetric product of the 2-roots of $\Psi(H)$ and $(W\Phi)_i =$ the ith symmetric product of the 2-weights of the representation $\Phi\colon \Psi(H) \subseteq G$. We recall the remark in §2 that the 2-torus $\Psi(Q)$ need not be maximal in $\Psi(H)$, so that the above vanishing condition holds for any sub-2-torus Q of H.

PROPOSITION 1. *Let* $\mathrm{SU}(n)$ *act smoothly on a manifold* M *such that the first three Pontrjagin classes and all Stiefel-Whitney classes vanish. Let* Ψ: $H \to \mathrm{SU}(m)$ *be an almost faithful complex representation of a compact connected Lie group* H. *Then for the following pairs of* (H, Ψ), $\Psi(H)$ *cannot be the connected principal isotropy type of the* $\mathrm{SU}(m)$ *action*:

1. *H is semisimple, connected, compact*, $\Psi = \mathrm{Ad}$.
2. $H = \mathrm{SU}(l) \times H_1$, *$l$ divides* 30, $\Psi = \mu_l \otimes \mu_l + \mathrm{Ad}\, H_1$.
3. $H = \mathrm{SU}(l)$, $\Psi = 2\mu_l$, *and l is odd.*
4. $H = \mathrm{SO}(l)$, $\Psi = \rho_l$, *l odd.*
5. $H = \mathrm{Sp}(l)$, $\Psi = \nu_l$, *l even.*
6. $H = G_2$, $\Psi = 2\Lambda_2$, *Λ_2 is the 7-dimensional representation of* G_2.
7. $H = G_2 \times G_2$, $\Psi = 2(\Lambda_2 + \Lambda_2')$.
8. $H = \mathrm{SU}(3) \times \mathrm{SU}(3)$, $\Psi = 2(\mu_3 + \mu_3')$, $2(\bar\mu_3 + \bar\mu_3')$, $(\mu_3 + \mu_3') + (\bar\mu_3 + \bar\mu_3')$.
9. $H = \mathrm{SU}(3)$ *or* $\mathrm{SU}(5)$, $\Psi = \mu_l + \bar\mu_l$.
10. $H = \mathrm{SU}(3)$, $\Psi = k\mu_3 + l\bar\mu_3$, $k + l = 6$.

Proof. 1. Since Ψ is allowed to be almost faithful, we may assume $H = H_1 \times \cdots \times H_k$, where H_i are simple normal factors of H. By reduction 2 in [**4**], one easily sees that we may assume H itself is simple.

(a) $H = \mathrm{SU}(l)$.

If l is even, then $Z(\mathrm{SU}(l))$ contains only one element of order 2 of the standard maximal 2-torus. Therefore we may performs the 2-weights calculation in $\mathrm{SU}(l)$. The vanishing condition is

$$WH_i \equiv 0 \mod\left(WH_2^2, \ldots, WH_l^2\right).$$

Now $WH_8 = ((\frac{1}{2})l + 1)\sigma_2^4$ while $WH_2^2 = 0$, $WH_4^2 = 0$. This forces $(\frac{1}{2})l$ to be odd. On the other hand,

$$WH_{16} = WSO(l)_8^2 = \sigma_4^4 + C\sigma_2^8 + \sigma_3\sigma_5,$$

$$WH_2^2 = 0 = WH_4^2 = WH_6^2 = WH_8^2$$

because $(\frac{1}{2})l$ is odd. Therefore, l must be odd. In this case $Z(SU(l))$ contains no element of order 2. Since $WH_4 = \sigma_2^2$ and $WH_2^2 = 0$, we conclude l cannot be odd either.

(b) $H = SO(2l + 1)$ or $\mathrm{Spin}(2l + 1)$.

$Z(\mathrm{Spin}(2l + 1)) \simeq Z/2$ and $Z(SO(2l + 1)) = 0$. Since calculations are done in the adjoint group, we need only consider the case of $SO(2l + 1)$. Here we simply notice that $WH_2 = \sigma_2$ and $\operatorname{Ker} i^*$ contains no element of this degree. So $H = B_l$ does not occur.

(c) $H = Sp(l)$.

$Z(Sp(l)) \simeq Z/2$ with generator $-I$, where I is the identity matrix. Now $WH_4 = (l + 1)\sigma_1^4$, $WH_2^2 = \dot{0}$, and $WH_1^2 = 0$. So if l is even, $WH_4 = \sigma_1^4$, which is not 0 even after removing the center. Hence l must be odd. In this case we compute 2-weights using the corank 1 2-torus in the standard maximal 2-torus as explained before. This 2-torus does not contain the generator of the center. Here we have $\tilde{W}H_8 = l\sigma_2^4$, while $\tilde{W}H_4^2 = 0$. So l cannot be odd either.

(d) $H = SO(2l)$ or $\mathrm{Spin}(2l)$.

Now $Z(\mathrm{Spin}(2l)) = Z/4$ or $Z/2 + Z/2$ and $Z(SO(2l)) = Z/2$. Again we perform our calculations on $SO(2l)$ since actually this should be performed on the adjoint group. $WH_4 = (l + 1)\sigma_2^2$ and $WH_1^2 = 0 = WH_2^2$. So l must be odd because $WH_4 \neq 0$ even if we stay away from the generator of $Z(SO(2l))$. For l odd, we examine $WH_8 = \sigma_4^2 + \sigma_3\sigma_5 + C\sigma_2^4$, where $C = 0, 1$. $WH_1 = WH_2 = WH_3 = 0 = WH_4$. Therefore, $WH_8 \not\equiv 0$.

(e) $H = G_2$.

$Z(G_2) = 0$. From $(WG_2)_*$ we see that $(WG_2)_8 \neq 0$.

(f) $H = F_4, E_6, E_7, E_8$.

For these cases we avoid specific computation by noticing the following:

If $P(\sigma_1,\dots,\sigma_l)$ is a homogeneous polynomial over a field k in the elementary symmetric polynomials $\{\sigma_i\}$ in the variables $y_1,\dots,y_l$, and $P = P_1 + \cdots + P_N$ is the decomposition of P into homogeneous polynomials of the same degree, then if, for all i, $P_i \equiv 0 \bmod P_1^2,\dots,P_{[i/2]}^2$, then actually $P_i = 0$ for all i.

In view of this, we need only show that $(WH)_*$ does not equal 1. We stress that by $P_i = 0$ we mean P_i is the zero polynomial and not a polynomial that evaluates to be 0. We have shown $(WF_4)_* \not\equiv 1$. For E_6 we use the maximal 2-torus of $A_1 \times A_5$ in E_6, which is contained in that of E_6. So the 2-weights of $\mathrm{Ad}\, E_6 | A_1 \times A_5 = \mathrm{Ad}\, A_1 + \mathrm{Ad}\, A_5 + \Lambda^3\mu_6$ are λ, $y_i + y_j$ with $i < j$ and multiplicity 2, $y_i + y_j + y_k + \lambda$ with $i < j < k$ and multiplicity 2. So

$$W_* = (1+\lambda)^2 \prod_{i<j} (1 + y_i + y_j)^2 \prod_{i<j<k} (1 + y_i + y_j + y_k + \lambda)^2.$$

Take $(\lambda, y_1, \ldots, y_6) = (0,1,1,0,0,0,0)$ and evaluate W_* at this point. We get 0; on the other hand, $W_* = 1 + W_1 + W_2 + \cdots + W_N$. So some W_i must be nonzero. Now $Z(E_6) = Z/3$, so it does not contain elements of order 2.

For E_8 we take the maximal 2-torus of $A_8 \subseteq E_8$. Since $\mathrm{Ad}\, E_8 | A_8 = \mathrm{Ad}\, A_8 + \Lambda^3\mu_9 + \overline{\Lambda}^3\mu_9$, the 2-weights are $y_i + y_j$ with $i < j$ and $y_i + y_j + y_k$ with $i < j < k$, with multiplicity 2. Hence

$$W_* = 1 + W_1 + W_2 + \cdots = \prod_{i<j} (1 + y_i + y_j)^2 \prod_{i<j<k} (1 + y_i + y_j + y_k)^2.$$

We evaluate at $(1,1,0,\ldots,0)$. Then $W_* = 0$ and so some W_i must be nonzero. Notice that $Z(E_8) = 0$. The argument for E_7 proceeds analogously. We use the maximal 2-torus of $A_7 \subseteq E_7$ and the fact that $\mathrm{Ad}\, E_7 | A_7 = \mathrm{Ad}\, A_7 + \Lambda^4\mu_8$. Although $Z(E_7) = Z/2$, we have enough points to choose from for the evaluation argument.

2. Again we may use reduction 2 in [**4**]. It follows that we are reduced to case 1, which cannot occur.

3. Ψ is injective and $WH_4 = l\sigma_2^2$, while the 2-weights of Ψ are y_i with multiplicity 2. Hence $((W\Psi)_2^2, \ldots, W\Psi_l^2) = (\sigma_2^4, \ldots, \sigma_l^4)$ implies $WH_4 \not\equiv 0$ mod the ideal if l is odd.

4. The 2-weights are $y_1, \ldots, y_l$ with multiplicity 1 and so $(W\Psi_2^2, \ldots, W\Psi_l^2) = (\sigma_2^2, \ldots, \sigma_l^2)$. For l odd, $WH_2 = \sigma_2 \not\equiv 0$.

5. The 2-weights are $y_1, \ldots, y_l$ with multiplicity 2. $(W\Psi_2^2, \ldots, W\Psi_l^2) = (\sigma_2^4, \ldots, \sigma_2^4)$. Since $WH_4 = (l+1)\sigma_1^4$, l cannot be even.

6. Let Λ_2 denote the 7-dimensional representation of G_2. Then $W\Lambda_{2*} = 1 + A + B + C$ (for notation see the proposition in §2) and so one sees that $(WG_2)_8 = (W2\Lambda_2)_8 \not\equiv 0 \bmod ((W2\Lambda_2)_1^2, \ldots, (W2\Lambda_2)_4^2)$.

7. Let A, B, C and A', B', C' be, respectively, the expressions given in the proposition in §2 for two copies of G_2. Then

$$WH_* = (1 + A^2 + B^2 + C^2)(1 + A'^2 + B'^2 + C'^2),$$

and if $\Psi = 2(\Lambda_2 + \Lambda_2')$, then

$$W\Psi_*^2 = (1 + A^4 + B^4 + C^4)(1 + A'^4 + B'^4 + C'^4),$$

so clearly this case is impossible.

8. If $H = \mathrm{SU}(3) \times \mathrm{SU}(3)$ and σ_i and τ_i are, respectively, symmetric polynomials corresponding to each factor of SU(3). A straightforward computation shows that

$$WH_* = 1 + \left(\sigma_2^2 + \tau_2^2\right) + \left(\sigma_3^2 + \tau_3^2\right) + \left(\sigma_2^2\tau_2^2\right) + \left(\sigma_3^2\tau_3^2 + \sigma_2^2\tau_3^2\right) + \sigma_3^2\tau_3^2.$$

If Ψ is one of the representations listed, then $W\Psi_*^2 = WH_*^2$ and we see that this case cannot occur.

9. This case is similar to (3).

10. $WH_* = 1 + \sigma_2^2 + \sigma_3^2$. If $k + l = 6$, the 2-weights are the same as those of $6\mu_3$, which are y_1, y_2, y_3 with multiplicity 6. Therefore,

$$W\Psi_*^2 = 1 + \sigma_2^4 + \sigma_3^4 + \sigma_2^8 + \left(\sigma_3^8 + \sigma_2^{12}\right) + \sigma_2^4\sigma_3^8 + \sigma_3^4\sigma_2^8 + \sigma_3^{12}.$$

Thus the vanishing condition is not satisfied. □

PROPOSITION 2. *Let* $\mathrm{Sp}(m)$ *act smoothly on a manifold such that the first three Pontrjagin classes and all Stiefel-Whitney classes vanish. Let* Ψ: $H \to \mathrm{Sp}(m)$ *be an almost faithful symplectic representation of a compact connected Lie group* H. *Then* (H, Ψ) *in the following cannot have* $\Psi(H)$ *as the connected principal isotropy type of the* $\mathrm{Sp}(m)$*-action*:

1. $H = \mathrm{SU}(n), n = 3, 4, 5, \Psi = \mu_n + \bar{\mu}_n$.
2. $H = \mathrm{SU}(3) \times \mathrm{SU}(3), \Psi = \mu_3 + \bar{\mu}_3 + \mu_3' + \bar{\mu}_3'$.
3. $H = G_2 \times G_2$, $\Psi = 2(\Lambda_2 + \Lambda_2')$, *where* Λ_2 *is the* 7*-dimensional representation of* G_2.
4. $H = G_2, \Psi = 2\Lambda_2$.

Proof. The calculations are exactly the same as those for the previous proposition. □

PROPOSITION 3. *Hypotheses as in Proposition* 2 *except with* $\mathrm{Sp}(m)$ *replaced by* $\mathrm{SO}(m)$. *Then the following pairs* (H, Ψ) *cannot have* $\Psi(H)$ *as the connected principal isotropy type of the* $\mathrm{SO}(m)$*-action*:

1. $H = G_2, \Psi = 4\Lambda_2$.
2. $H = \mathrm{SU}(3), \mathrm{SU}(5), \Psi = 2(\mu_n + \bar{\mu}_n)$.
3. $H = \mathrm{SU}(3) \times \mathrm{SU}(3), \Psi = 2(\mu_3 + \mu_3' + \bar{\mu}_3 + \bar{\mu}_3')$.
4. $H = \mathrm{Sp}(n)$, *n even*, $\Psi = 2\nu_n$.
5. $H = \mathrm{SO}(n)$, *n odd*, $\Psi = 2\rho_n$.
6. *H is semisimple and contains simple normal factors of type* B_n, C_{2n}, D_{2n}, $\Psi = 2\,\mathrm{Ad}\,\Psi(H)$.

Proof. Again the calculations are similar to those of Proposition 1 and so are omitted. □

As an application of these propositions, let us prove

PROPOSITION 4. *Let* $G = \mathrm{Sp}(m)$ *act smoothly on* $W_{n,n-k}$ = *complex Stiefel manifold of orthonormal n-k planes in* C^n *with* $k \geq n/2$, $n \geq 11$, *and* $\dim \mathrm{Sp}(m) \geq \dim \mathrm{SU}(n)$. *Then the action is trivial.*

Proof. Combining Proposition 2 in this section with Theorem 2 in [**4**] we conclude that the connected principal isotropy type of the action must be one of the following:

1. any torus T,
2. $\nu_l(\mathrm{Sp}(l))$,
3. $r(\nu_1^{(1)} + \cdots + \nu_1^{(l)})(\mathrm{Sp}(1)^{(1)} \times \cdots \times \mathrm{Sp}(1)^{(l)})$, $r = 1,2,4$. In case 1,

if (H^0) is the connected principal isotropy type, then $\dim H^0 \leq m$. One easily sees that $m \leq \sqrt{3/8}\; n < 0.615n$. In case 3, $\dim H^0 \leq 3rl \leq 3m$ and one checks that $m < 0.615n + 1$. On the other hand, $n \geq 11$ and $k \geq n/2$ together with $\dim \mathrm{Sp}(m) \geq \dim \mathrm{SU}(n)$ imply $m \geq 0.615n + 1$. Hence case 2 must occur and, by Theorem C1 of §8 of [**4**], all connected isotropy groups are of the type $\nu_{l_x}(\mathrm{Sp}(l_x))$, $l_x \geq l$, where l is the rank of the connected principal isotropy type. If $G(x)$ is any orbit, then, since it is covered by a quaternionic Stiefel manifold, $\pi_i(G(x)) = 0$ whenever $i \leq 4l + 2$. Consequently, $\tilde{H}(G(x);\ Q) = 0$ for $i \leq 4l + 2$. By the Vietoris mapping theorem (see for example, §6.4, p. 142 of [3]) we have

$$H^i(W_{n,n-k}/G;\ Q) \simeq^{p^*} H^i(W_{n,n-k};\ Q) \quad \text{if } i \leq 4l + 2,$$

where p is the orbit map. Therefore, there are nonzero elements $a_k, \ldots, a_q$ in $H^{2i+1}(W_{n,n-k}/G;\ Q)$, $q = \min(2l, n-1)$, $\deg a_i = 2i + 1$, so that $p^* a_i = x_i$ in $H^{2i+1}(W_{n,n-k};\ Q)$, where $H^*(W_{n,n-k};\ Q) \simeq \Lambda(x_k, \ldots, x_{n-1})$. Hence $a_k \cdots a_q \neq 0$. But note that

$$\begin{aligned}\dim(W_{n,n-k}/G) &= n^2 - k^2 - (2m^2 + m - 2l^2 - l)\\ &\leq (4l^2 - k^2) + (-2l^2 + l + 1).\end{aligned}$$

If $l > 1$, $-2l^2 + l + 1 < 0$, so

$$\dim(W_{n,n-k}/G) < 4l^2 - k^2 < \dim(a_k \cdots a_q) \quad \text{if } 2l = q,$$

or equivalently, $n - 1 \geq 2l$, which is a contradiction. So $n - 1 < 2l$, in which case $\dim(W_{n,n-k}/G) = n^2 - k^2$, implying $l = m$, or, equivalently, the action is trivial.

Added in Proof. This paper was written in 1979 before the author's thesis. Circumstances have delayed its publication. Meanwhile actions on the complex Stiefel manifolds $W_{n,2}$, n odd, were studied more extensively in the author's papers listed below.

[**6**] Trans. Amer. Math. Soc., **272** (1982), 589–610, 611–628.

[7] Canad. Math. Soc. Conf. Proc., Vol. 2, Part 2, (1982), 303–311.

References

[1] A. Borel and F. Hirzebruch, *Characteristic Classes and Homogeneous Spaces I*, Amer. J. Math., **80** (1958), 458–538; II, ibid. **81** (1959), 315–382; III, ibid. **82** (1960), 491–504.

[2] ———, *Sous-groupes commutatifs et torsion des groupes de Lie compacts connexes*, Tohoku Math. J., **13** (1961), 216–240.

[3] G. Bredon, *Sheaf Theory*, McGraw Hill (1967).

[4] Wu-Yi Hsiang, *On Characteristic Classes of Compact Homogeneous Spaces and Their Application in Compact Transformation Groups I*, Bol. Soc. Brasil. Mat., **10** (1979), 87–161.

[5] ———, *Cohomology Theory of Topological Transformation Groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 85, Springer-Verlag (1970).

Received October 16, 1981.

Department of Mathematical Sciences
McMaster University
Hamilton, Ontario, Canada L8S 4K1

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph must be capable of being used separately as a synopsis of the entire paper. In particular it should contain no bibliographic references. Please propose a heading for the odd numbered pages of less than 35 characters. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. 39. Supply name and address of author to whom proofs should be sent. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California 90024.

There are page-charges associated with articles appearing in the Pacific Journal of Mathematics. These charges are expected to be paid by the author's University, Government Agency or Company. If the author or authors do not have access to such Institutional support these charges are waived. Single authors will receive 50 free reprints; joint authors will receive a total of 100 free reprints. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: $132.00 a year (6 Vol., 12 issues). Special rate: $66.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Old back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

The Pacific Journal of Mathematics ISSN 0030-8730 is published monthly by the Pacific Journal of Mathematics at P.O. Box 969, Carmel Valley, CA 93924. Application to mail at Second-class postage rates is pending at Carmel Valley, California, and additional mailing offices. Postmaster: Send address changes to Pacific Journal of Mathematics, P. O. Box 969, Carmel Valley, CA 93924.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS. A NON-PROFIT CORPORATION

Pacific Journal of Mathematics

Vol. 112, No. 2 February, 1984