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We prove that a finite-dimensional Jordan triple system over a field k of characteristic zero has a completely reducible structure algebra iff it is a direct sum of a trivial and a semisimple ideal. This theorem depends on a classification of Jordan triple systems with completely reducible derivation algebra in the case where k is algebraically closed. As another application we characterize real Jordan triple systems with compact automorphism group.

The main topic of this paper is finite-dimensional Jordan triple systems over a field of characteristic zero which have a completely reducible derivation algebra.

The history of the subject begins with [7] where G. Hochschild proved, among other results, that for an associative algebra $\mathcal{C}$ the derivation algebra is semisimple iff $\mathcal{C}$ itself is semisimple. Later on R. D. Schafer considered in [18] the case of a Jordan algebra $\mathcal{J}$. His result was that Der $\mathcal{J}$ is semisimple if and only if $\mathcal{J}$ is semisimple with each simple component of dimension not equal to 3 over its center. This theorem was extended by K.-H. Helwig, who proved in [6]:

Let $\mathcal{J}$ be a Jordan algebra which is finite-dimensional over a field of characteristic zero. Then the following are equivlent:

(1) Der $\mathcal{J}$ is completely reducible and every derivation of $\mathcal{J}$ has trace zero,

(2) $\mathcal{J}$ is semisimple,

(3) the bilinear form on Der $\mathcal{J}$ given by $(D_1, D_2) \rightarrow \text{trace}(D_1 D_2)$ is non-degenerate and every derivation of $\mathcal{J}$ is inner.

After some preparations in §§1—3 we will show in §4 that the same theorem holds for Jordan triple systems. The proof in this case is different from the Jordan algebra case. It relies on a classification of Jordan triple systems whose derivation algebras are completely reducible. It is easy to see that V is an example for such a triple system, if

(a) V is semisimple or if

(b) V is trivial, i.e. all products vanish.

Surprisingly there occurs another type of example:

(c) $V = X \oplus M$ with quadratic representation

$$P(x \oplus m)(y \oplus n) = (2(x, y)x - (x, x)y) \oplus ((x, x)n + 2(x, y)m)$$

where $(\cdot, \cdot)$ is a symmetric non-degenerate bilinear form on X. Our main result says that these are essentially all examples. To be more precise, let V be a Jordan triple system over an algebraically closed field. Then the following are equivalent:

(I) Der V is completely reducible.

(II) $V = V^1 \oplus V^2$ is a direct sum of two ideals V^i where—up to enumeration—one of the following two cases occurs:

(α) V^1 is semisimple and V^2 is trivial,

(β) V^1 is semisimple and V^2 is a direct sum of ideals which are isomorphic to example (c).

In §5 we apply our results to characterize Jordan triple systems which have a compact automorphism group. Another application is given in §6 where we prove that a Jordan triple system V has a completely reducible structure algebra iff V is a direct sum of a semisimple and a trivial ideal.

The author thanks M. Koecher and K. McCrimmon for various useful suggestions and H. P. Petersson for having drawn his attention to [6].

1. A review of known results.

1.1. In this section we recall some known facts from the theory of Lie algebras and derive a few consequences. Throughout, M denotes a finite-dimensional vector space over a field k of characteristic zero and $\mathfrak{g}$ is a subalgebra of $\mathfrak{gl}(M)$, the Lie algebra of all endomorphisms of M. By definition, $\mathfrak{g}$ is *completely reducible* (in M), if every $\mathfrak{g}$-invariant subspace has a $\mathfrak{g}$-invariant complement.

(1.1) ([2] §6.5, Théorème 4). $\mathfrak{g}$ is completely reducible iff $\mathfrak{g}$ is reductive and the center of $\mathfrak{g}$ consists of semisimple elements.

As a corollary we get

(1.2) If $\mathfrak{g}$ is completely reducible, then every ideal of $\mathfrak{g}$ is completely reducible, too.

(1.3) ([1] §9.2, Prop. 3). Let K be an extension field of k. Then $\mathfrak{g}$ is completely reducible in M iff $K \otimes \mathfrak{g}$ is completely reducible in $K \otimes M$.

(1.4) $\mathfrak{g}$ is called almost algebraic ("scindable" in [3]) if $\mathfrak{g}$ contains the semisimple and nilpotent components of all of its elements. The Lie algebra of an algebraic subgroup of $GL(M)$ is almost algebraic ([5]. In particular, the derivation algebra of an algebra or a triple system is almost algebraic (for a direct proof see [3] Chap. 7, §1, Prop. 4).

The following lemma is contained in [6] p. 28 for the case where $\mathfrak{g}$ is the derivation algebra of an algebra:

LEMMA 1.1. *Let $\mathfrak{g}$ be almost algebraic and assume that the bilinear form ρ on $\mathfrak{g}$ defined by $\rho(X, Y) = \text{trace}(XY)$ is non-degenerate. Then $\mathfrak{g}$ is completely reducible.*

Proof. By [2] §6.4, Prop. 5 we know that $\mathfrak{g}$ is reductive. Hence, using (1.1), the assertion follows, if we can show that the center $\mathfrak{z}$ of $\mathfrak{g}$ consists of semisimple elements. But $\mathfrak{z}$ is again almost algebraic ([3] §5.1, Corollaire 1 of Prop. 3) and thus it is enough to show that $\mathfrak{z}$ contains no nilpotent elements. Let $Z \in \mathfrak{z}$ be nilpotent, then ZX is nilpotent for every $X \in \mathfrak{g}$ and therefore trace(ZX) = 0, forcing Z to be zero.

The following lemma is a trivial generalization of [2] §6.1 Proposition 1:

LEMMA 1.2. *Let $\mathfrak{g}$ be completely reducible. Define ρ as in Lemma 1.1. Then ρ is non-degenerate iff the restriction of ρ to the center of $\mathfrak{g}$ is non-degenerate.*

Proof. By (1.1) $\mathfrak{g}$ is reducitve, i.e. $\mathfrak{g} = \mathfrak{z} \oplus [\mathfrak{g}, \mathfrak{g}]$, where $[\mathfrak{g}, \mathfrak{g}]$ is semisimple and $\mathfrak{z}$ is the center of $\mathfrak{g}$. Since ρ is an invariant bilinear form, this decomposition is orthogonal relative to ρ. But the restriction of ρ to $[\mathfrak{g}, \mathfrak{g}]$ is non-degenerate by [2] §6.1 Prop. 1. Hence ρ is nondegenerate iff the restriction of ρ to $\mathfrak{z}$ is non-degenerate.

1.2. In this section we recall some well-known results about Jordan triple systems which are needed in the sequel. Thereby we also fix our notation.

A *Jordan triple system* as it is considered in this paper is a *finite-dimensional vector space V over a field k of characteristic zero* together with a trilinear map

$$\{\cdots\}: V \times V \times V \to V: (u, v, w) \to \{uvw\} =: L(u, v)w$$

which satisfies the following identities

(1.5) $$\{u v w\} = \{w v u\}$$

(1.6) $$[L(u, v), L(x, y)] = L(\{u v x\}, y) - L(x, \{v u y\})$$

for all $u, v, w, x, y \in V$. In what follows we just speak of V as the Jordan triple system without exhibiting the map $\{ \cdots \}$.

The theory of Jordan triple systems is developed in [10], [11] and [14] for a more general class of triple systems. Since the proofs of our main results are only valid in the situation as defined above we restrict ourselves to this case from the very beginning.

For a Jordan triple system V one defines the *quadratic representation* $P \colon V \to \text{End } V$ by $P(x) = \frac{1}{2}\{xyx\}$. This is in fact a quadratic map since its linearization $P(x, z) = P(x + z) - \{(x) - P(z)$ satisfies $P(x, z)y = \{xyz\}$. As a result, V is uniquely determined by P.

There are many identitites valid in V, see e.g. [11] §2. We need the following which is a translation of (JP12) in [11]:

(1.7) $$L(x, y)P(z) + P(z)L(y, x) = P(z, \{xyz\}).$$

There are several Lie algebras associated with Jordan triple systems. In this paper we are concerned with the structure algebra and the derivation algebra of V. The structure algebra appears in connection with the *derivation algebra of the Jordan pair* (V, V) associated to V: By definition, $(A, B) \in \text{Der}(V, V)$ if

$$[A, L(x, y)] = L(Ax, y) + L(x, By)$$

and

$$[B, L(x, y)] = L(Bx, y) + L(x, Ay)$$

holds for all $x, y \in V$. $\text{Der}(V, V)$ is a subalgebra of $(\mathfrak{gl}(V), \mathfrak{gl}(V))$ with respect to componentwise multiplication. The *structure algebra* $\mathfrak{I}(V)$ of V is the image of $\text{Der}(V, V)$ under the projection map onto the first component:

$$\text{Der}(V, V) \to \mathfrak{I}(V) \colon (A, B) \to A.$$

We point out that this map is in general not injective. It is an immediate consequence of (1.6) that

$$(L(u, v), -L(v, u)) \in \text{Der}(V, V)$$

for all $u, v \in V$, correspondingly $L(u, v) \in \mathfrak{I}(V)$. The subspace of $\mathfrak{I}(V)$ spanned by $\{L(u, v); u, v \in V\}$ is an ideal of $\mathfrak{I}(V)$ called the *inner structure algebra* and denoted by inn $\mathfrak{I}(V)$.

A *derivation* of V is a k-linear map $D: V \to V$ satisfying $D\{xyz\} = \{Dz, y, z\} + \{x, Dy, z\} + \{x, y, Dz\}$ for all $x, y, z \in V$, i.e. an endomorphism D with $(D, D) \in \text{Der}(V, V)$. The space $\text{Der}\, V$ of all derivations of V is a Lie subalgebra of $\mathfrak{T}(V)$. Since $(L(u, v), -L(v, u)) \in \text{Der}(V, V)$ it follows easily that

$$\Delta(u, v) := L(u, v) - L(v, u) \in \text{Der}\, V.$$

The span of $\{\Delta(u, v); u, v \in V\}$ is an ideal of $\text{Der}\, V$ denoted by $\text{inn Der}\, V$. The elements of $\text{inn Der}\, V$ are called *inner derivations*.

If K is an extension field of k and V a Jordan triple system over k, the triple product of V can be naturally extended to $K \otimes V$. In this way $K \otimes V$ becomes a Jordan triple system. As for algebras one proves

$$(1.8) \quad \mathfrak{T}(K \otimes V) = K \otimes \mathfrak{T}(V), \qquad \text{Der}(K \otimes V) = K \otimes \text{Der}\, V.$$

A·*subsystem* of V resp. an *ideal* of V is a subspace U of V such that $\{UUU\} \subset U$ resp. $\{VVU\} + \{VUV\} \subset U$. One calls V *simple* if $\{VVV\} \neq 0$ and V contains only the trivial ideals 0 and V. Since $\{VVV\}$ is an ideal, we have in this case $\{VVV\} = V$.

The k-linear span of all products in V of degree $> n$ is denoted by $V_{(n)}$, where $\{xyz\}$ resp. $\{xy\{uvw\}\}$ or $\{x\{yuv\}w\}\ldots$ is considered to be of degree 3 resp. 5.... Obviously, $V = V_{(1)} \supset V_{(3)} \supset V_{(5)} \supset \cdots$ and each $V_{(n)}$ is an ideal of V. If $V_{(n)} = 0$ for some n, then V is said to be *nilpotent*.

In the situation considered here every Jordan triple system V contains a unique maximal nilpotent ideal $\text{Rad}\, V$ which is called the *radical of V*. Since $(\text{Rad}\, V, \text{Rad}\, V)$ is the radical of the Jordan pair (V, V), it is invariant under the automorphism group of (V, V) and thus also under its Lie algebra (see e.g. [5] II §8 Prop. 6) which is $\text{Der}(V, V)$. Hence

$$(1.9) \qquad\qquad \text{Rad}\, V \text{ is invariant under } \mathfrak{T}(V).$$

We will need another property of $\text{Rad}\, V$ which follows from [11] Theorem 14.10:

(1.10) The subspace $L(V, \text{Rad}\, V) + L(\text{Rad}\, V, V)$ of $\text{End}\, V$ consists of nilpotent endomorphisms.

A Jordan triple V is called *semisimple*, if its radical vanishes. It is known ([14] §11) that the following properties are equivalent:

$(1.11.\text{a})$ V is semisimple,

$(1.11.\text{b})$ the trace form σ of V defined by $\sigma(u, v)$
$= \frac{1}{2} \text{trace}(L(u, v) + L(v, u))$ is non-degenerate,

$(1.11.\text{c})$ $K \otimes V$ is semisimple for every extension field K of k,

(1.11.d) V is the direct sum of its simple ideals.

It is also proved in [14] §11.6:

(1.12)   If V is semisimple, then $\mathfrak{T}(V) = \mathrm{inn}\,\mathfrak{T}(V)$ and $\mathrm{Der}\,V = \mathrm{inn}\,\mathrm{Der}\,V$. Moreover, the projection $\mathrm{Der}(V, V) \to \mathfrak{T}(V)$ is an isomorphism.

The existence of a Wedderburn decomposition of V was show in [9]:

(1.13)   For every Jordan triple system V (as defined above) there exists a semisimple subsystem S of V such that $V = \mathrm{Rad}\,V \oplus S$.

It follows from [14] 11.3 Theorem 4:

(1.14)   If V is a simple Jordan triple system, then either $\mathfrak{T}(V)$ operates irreducibly on V or $V = V^+ \oplus V^-$ is the direct sum of two $\mathfrak{T}(V)$-irreducible subspaces V^+ and V^- satisfying

$$\{V^\varepsilon V^{-\varepsilon} V^\varepsilon\} \subset V^\varepsilon, \quad \{V^\varepsilon V^\varepsilon V\} = 0 \qquad \text{for } \varepsilon = \pm.$$

Using (1.11) we get as an easy corollary:

(1.15)   If V is semisimple, then $\mathfrak{T}(V)$ is completely reducible.

LEMMA 1.3. *Let V be semisimple and define the bilinear form ρ on $\mathfrak{T}(V)$ by $\rho(X, Y) = \mathrm{trace}(XY)$. Then ρ is non-degenerate.*

Proof. First we make two reductions:

(1) Let K be an extension field of k. Then $K \otimes V$ is a semisimple Jordan triple system by (1.11.c), $D \otimes \mathfrak{T}(V) = \mathfrak{T}(V \otimes K)$ and ρ defined for $K \otimes V$ is the bilinear extension of $\rho \,|\, \mathfrak{T}(V)$. Without loss of generality we may therefore only consider the case where k is algebraically closed.

(2) If $V = V^1 \oplus \cdots \oplus V^n$ is the decomposition of V as sum of its simple ideals, then we have the corresponding decomposition $\mathfrak{T}(V) = \mathfrak{T}(V^1) \oplus \cdots \oplus \mathfrak{T}(V^n)$ where $\mathfrak{T}(V^i)$ is canonically imbedded in $\mathfrak{T}(V)$. Since this decomposition is orthogonal relative to ρ it is enough to consider the case where V is simple. Because $\mathfrak{T}(V)$ is completely reducible we can apply Lemma 1.2 and see that we have to show that the restriction of ρ to the center $\mathfrak{z}$ of $\mathfrak{T}(V)$ is non-degenerate.

If $\mathfrak{T}(V)$ is irreducible, then the centralizer of $\mathfrak{T}(V)$ in $\mathrm{End}(V)$ equals $k\,\mathrm{Id}$ and contains $\mathfrak{z}$. But $\mathrm{Id} \in \mathfrak{T}(V)$ since $(\mathrm{Id}, -\mathrm{Id}) \in \mathrm{Der}(V, V)$ and thus $\mathfrak{z} = k\,\mathrm{Id}$. Obviously, $\rho \,|\, \mathfrak{z}$ is nondegenerate.

If $\mathfrak{I}(V)$ is not irreducible, then, by (1.14), we know that $V = V^+ \oplus V^-$ is a direct sum of $\mathfrak{I}(V)$-irreducible subspaces V^+ and V^-. Again, we have $\mathrm{Id} \in \mathfrak{I}(V)$, i.e.

$$\mathrm{Id} = \sum_i L(u_i, v_i) = \sum_i L(u_i^+, v_i^-) + \sum_i L(u_i^-, v_i^+),$$

thus

$$\mathrm{Id}\,|\,V^+ = \sum_i L(u_i^+, v_i^-) \in \mathfrak{I}(V), \qquad \mathrm{Id}\,|\,V^- = \sum_i L(u_i^-, v_i^+) \in \mathfrak{I}(V).$$

Every endomorphism of V which commutes with $\mathrm{Id}\,|\,V^\varepsilon$, $\varepsilon = \pm$, leaves invariant V^ε. This implies $\mathfrak{z}(\mathfrak{I}(V)) = k\,\mathrm{Id}\,|\,V^+ \oplus k\,\mathrm{Id}\,|\,V^-$. Obviously, $\rho\,|\,\mathfrak{z}(\mathfrak{I}(V))$ is non-degenerate.

COROLLARY 1.4. *Let V be semisimple. Then* $\mathrm{Der}\,V$ *is completely reducible and the bilinear form ρ on* $\mathrm{Der}\,V$ *defined by $\rho(X, Y) = \mathrm{trace}(XY)$ is non-degenerate.*

Proof. Since $\mathrm{Der}\,V$ is almost algebraic it is, by Lemma 1.1, enough to show the second assertion. Let T^* denote the adjoint of $T \in \mathrm{End}\,V$ relative to the trace form σ of V. Taking traces in the defining identities for $(A, B) \in \mathrm{Der}(V, V)$ shows $B = -A^*$. Because also $(B, A) \in \mathrm{Der}(V, V)$ we see that $\mathfrak{I}(V)$ is invariant under $*$ and thus $\mathfrak{I}(V) = \mathfrak{I}_+(V) \oplus \mathfrak{I}_-(V)$ where $\mathfrak{I}_\pm(V) = \{T \in \mathfrak{I}(V); T^* = \pm T\}$. since $D \in \mathrm{Der}\,V$ iff $(D, D) \in \mathrm{Der}(V, V)$ we obtain $\mathrm{Der}\,V = \mathfrak{I}(V)$. Let $S \in \mathfrak{I}_+(V)$ and $T \in \mathfrak{I}_-(V)$. Then

$$\mathrm{trace}(ST) = \mathrm{trace}(ST)^* = \mathrm{trace}(T^*S^*) = -\mathrm{trace}(TS) = -\mathrm{trace}(ST)$$

shows that $\mathfrak{I}_+(V)$ and $\mathfrak{I}_-(V)$ are orthogonal relative to ρ, in particular $\rho\,|\,\mathrm{Der}\,V$ is non-degenerate.

An element $e \in V$ such that $P(e)e = e$ is called a *tripotent*. Every tripotent of V induces a *Peirce decomposition*

$$V = V_2(e) \oplus V_1(e) \oplus V_0(e)$$

where $V_i(e) = \{x \in V; \{eex\} = ix\}$ for $i = 0, 1, 2$. The Peirce spaces have the following multiplication rules:

(1.16) $\{V_i(e)V_j(e)V_k(e)\} \subset V_{i-j+k}(e)$, in particular, $V_i(e)$ is a subsystem of V.

(1.17) $V_2(e) = \mathrm{im}\,P(e)$ and $P(e)\,|\,V_2(e)$ is an involutorical automorphism of $V_2(e)$, hence

(1.18) $V_2(e) = V_2^+(e) \oplus V_2^-(e)$ with $V_2^\varepsilon(e) = \{x \in V_2(e); \bar{x} = \varepsilon x\}$ for $\varepsilon = \pm$, where we are using the abbreviation $P(e)x = \bar{x}$.

Together with the algebra product $xy = \frac{1}{2}\{xey\}$ the subspace $V_2(e)$ becomes a Jordan algebra denoted by $V_2(e)^{(e)}$. The triple product of $V_2(e)$ can be expressed by the algebra product

$$(1.19) \quad \{xyz\} = 2[x(\bar{y}z) - \bar{y}(xz) + (x\bar{y})z] \quad \text{for } x, y, z \in V_2(e).$$

Two nonzero tripotents e and f are called *orthogonal* if $f \in V_0(e)$ which is equivalent to $e \in V_0(f)$. A nonzero tripotent e is called *minimal*, if e cannot be written as the sum of two orthogonal tripotents. We remark that in this case e is a primitive idempotent of the Jordan algebra $[V_2^+(e)]^{(e)}$. Using well-known results for Jordan algebras (see e.g. [4] IV §5.5) we get

(1.20) If k is algebraically closed and $e \in V$ is a minimal tripotent, then $V_2^+(e) = ke \oplus \text{Rad } V_2^+(e)$.

An *orthogonal system* is a tuple $(e_1, \ldots, e_r)$ of pairwise orthogonal tripotents. In this case V has a *Peirce decomposition relative* to $(e_1, \ldots, e_r)$, i.e. a decomposition

$$(1.21) \quad V = \bigoplus_{0 \le i \le j \le r} V_{ij} \quad \text{where}$$

$$V_{00} = \bigcap_{i=1}^r V_0(e_i), \quad V_{ii} = V_2(e_i), \quad 1 \le i \le r,$$

$$V_{ij} = V_{ji} = V_1(e_i) \cap V_1(e_j) \quad \text{for } 1 \le i < j \le r,$$

$$V_{0j} = V_{j0} = V_1(e_j) \cap \bigcap_{i \ne j} V_0(e_i) \quad \text{for } 0 < j \le r.$$

We also put for $1 \le i \le r$

$$(1.22) \quad V_{ii} = V_{ii}^+ \oplus V_{ii}^- \quad \text{with } V_{ii}^\varepsilon = V_2^\varepsilon(e_i).$$

The Peirce spaces satisfy the following multiplication rules

(1.23) $\{V_{jk}V_{kl}V_{lm}\} \subset V_{jn}$, and all other types of products are zero.

The radical of V splits relative to $(e_1, \ldots, e_r)$. In particular, one derives from [17] Lemma 6:

(1.24) If V is semisimple, then all the Peirce spaces V_{ij} are semisimple, too.

In general, V need not contain any nonzero tripotents. However, one knowns ([**10**] 3.3):

(1.25) If k is algebraically closed and V is semisimple, then V contains nonzero tripotents.

By induction, we get:

(1.26) Let k be algebraically closed and V semisimple. Then V contains a maximal set of orthogonal minimal tripotents. The corresponding Peirce space V_{00} is zero.

2. Examples of Jordan triple systems with completely reducible derivation algebra.

2.1. As in the preceding section let k be a field of characteristic 0. Assume X is a finite-dimensional vector space over k and $(\cdot, \cdot): X \to k$ is a symmetric bilinear form. We put

$$(2.1) \qquad \{xyz\} = 2[(x, y)z + (z, y)x - (x, z)y].$$

It is well-known ([**10**]), that this triple system is in fact a Jordan triple system, called the *Jordan triple (system) of the quadratic form* $(\cdot, \cdot)$ and denoted by $[X; (\cdot, \cdot)]$. Obviously, its quadratic representation is given by

$$(2.2) \qquad P(x) = 2(x, y)x - (x, x)y.$$

From (2.1) one easily derives that the trace form σ is given by $\sigma(x, y) = 2(x, y) \dim X$. Hence, by (1.11.b), we get

(2.3) A Jordan triple system of a quadratic form is semisimple iff the form is non-degenerate.

Also, the next assertion trivially follows from (2.2) and (2.1):

(2.4) A nonzero element e of X is a tripotent iff $(e, e) = 1$. In this case $X = X_2(e) = ke \oplus X_2^-(e)$ with $X_2^-(e) = \{x \in X; (e, x) = 0\}$.

Jordan triples of quadratic forms occur in the following connection:

LEMMA 2.1. *Let V be a Jordan triple and $e \in V$ a tripotent with $V = V_2(e)$ and $V_2^+(e) = ke$. Then there exists a unique quadratic form $(\cdot, \cdot)$ on V such that V is the Jordan triple of $(\cdot, \cdot)$.*
The form is given by

$$(\alpha e + x, \beta e + y) = \alpha\beta - \tfrac{1}{2}\{xey\}, \qquad x, y \in V_2^-(e).$$

Proof. Since $P(e)$ is an involutorical automorphism of V we get $\{xey\} \in V_2^+ (e) = ke$ for every $x, y \in V_2^- (e)$. This shows that our form is —up to a trivial identification—well-defined. From (1.19) we derive $P(u)v = 2u(u\bar{v}) - u^2\bar{v}$, $uv = \frac{1}{2}\{uev\}$. Since

$$uv = (\alpha e + u^-)(\beta e + v^-) = (\alpha\beta - (u^-, v^-))e + \alpha v^- + \beta u^-$$

it is easily seen that (2.2) is valid in V, i.e. V is the Jordan triple of $(\cdot, \cdot)$. The uniqueness follows form (2.4) and (2.1).

There is a general method for constructing new Jordan triple systems out of old ones which is stated in the following

THEOREM 2.2 ([14]10 Theorem 2). *Let V be a Jordan triple system with quadratic representation P and $\varphi \in \text{End } V$ such that $P(\varphi x) = \varphi P(x)\varphi$ for all $x \in V$. Then the quadratic map $P_{(\varphi)}$ where $P_{(\varphi)}(x) = P(x)\varphi$ defines on V the structure of a Jordan triple system.*

We use this theorem in the following special case:

LEMMA 2.3. *Let X, M be nonzero finite dimensional vector spaces over k and $(\cdot, \cdot)$ a symmetric non-degenerate bilinear form on X. We define a quadratic representation on $V = X \oplus M$ by*

(2.5)
$$\begin{aligned} P(x \oplus m)(y \oplus n) = (2(x, y)x - (x, x)y) \\ \oplus ((x, x)n + 2(x, y)m) \end{aligned}$$

where $x, y \in X$ and $m, n \in M$. Then
    (a) *(V, P) is a Jordan triple system,*
    (b) *the decomposition $V = X \oplus M$ is a Wedderburn decomposition with the semisimple part X and $M = \text{Rad } V$,*
    (c) *$\{VVV\} = V$.*
    (d) *the derivation algebra of V consists of the mappings $D(x \oplus m) = D_1 x \oplus D_2 m$ where $D_1 \in \text{Der}[X; (\cdot, \cdot)]$ and $D_2 \in \mathfrak{gl}(M)$ is arbitrary. In particular, $\text{Der } V$ is completely reducible.*
    (e) *$\mathfrak{I}(V)$ and $\text{inn } \mathfrak{I}(V)$ are not completely reducible.*

Proof. (a) We extend $(\cdot, \cdot)$ to a bilinear form on V by setting $(x \oplus m, y \oplus n) = (x, y)$ and define $\varphi \in \text{End } V$, $\varphi(x \oplus m) = x - m$. Let Q denote the quadratic representation of the Jordan triple system $[V; (\cdot, \cdot)]$.

Then, using (2.2), we derive

$$Q(\varphi(x+m))(y+n) = Q(x-m)(y+n)$$
$$= 2(x,y)(x-m) - (x,x)(y+n)$$
$$= \varphi[2(x,y)(x+m) - (x,x)(y-n)] = \varphi Q(x+m)\varphi(y+n).$$

Thus, by Theorem 2.2, we conclude that $P = Q_{(\varphi)}$ defines on V a Jordan triple system. But

$$P(x+m)(y+n) = Q(x+m)(y-n)$$
$$= 2(x,y)(x+m) - (x,x)(y-n)$$

which is just (2.5).

(b) From (2.5) and (2.2) it follows that the subsystem X of V is the Jordan triple system of the quadratic form $(\cdot, \cdot)\,|\,X$. Hence, by (2.3), it is semisimple. On the other hand, $\{VVM\} = M = \{VMV\}$ and $\{MMM\} = 0$ shows that M is a nilpotent ideal of V and therefore $M \subset \operatorname{Rad} V$. Now (b) easily follows.

(c) Since X is semisimple we have $\{XXX\} = X$. Moreover, there exists $x \in X$ with $(x,x) \neq 0$. Then $P(x)M = M$. Altogether, $\{VVV\} = V$ follows.

(d) Let D be a derivation, $x \in X$. So $Dx = y \oplus m$ for some $y \in X$ and $m \in M$. Because $DP(x)x = P(x)Dx + \{xxDx\}$ we derive from (2.5)

$$(x,x)(y+m) = P(x)y + \{xxy\} + (x,x)m + 2(x,x)m,$$

hence $(x,x)m = 0$. But since the x with $(x,x) \neq 0$ form an algebraically dense subset of X we have $DX \subset X$. By (1.9) we also know $DM \subset M = \operatorname{Rad} V$. A straightforward computation using $(D_1 x, x) = 0$ for every $D_1 \in \operatorname{Der}[X;(\cdot,\cdot)]$ and $x \in X$ shows now that $D \in \operatorname{Der} V$ iff $D\,|\,X \in \operatorname{Der}[X;(\cdot,\cdot)]$ and $D\,|\,M \in \mathfrak{gl}(M)$. Since $\operatorname{Der}[X;(\cdot,\cdot)]$ is completely reducible in X by Corollary 1.4 it follows that $\operatorname{Der} V$ is completely reducible in V.

(e) Since $\operatorname{inn} \mathfrak{T}(V)$ is an ideal of $\mathfrak{T}(V)$ it follows from (1.2) that is enough to show: $\operatorname{inn} \mathfrak{T}(V)$ is not completely reducible. Putting $(xy^*)(z) := (y,z)x$ we get

$$\tfrac{1}{2}L(x \oplus m, y \oplus n)$$
$$= ((x,y)\operatorname{Id}_X + xy^* - yx^*) \oplus (nx^* + my^* + (x,y)\operatorname{Id}_M).$$

This easily implies Id_V, $xy^* - yx^*$ and $nx^* \in \operatorname{inn} \mathfrak{T}(V)$ for all $x, y \in X$, $n \in M$. Since the skew adjoint endomorphisms of X are spanned by $xy^* - yx^*$, $x, y \in X$, and $\operatorname{Hom}(X, M)$ is spanned by nx^*, $n \in M$, $x \in X$

we get

$$\operatorname{inn} \mathfrak{T}(V) = k \operatorname{Id} \oplus \operatorname{so}(X) \oplus \operatorname{Hom}(X, M)$$

as direct sum of vector spaces. It follows that $0 \neq \operatorname{Hom}(X, M) \subset \operatorname{inn} \mathfrak{T}(V)$ is an abelian ideal. If $\operatorname{inn} \mathfrak{T}(V)$ is completely reducible, then $\operatorname{Hom}(X, M)$ is also completely reducible and (1.1) implies that every element of $\operatorname{Hom}(X, M)$ is semisimple, but $\operatorname{Hom}(X, M)$ consists of nilpotent endomorphisms.

REMARK 2.4. (a) We will see later (Theorem 2.7) that there are other types of Jordan triple systems whose derivation algebras are completely reducible. In the sequel we will refer to a Jordan triple system considered in Lemma 2.3 as Example 2.3.

(b) For an Example 2.3 with $V = X \oplus M$ it is easy to see that M is a bimodule for X and V is the split-null-extension of X by M.

(c) One can show $\mathfrak{T}(V) = k \operatorname{Id} \oplus \operatorname{so}(X(\cdot, \cdot)) \oplus \operatorname{Hom}(V, M)$, but we don't need this in the sequel.

2.2. We consider the situation where the Jordan triple system V is a direct sum of two ideals V^1 and V^2. We put

$$\mathfrak{gl}_0(V) := \{A \in \mathfrak{gl}(V); AV^i \subset V^i, i = 1, 2\},$$
$$\mathfrak{gl}_1(V) := \{A \in \mathfrak{gl}(V), AV^1 \subset V^2, AV^2 \subset V^1\}$$

Then $\mathfrak{gl}(V) = \mathfrak{gl}_0(V) \oplus \mathfrak{gl}_1(V)$ is a $\mathbf{Z}_2$-grading of the Lie algebra $\mathfrak{gl}(V)$. Correspondingly,

$$(\mathfrak{gl}(V), \mathfrak{gl}(V)) = \mathfrak{gl}_0(V, V) \oplus \mathfrak{gl}_1(V, V)$$

where $\mathfrak{gl}_i(V, V) = (\mathfrak{gl}_i(V), \mathfrak{gl}_i(V))$.

LEMMA 2.5. *Let $V = V^1 \oplus V^2$ be a direct sum of two ideals V^i. Then*

(a) $\operatorname{Der}(V, V) = \operatorname{Der}_0(V, V) \oplus \operatorname{Der}_1(V, V)$ *where* $\operatorname{Der}_i(V, V) = \operatorname{Der}(V, V) \cap \mathfrak{gl}_i(V, V)$

(b) $\operatorname{Der}_0(V, V) = \operatorname{Der}(V^1, V^1) \oplus \operatorname{Der}(V^2, V^2)$ *is a direct sum of the ideals* $\operatorname{Der}(V^i, V^i) \subset \operatorname{Der}_0(V, V)$.

(c) *if* $(A, B) \in \mathfrak{gl}_1(V, V), (A, B) \in \operatorname{Der}_1(V, V)$ *iff*

$$A\{V^i V^i V^i\} = \{AV^i, V^j, V^j\} = \{V^j, AV^i, V^j\} = 0 \quad and$$
$$B\{V^i V^i V^i\} = \{BV^i, V^j, V^j\} = \{V^j, BV^j, V^j\} = 0$$

for $(i, j) = (1, 2), (2, 1)$.

Proof (a) Since $\pi\colon x^1 \oplus x^2 \to x^1 \oplus -x^2$ defines an automorphism of V we have $(\pi, \pi) \in \mathrm{Aut}(V, V)$, thus

$$(A, B) \to (\pi A \pi, \pi B \pi)$$

is an involutive automorphism of $\mathrm{Der}(V, V)$ and the decomposition in the assertion is just the eigenspace decomposition of this automorphism.

(b) is obvious and (c) is a straightforward verification.

We will need the following corollary later:

COROLLARY 2.6. *Let* $V = V^1 \oplus V^2$ *be a direct sum of two ideals* V^i, $i = 1, 2$.

(a) $\mathrm{Der}\, V = \mathrm{Der}_0\, V \oplus \mathrm{Der}_1 V$ *where* $\mathrm{Der}_i\, V = \mathrm{Der}\, V \cap \mathfrak{gl}_i(V)$

(b) $\mathrm{Der}_0\, V = \mathrm{Der}\, V^1 \oplus \mathrm{Der}\, V^2$ *is a direct sum of the two ideals* $\mathrm{Der}\, V^i$ $\subset \mathrm{Der}_0\, V$.

(c) *Let* $D \in \mathfrak{gl}_1(V)$. *Then* $D \in \mathrm{Der}_1 V$ *iff*

$$D\{V^i V^i V^i\} = \{DV^i, V^j, V^j\} = \{V^j, DV^i, V^j\} = 0$$

for $(i, j) = (1, 2), (2, 1)$.

Proof. This is an immediate consequence of Lemma 2.5, if one notes that the automorphism $(A, B) \to (\pi A \pi, \pi B \pi)$ leaves $\{(D, D) \in \mathrm{Der}(V, V)\} \sim \mathrm{Der}\, V$ invariant.

THEOREM 2.7. *Let* $V = V^1 \oplus V^2$ *be a direct sum of two ideals* V^i.

(a) *If* V^1 *is semisimple and* V^2 *is trivial, i.e.* $\{V^2 V^2 V^2\} = 0$, *then* $\mathfrak{I}(V)$ *and* $\mathrm{Der}(V)$ *are completely reducible.*

(b) *If* V^1 *is semisimple and* V^2 *is nonzero and a direct sum of ideals each of which is isomorphic to an Example 2.3, then* $\mathrm{Der}\, V$ *is completely reducible, but* $\mathfrak{I}(V)$ *is not.*

Proof. (a) Since $V^1_{(3)} = V^1$ by (1.11.d) and since V^1 contains no subspace I with $\{IV^1 V^1\} = 0 = \{V^1 I V\}$, it follows from Lemma 2.5 that $\mathrm{Der}_1(V, V) = 0$. Consequently $\mathfrak{I}(V) = \mathfrak{I}(V^1) \oplus \mathfrak{I}(V^2)$ is a direct sum of ideals $\mathfrak{I}(V^i) \subset \mathfrak{I}(V)$. But $\mathfrak{I}(V^1)$ is completely reducible by (1.15) and $\mathfrak{I}(V^2) = \mathfrak{gl}(V^2)$ is also completely reducible, which implies that $\mathfrak{I}(V)$ is completely reducible. Since $\mathrm{Der}\, V$ is a subalgebra of $\mathfrak{I}(V)$ we have $\mathrm{Der}\, V = \mathrm{Der}\, V^1 \oplus \mathrm{Der}\, V^2$ with $\mathrm{Der}\, V^2 = \mathfrak{gl}(V^2)$. Thus the remaining assertion is a consequence of Corollary 1.4.

(b) Here V is a direct sum of ideals I^i which are semisimple or isomorphic to an Example 2.3. By (1.11.d) and Lemma 2.3.c we know $I^i_{(3)} = I^i$ for all ideals I^i. Thus, by induction $\mathrm{Der}_1(V, V) = 0$ and $\mathfrak{T}(V) = \oplus \mathfrak{T}(I^i)$, $\mathrm{Der}\, V = \oplus_i \mathrm{Der}\, I^i$.

Since at least one I^i is an Example 2.3, the ideal $\mathfrak{T}(I^i)$ of $\mathfrak{T}(V)$ is not completely reducible (Lemma 2.3.e) and hence $\mathfrak{T}(V)$ is not completely reducible by (1.2). However, all ideals $\mathrm{Der}\, I^i$ are completely reducible and therefore $\mathrm{Der}\, V$ is completely reducible.

3. The center of a Jordan triple system.

The *center* of a Jordan triple system V is defined as the subspace

$$C(V) = \{c \in V;\ L(c, v) = L(v, c) \text{ for all } v \in V\}.$$

LEMMA 3.1. (a) *For $u, v \in V$ and $c, c_1, c_2 \in C(V)$ we have*

$$\{cvu\} = \{vcu\} = \{cuv\},$$

$$[L(u, v), L(c_1, c_2)] = 0 = [P(u, v), L(c_1, c_2)].$$

(b) $C(V)$ *is a subsystem of V.*

Proof. (a) Every $c \in C(V)$ satisfies $\{cvu\} = \{vcu\}$ for all $u, v \in V$. Since the right side of this equation is symmetric in u and v we also get $\{cvu\} = \{cuv\}$ for all $u, v \in V$. From (1.6) we derive for $c \in C(V)$: $[L(u, v), L(c, c)] = L(\{uvc\}, c) - L(c, \{vuc\}) = 0$ because $\{uvc\} = \{vuc\}$. Now $[L(u, v), L(c_1, c_2)] = 0$ follows by linearization since

$$2L(c_1, c_2) = L(c_1 + c_2, c_1 + c_2) - L(c_1, c_1) - L(c_2, c_2).$$

To prove the last equation we apply formula (1.7) with $y = c_1$, $x = c_2$ and $z, w \in V$ arbitrary. We get

$$P(z)\{c_1 c_2 w\} = -\{c_2 c_1 P(z)w\} + \{zw\{c_2 c_1 z\}\}$$
$$= \{c_2, c_1, -P(z)w + \{zwz\}\} = \{c_2 c_1 P(z)w\}$$

which shows $[P(z), L(c_1, c_2)] = 0$. Now the last equation follows by linearization.

(b) We have for $v \in V$ and $c_1 \in C(V)$ by applying (1.6) and (a):

$$L(\{c_1 c_2 c_3\}, v) = [L(c_1, c_2), L(c_3, v)] + L(c_3, \{c_2 c_1 v\})$$
$$= L(\{c_2 c_1 v\}, c_3)$$
$$= [L(c_2, c_1), L(v, c_3)] + L(v, \{c_1 c_2 c_3\})$$
$$= L(v, \{c_1 c_2 c_3\}),$$

i.e. $\{c_1 c_2 c_3\} \in C(V)$.

The following lemma will be used in the next section:

LEMMA 3.2. *Let $V = S \oplus R$ be a decomposition where S is a semisimple subsystem and R is an ideal of V contained in the center of V.*
   (a) *Every derivation of V leaves S invariant.*
   (b) *For $T \in \operatorname{End} R$ we define $D_T \in \operatorname{End} V$ by $D_T(s \oplus m) = Tm$, where $s \in S$, $m \in R$. Then $D_T \in \operatorname{Der} V$ iff*
   (1) $T \in \operatorname{Der} R$
   (2) $T\{stm\} = \{stTm\}$ *for all $s, t \in S$, $m \in R$ and*
   (3) $T\{smn\} = \{sTmn\} + \{smTn\}$ *for all $s \in S$, $m, n \in R$.*

Proof. (a) We consider V as a bimodule for S by setting $q(x) := P(x)$, $l(x, y) = L(x, y)$, $x, y \in S$, and denote by $\hat{V}$ the split-Null-extension of S by V (see [9] for details). Let D be a derivation of V. Then $D: S \to V$ is a derivation of S into the bimodule V. Since S is semisimple it follows from the Remark on page 26 in [9] that there are $s_1, \ldots, s_n \in S$ and $v_1, \ldots, v_n \in V$ such that $Ds = \sum_i \hat{\Delta}(s_i, v_i)s$ for $s \in S$ where $\hat{\Delta}(s_i, v_i)$ is an inner derivation of $\hat{V}$. Since the product in $\hat{V} = S \oplus V$ satisfies

$$\{s_1 \oplus v_1, s_2 \oplus v_2, s_3 \oplus v_3\}$$
$$= \{s_1 s_2 s_3\} \oplus \{s_1 s_2 v_3\} + \{s_1 v_2 s_3\} + \{v_1 s_2 s_3\}$$

it follows that

$$\hat{\Delta}(s_1 \oplus v_1, s_2 \oplus v_2)(s_3 \oplus v_3)$$
$$= \Delta(s_1, s_2)s_3 \oplus \Delta(s_1, s_2)v_3 + \Delta(s_1, v_2)s_3 + \Delta(v_1, s_2)s_3.$$

Hence $\hat{\Delta}(s_i, v_i)s = 0 \oplus \Delta(s_i, v_i)s$. We put $v_i = t_i + r_i$ with $t_i \in S$ and $r_i \in R$. Then $\Delta(s_i, v_i) = \Delta(s_i, t_i)$ and thus

$$Ds = \sum_i \Delta(s_i, v_i)s = \sum_i \Delta(s_i, t_i)s \in S.$$

   (b) A straightforward computation shows that $D_T \in \operatorname{Der} V$ is equivalent to (1), (2), (3) and the following two equations: $T\{smt\} = \{sTmt\}$, $T\{msn\} = \{Tmsn\} + \{msTn\}$ for all $s, t \in S$, $m, n \in R$. But since $R \subset C(V)$ these equations are consequences of (2) and (3).

4. Jordan triple systems with completely reducible derivation algebras.
For the case where the ground field is algebraically closed we will prove the converse of Theorem 2.7 inasmuch as the derivation algebras are concerned. Before stating the theorem we want to recall that we are only dealing with Jordan triple systems which are finite-dimensional over a field of characteristic zero.

THEOREM 4.1. *Let V be a Jordan triple system over an algebraically closed field. Then the following are equivalent*:

(a) Der V *is completely reducible*,

(b) V *is a direct sum of two ideals V^i, $i = 1, 2$, where after a possible change of numeration one of the following two cases occurs*:

(1) V^1 *is semisimple and V^2 is trivial*,

(2) V^1 *is semisimple and V^2 is a direct sum of ideals which are isomorphic to an Example* 2.3.

Proof. The implication (b) $\Rightarrow$ (a) was already proved in Theorem 2.7. To prove the converse, we proceed in several steps ((4.1) to (4.11)). We put $R = \mathrm{Rad}\, V$.

(4.1) $\Delta(V, R) = 0, \quad \text{i.e. } R \subset C(V).$

Proof. By (1.9) R is invariant under every derivation, hence the formula $[D, \Delta(u, v)] = \Delta(Du, v) + \Delta(u, Dv)$ for $D \in \mathrm{Der}\, V$ implies that $\Delta(V, R)$ is an ideal of Der V. Now (1.1) and (1.2) show that $\Delta(V, R)$ is reductive. On the other hand, $\Delta(V, R)$ consists of nilpotent endomorphisms by (1.10) and therefore is by Engel's theorem a nilpotent Lie algebra. It follows that $\Delta(V, R)$ is abelian and, by applying (1.1) once more, every element of $\Delta(V, R)$ is semisimple. Thus $\Delta(V, R) = 0$.

(4.2)   If $V = R$, then V is trivial, i.e. $\{VVV\} = 0$.

Proof. Since V is nilpotent, there exists an $m \in \mathbf{N}$ such that $V_{(2m-1)} \neq 0$ but $V_{(2m+1)} = 0$. We want to prove $m = 1$, so we assume $m \geq 2$. In this case the linear span of $\{L(x, y); x \in V_{(i)}, y \in V_{(j)}, i + j \geq 2m - 2\}$ is abelian by Lemma 3.1.a, contained in Der V because of $\{xy\{uvw\}\} \in V_{(i+j+3)} = 0$, and in fact is an ideal of Der V since $DV_{(i)} \subset V_{(i)}$ for every $D \in \mathrm{Der}\, V$. From (1.1) and (1.2) we derive that $L(x, y)$ for $x \in V_{(i)}$, $y \in V_{(j)}$, $i + j \geq 2m - 2$, is semisimple and from (1.10) we see that $L(x, y)$ is nilpotent, hence $L(x, y) = 0$. Since $V = C(V)$ we have

$$\{tu\{vwz\}\} = \{t\{uvw\}x\} = \{\{tuv\}wz\}$$

for every $t, u, v, w, z \in V$, which implies $V_{(2m-1)} = \{V_{(2m-3)}VV\} = 0$ because $L(V_{(2m-3)}, V) = 0$. This is a contradiction and shows $m = 1$.

Because of (4.2) we assume in the sequel $V \neq R$. We now come back to the general situation and choose a semisimple subsystem S of V such

that $V = R \oplus S$, which is possible thanks to (1.13). Then we are in the situation considered in Lemma 3.2.

(4.3)
$$\text{Der } V = \Delta(V, V) \oplus \text{ex Der } V \quad \text{(direct sum of ideals)}$$
where $\Delta(V, V) = \Delta(S, S) = \{D \in \text{Der } V; \quad DS \subset S,$
$D \,|\, R = 0\}$ and $\text{ex Der } V := \{D \in \text{Der } V; \quad DR \subset R,$
$D \,|\, S = 0\}.$

Proof. By Lemma 3.2 every $D \in \text{Der } V$ leaves S and R invariant; hence $D \,|\, S$ is a derivation which is inner by (1.12), i.e. $D \,|\, S = \Sigma_i \Delta(s_i, t_i) \,|\, S$ for some $s_i, t_i \in S$. This shows $D = \Sigma_i \Delta(s_i, t_i) + (D - \Sigma_i \Delta(s_i, t_i))$ with $\Sigma_i \Delta(s_i, t_i) \in \Delta(V, V)$ and $D - \Sigma_i \Delta(s_i, t_i) \in \text{ex Der } V$. Because $\Delta(s_i, t_i)R = 0$, we have $\Delta(S, S) = \{D \in \text{Der } V; DS \subset S, D \,|\, R = 0\}$. The equality $\Delta(S, S) = \Delta(V, V)$ is obvious from (4.1). It is also clear that $\Delta(V, V)$ and $\text{ex Der } V$ are subalgebras which annihilate each other, thus they are ideals of $\text{Der } V$.

In order to extend derivations, Lemma 3.2.b shows that we need to study the operation of the structure algebra of S on R. To this end we choose a maximal orthogonal system $(e_1, \ldots, e_r)$ of minimal tripotents of S. This is possible because the groundfield k is algebraically closed (1.26). We prove next:

(4.4)
$$L(S, S) \,|\, S = \sum_{j=1}^{r} kL(e_i, e_i) \,|\, R.$$

Proof. Let V_{ij} resp. S_{ij} be the Peirce spaces of V resp. S relative to $(e_1, \ldots, e_r)$. We know $S_{ii}^+ = ke_i$ by (1.20) and $S_{00} = 0$ by (1.26). Moreover, for $m \in R$ we have $\{e_i e_i m\} = \{e_i m e_i\}$. Hence, if $m = m_2^+ + m_2^- + m_1 + m_0$ is the Peirce decomposition of m relative to e_i, we see $2m_2^+ + 2m_2^- + m_1 = 2m_2^+ - 2m_2^-$, which implies $m_1 = 0 = m_2^-$. Therefore

$$R \subset \bigcap_{i=1}^{r} V_2^+ (e_i) \oplus V_0(e_i).$$

The definition of the Peirce spaces now implies $R = \oplus_{i=0}^{r} R_i$ with $R_i = R \cap V_{ii}^+$ where we put $V_{00}^+ = V_{00}$. This shows $V_{ii}^+ = ke_i \oplus R_i$, $V_{ii}^- = S_{ii}^-$ for $1 \leq i \leq r$ and $V_{ij} = S_{ij}$ for $i \neq j$.

We want to investigate $L(S, S)R$ which is a sum of products $\{u_{ij} v_{kl} r_{mm}\}$, $u, v \in S$, $r \in R$. Such a product is zero whenever $k, l \neq m$ and $\{i, j\} \cap \{k, l\} = \emptyset$ set. The remaining cases are products of type $\{u_{ij} v_{jm} r_{mm}\} \in V_{im} \cap R = \delta_{im} R_m$. Therefore only $L(u_{jm}, v_{jm})$ can operate non-trivially on R. Using $R \subset C(V)$ shows that the product $\{u_{jm} v_{jm} r_{mm}\} \in R_m$ also lies in R_j which implies $L(u_{jm}, v_{jm})R = 0$ unless $j = m$. In the

case $j = m$ we only need to look at a product of the type $\{u_{jj}v_{jj}r_{jj}\}$. We decompose $u_{jj} = \alpha e_j + u_{jj}^-$, $v_{jj} = \beta e_j + v_{jj}^-$ and get

$$\{u_{jj}v_{jj}r_{jj}\} = 2\alpha\beta r_{jj} + \alpha\{e_j v_{jj}^- \, r_{jj}\} + \beta\{u_{jj}^- \, e_j r_{jj}\} + \{u_{jj}^- \, v_{jj}^- \, r_{jj}\}.$$

But $\{e_j v_{jj}^- \, r_{jj}\}$ and $\{u_{jj}^- \, e_j r_{jj}\} \in V_{jj}^- \cap R = 0$ which shows $V_{jj}^- R_j = 0$, where the left-hand side is the product in the Jordan algebra $V_2(e_j)^{(e_j)}$. Now (1.19) implies $\{u_{jj}^- \, v_{jj}^- \, r_{jj}\} = -2(u_{jj}^- \, v_{jj}^- \,)r_{jj}$. But $u_{jj}^- \, v_{jj}^- \in ke_j$ and therefore $L(u_{jj}^-, v_{jj}^-\,) \, | \, R \in kL(e_j, e_j) \, | \, R$.

(4.5)   ex Der $V = \mathrm{Der}\, R_0 \oplus (\oplus_{j=1}^r \mathrm{Der}_2 R_j)$ is a direct sum of ideals where $D_j \in \mathrm{Der}_2 R_j$ iff $D_j \in \mathrm{Der}\, R_j$ and for all $m, n \in R_j$ $D_j\{me_j n\} = \{D_j me_j n\} + \{me_j D_j n\}$.

Proof. From (4.3) we know ex Der $R = \{D \in \mathrm{Der}\, V; \, DR \subset R, \, D \, | \, S = 0\}$. Applying Lemma 2.3.b and (4.4) shows $D \in \mathrm{ex\, Der}\, R$ iff $D = D_T$ where

$$T \in \mathrm{Der}\, R, \, \big[T, L(e_j, e_j)\big] = 0 \quad \text{for } 1 \le j \le r$$

and

$$T\{smn\} = \{sTmn\} + \{smTn\} \quad \text{for all } s \in S, \, m, n \in R.$$

First of all, $[T, L(e_j, e_j)] = 0$ for $1 \le j \le r$ is equivalent to $TR_j \subset R_j$ for $0 \le j \le r$, i.e. $T = \Sigma_{j=0}^r T_j$ where $T_j = T \, | \, R_j$. Since $R = \oplus_{j=0}^r R_j$ is a direct sum of ideals we get $T \in \mathrm{Der}\, R$ iff $T_j \in \mathrm{Der}\, R_j$ for $0 \le j \le r$. Finally, since $L(R_j, R_k) = 0$ for $j \ne k$ we have $\{smn\} = \Sigma_{j=0}^r\{sm_j n_j\} = \Sigma_{j=0}^r\{m_j s_{jj} n_j\}$. The summand $j = 0$ vanishes since $s_{00} = 0$, moreover $\{m_j s_{jj}^- \, n_j\} \in R_j \cap V_{jj}^- = 0$ and thus $\{smn\} = \Sigma_{j=1}^r\{s_{jj}^+ \, m_j n_j\}$. This shows that $T\{smn\} = \{sTmn\} + \{smTn\}$ is equivalent to

$$T_j\{me_j n\} = \{T_j me_j n\} + \{me_j T_j n\} \quad \text{for } 1 \le j \le r$$

and all $m, n \in R_j$. The assertion now easily follows.

(4.6) R_0 is an ideal of V with $\{R_0 VV\} = 0$.

Proof. We have

$$\{VR_0 V\} = \{VVR_0\} = \{SSR_0\} + \{SRR_0\} + \{RRR_0\}.$$

Here $\{SSR_0\} = 0$ by (4.4). Further

$$\{RR_0 S\} = \{R_0 R_0 S\} = \sum_{j=1}^r \{R_0 R_0 S_{0j}\} = \sum_{j=1}^r \{R_0 S_{0j} R_0\} = 0$$

and

$$\{RRR_0\} = \{R_0R_0R_0\}$$

Hence (4.6) follows if we can show $\{R_0R_0R_0\} = 0$. To this end we note, using (1.2), that Der R_0 is completely reducible in R. Therefore $\{R_0R_0R_0\} = 0$ is a consequence of (4.2).

(4.7) $$\{RRV\} = 0.$$

Proof. Because of (4.6) and $L(R_j, R_k) = 0$ for $j \neq k$ it is enough to show $\{R_jR_jV\} = 0$ for $1 \leq j \leq r$. Here

$$\{R_jR_jS\} = \{R_jSR_j\} = \{R_jS_{jj}R_j\} = \{R_jS_{jj}^+ R_j\}$$

since $\{R_jS_{jj}^- R_j\} \subset R \cap V_{jj}^- = 0$. Hence $\{R_jR_jS\} = 0$ iff $\{R_je_jR_j\} = 0$. But $R_j \subset V_2^+ (e_j)$ and thus (1.19) shows that $\{R_je_jR_j\} = 0$ also implies $\{R_jR_jR_j\} = 0$. We therefore have to prove $\{R_je_jR_j\} = 0$. We can assume $R_j \neq 0$ and for brevity let $N = R_j$, $c = e_j$. We define

$$N^{[0]} = N, \qquad N^{[k]} = \{NcN^{[k-1]}\} \quad \text{for } k \geq 1.$$

Since the associative algebra generated by $\{L(a,c)|N; a \in N\}$ is nilpotent ([11] Theorem 14.10), there is an $m \in \mathbf{N}$ such that $N^{[m]} = 0$, but $N^{[m-1]} \neq 0$. We assume $m \geq 2$. Then the vector space $l = \{L(a,c)|N; a \in N^{[m-2]}\}$ is well defined. To prove that l consists of derivations of N we first note that the subsystem $ke \oplus N$ coincides with its center, hence, by Lemma 3.1.a and (1.19) we have for $m, n, p \in N$

(∗) $$2\{mnp\} = \{mc\{ncp\}\} = \{\{mcn\}cp\} = \{nc\{mcp\}\}.$$

This implies for $a \in N^{[m-2]}$ that $\{ac\{mnp\}\} \in R^{[m+1]} = 0$ and $\{\{acm\}np\} = 0$. Thus $L(a,c)|N \in \text{Der } N$. Further, by (∗), $\{ac\{mcn\}\} = 0 = \{\{acm\}cn\} = \{mc\{acn\}\}$, which shows $L(a,c)|N \in \text{Der}_2 N = \text{Der}_2 R_j$.

Let $D \in \text{Der}_2 N$. Then $DN^{[k]} \subset N^{[k]}$ and $[D, L(a,c)]|N = L(Da,c)$ $|N$. Therefore l is an ideal of $\text{Der}_2 N$ which is abelian, because for $b \in N^{[m-2]}$ and $n \in N$ we have

$$[L(b,c), L(a,c)]n = L(\{bca\},c)n = \{nc\{acb\}\} \in N^{[m]} = 0.$$

Since $\text{Der}_2 N$ is completely reducible in N and $L(a,c)|N$ is nilpotent, we conclude from (1.1) that $l = 0$, i.e. $N^{[m-1]} = 0$ which is a contradiction to our definition of m. Thus $m = 1$, i.e. $\{R_je_jR_j\} = \{NcN\} = 0$.

If $R_0 \neq 0$, then $\bigoplus_{j=1}^r R_j = 0$ and $V = S \oplus R_0$ is a
(4.8) direct sum of the semisimple ideal S and the trivial ideal R_0.

Proof. We put $R^+ = \oplus_{j=1}^r R_j$ and compute $\{V, V, S \oplus R^+\}$ using (4.7). We get

$$\{VVS\} + \{VVR^+\} = \{SSS\} + \{RSS\} + \{SSR^+\} \subset S \oplus R^+$$

by (4.4). Similarly, $\{V, S \oplus R^+, V\} \subset S \oplus R^+$. Taking into account (4.6) we see that $V = R_0 \oplus (R^+ \oplus S)$ is a direct sum of two ideals. By (4.4) we have $(R^+ \oplus S)_{(3)} = R^+ \oplus S$. Since R^+ is the radical of $R^+ \oplus S$, we derive from Corollary 2.6, (4.6) and (4.7) that

$$\mathrm{Der}\, V = \mathrm{Der}(R^+ \oplus S) \oplus \mathrm{Der}\, R_0 \oplus \mathrm{Hom}(R_0, R^+).$$

Now a comparison with (4.3) and (4.5) shows $\mathrm{Hom}(R_0, R^+) = 0$. Hence $R^+ = 0$ in case $R_0 \neq 0$ and (4.8) follows.

In what follows we assume $R_0 = 0$ and show next

(4.9)   If $R_j \neq 0, j \neq 0$, then $V_{jk} = 0$ for $0 \leq k \leq r$ with $k \neq j$.

Proof. First, we look at the case $k = 0$ and assume that $V_{0j} = S_{0j} \neq 0$. Since S_{0j} is semisimple by (1.24), it contains a tripotent f, (1.25). By (1.7) we have

$$\{ffe_j\} = \{ffP(e_j)e_j\} = -P(e_j)\{ffe_j\} + \{e_j e_j \{ffe_j\}\}$$
$$= -P(e_j)\{ffe_j\} + 2\{ffe_j\},$$

because $\{ffe_j\} \in V_2(e_j)$. Thus $P(e_j)\{ffe_j\} = \{ffe_j\} \in S_2^+(e_j) = ke_j$. Since $f \in V_1(e_j)$, we know $\{ffe_j\} \neq 0$, and $P(f)e_j \in S_{00} = 0$ shows $e_j \in V_1(f) \oplus V_0(f)$. Altogether, this gives $\{ffe_j\} = e_j$. Now, for $n \in R_j$ we get $\{ffn\} = \{fnf\} = 0$, but $2n = \{e_j e_j n\} = \{e_j n e_j\} \in V_2(f)$, a contradiction. Therefore $S_{0j} = 0$.

Let $1 \leq k \leq r$. Then $V_{jk} = V_{jk}^+ \oplus V_{jk}^-$ where $V_{jk}^\pm = \{x \in V_{jk}; \{e_j x e_k\} = \pm x\}$. It is obviously enough to prove $V_{jk}^\varepsilon = 0$. We define the Jordan algebras $\mathcal{Q} = [V_2^+(e_j + \varepsilon e_k)]^{(e_k + \varepsilon e_k)}$ and $\mathcal{B} = \{S_2^+(e_j + \varepsilon e_k)]^{(e_j + \varepsilon e_k)}$. Then $\mathcal{B}$ is a subalgebra of the Jordan algebra $\mathcal{Q}$ with the same unit element. We assume $V_{jk}^\varepsilon = S_{jk}^\varepsilon \neq 0$. Then S_{jk}^ε contains an element which is invertible in $\mathcal{B}$ (by [4] VIII Lemma 3.3) and hence also in $\mathcal{Q}$. Thus $P(x)|\mathcal{Q}$ is bijective. On the other hand, we derive for $n \in R_j$ that $P(x)n = \frac{1}{2}\{xxn\} \in V_{kk} \cap V_{jj} = 0$, which forces n to be zero, a contradiction.

(4.10)   If $R_j \neq 0$, then $V_2(e_j)$ is isomorphic to an Example 2.3.

Proof. We know $V_2(e_j) = S_2(e_j) \oplus R_j$. Let s, $t \in S_2(e_j)$ and m, $n \in R_j$. Then, using $\{RRV\} = 0$ by (4.7), we derive

$$P(s \oplus m)(t \oplus n) = P(s)t \oplus P(s)n + \{stm\}.$$

Because $S_2(e_j) = ke_j \oplus S_2^-(e_j)$ we can apply Lemma 2.1 and (2.3) and conclude that $S_2(e_j)$ is the Jordan triple of a non-degenerate symmetric bilinear form $(\cdot, \cdot)$. A comparison with Lemma 2.3 now shows that (4.10) follows if we can prove $P(s)n = (s, s)n$. To this end we decompose $s = \alpha e_j + s^-$ with $\alpha \in k$ and $s^- \in S_2^-(e_j)$. Since $\{s^-ne_j\} = 0$, we get

$$P(s)n = \alpha^2 P(e_j)n + P(s^-)n = \alpha^2 n + P(s^-)n.$$

From (1.19) we derive

$$P(s^-)n = \tfrac{1}{2}\{s^-s^-n\} = -\tfrac{1}{4}\{\{s^-e_js^-\}e_jn\}.$$

But $\{s^-e_js^-\} = 2P(s^-)e_j = -2(s^-, s^-)e_j$ implies

$$P(s^-)n = \tfrac{1}{2}(s^-, s^-)\{e_je_jn\} = (s^-, s^-)n$$

and $P(s)n = (s, s)n$ follows. Finally, we complete the proof of Theorem 4.1 by showing

(4.11) Let $A := \{j;\ R_j \neq 0\}$, $c = \Sigma_{j \notin A} e_j$, $V^1 := V_2(c) \oplus V_1(c)$ and $V^2 = \oplus_{j \in A} V_2(e_j)$. Then $V = V^1 \oplus V^2$ is a direct sum of ideals where V^1 is semisimple.

Proof. By (4.9) and the assumption $R_0 = 0$ we get $V = V^1 \oplus V^2$ which is obviously a direct sum of ideals. Since $\mathrm{Rad}\, V^1 = V^1 \cap \mathrm{Rad}\, V = 0$, we see that V^1 is semisimple.

We come back to the general situation considered in the paper: V is a finite-dimensional Jordan triple system over a field of characteristic zero. In this situation we have

COROLLARY 4.2. *If* $\mathrm{Der}\, V$ *is completely reducible, then* $\mathrm{Rad}\, V$ *is contained in the center of* V *and* $\{V, \mathrm{Rad}\, V, \mathrm{Rad}\, V\} = 0$.

Proof. Let K be an extension field of k. Then $\mathrm{Der}(K \otimes V) = K \otimes \mathrm{Der}\, V$ by (1.8) and $\mathrm{Der}(K \otimes V)$ is completely reducible in $K \otimes V$ by (1.3). Since $\mathrm{Rad}(K \otimes V) = K \otimes \mathrm{Rad}\, V$ and $C(K \otimes V) = K \otimes C(V)$ we may assume that k is algebraically closed. But in this case the assertion is just (4.1) and (4.7).

The following theorem is a generalization of a theorem of K.-H. Helwig ([6]) which was quoted in the introduction.

THEOREM 4.3. *For a Jordan triple system V the following statements are equivalent*:

(a) Der V *is completely reducible and every derivation of V has trace zero*,

(b) V *is semisimple*,

(c) *the bilinear form* $(D_1, D_2) \to \text{trace}(D_1 D_2)$ *of* Der V *is non-degenerate, and every derivation of V is inner*.

Proof. We put $R = \text{Rad } V$ and choose a semisimple subsystem S of V such that $V = S \oplus R$, which is possible by (1.13). If (a) or (c) holds, then, using Lemma 1.1 in case (c), Der V is completely reducible and we can apply Corollary 4.2 to conclude $R \subset C(V)$ and $\{RRV\} = 0$. Let $T = \text{Id}_R$. Then $D_T \in \text{Der } V$ by Lemma 3.2.(b). But obviously D_T has trace zero or is inner iff $R = 0$, i.e. V is semisimple. Therefore both (a) and (c) imply (b).

If (b) holds, then Der V is completely reducible and the trace form of Der V is non-degenerate by Corollary 1.4. Moreover let σ be the trace form of the Jordan triple V. Then σ is non-degenerate and every derivation of V is skew relative to σ and therefore has trace zero. It is also known that every derivation is inner (1.12). Hence (b) implies (a) and (c). ∎

COROLLARY 4.4. *If* Der V *is semisimple, then V is semisimple too*.

5. Real Jordan triple systems with compact automorphism group.

We apply Theorem 4.3 to classify real Jordan triple systems with compact automorphism group.

We first recall that a real Jordan triple system is called *compact* (resp. of *non-compact type*) if its trace form σ is positive-definite (resp. negative-definite).

THEOREM 5.1. *Let V be a real Jordan triple system. Then* Aut V *is compact iff V is semisimple and for every simple component V^i of V one of the following three cases occurs*:

(1) V^i *is compact*,

(2) V^i *is of non-compact type*,

(3) V^i *is isomorphic to the Jordan triple system defined on an Euclidean space* $(X, \langle \cdot, \cdot \rangle)$ *by*

$$(5.1) \qquad\qquad P(x)y = 2\langle x, \bar{y} \rangle x - \langle x, x \rangle \bar{y}$$

where $\bar{}$ is a non-trivial orthogonal reflection.

Proof. We assume that Aut V is compact and proceed in several steps:

(5.2) V is semisimple.

Proof. If Aut V is compact, it is well-known that there exists a positive-definite form on V such that Der V consists of skew-adjoint endomorphisms. In particular, Der V is completely reducible and every derivation of V has trace zero. Hence (5.2) is a consequence of Theorem 4.3.

(5.3)   We can assume: V is simple.

Proof. Let $V = V^1 \oplus \cdots \oplus V^q$ be the decomposition of V into its simple components. Then Aut V^j is canonically imbedded into Aut V as a closed subgroup, hence Aut V^j is compact.

By [15] Satz 3.3 we know that V has a Cartan involution α. Let $V = V_+ \oplus V_-$ be the corresponding eigenspace decomposition. Then V_+ is compact and V_- is of non-compact type. Obviously we may assume $V_+ \neq 0$ and $V_- \neq 0$.

(5.4) $L(x_+, x_-) = L(x_-, x_+) = P(x_+, x_-)$ for every $x_\pm \in V_\pm$.

Proof. We can apply [15] Theorem 2.3.(b) (taking $\lambda = $ trace form) and see that $[D, \alpha] = 0$ for every derivation D of V. In particular, we get $\alpha\Delta(x_+, x_-)\alpha = \Delta(x_+, x_-)$. But $\alpha\Delta(x_+, x_-)\alpha = -\Delta(x_+, x_-)$ and hence $\Delta(x_+, x_-) = 0$, which is equivalent to the first equation of (5.4). To prove the second, let $u = u_+ + u_- \in V$. Then, by applying the first equation (5.4), we get

$$\{x_+ x_- u\} = \{x_+ x_- u_+\} + \{x_+ x_- u_-\} = \{x_+ u_+ x_-\} + \{x_- x_+ u_-\}$$
$$= \{x_+ u_+ x_-\} + \{x_- u_- x_+\} = \{x_+ u x_-\},$$

i.e. $L(x_+, x_-) = P(x_+, x_-)$.

We recall from [12] §1.1 that every element $x \in V_+$ has a minimal decomposition, i.e. $x = \sum_{j=1}^n \lambda_j e_j$ where $\lambda_j \in \mathbf{R}$ and $(e_1, \ldots, e_n)$ is an orthogonal system of minimal tripotents. In particular, V_+ has many tripotents.

(5.5)   If c is a tripotent in V_+, then $V_- = (V_- \cap V_2^+(c)) \oplus (V_- \cap V_0(c))$.

Proof. Let $x_- \in V_-$ have the Peirce decomposition $x_- = y_2^+ + y_2^- + y_1 + y_0$ with $y_2^\pm \in V_2^\pm(c)$ and $y_i \in V_i(c)$, $i = 1, 0$. Then $\{cx_-c\} = \{x_- cc\}$

by (5.4). But the left hand side equals $2y_2^+ - 2y_2^-$, whereas the right hand side is $2y_2^+ + 2y_2^- + y_1$. Hence $y_2^- = 0 = y_1$. Since $L(c, c)V_- \subset V_-$ and $P(c)V_- \subset V_-$, we get the claimed splitting of V_-.

(5.6) $V = V_2(c)$ where c is a tripotent of V_+ which is minimal in V_+.

Proof. First, we assume: For every minimal tripotent c of V_+ the Peirce space $V_1(c)$ is non-zero.

Let $x \in V_- \cap V_2(c)$, c a minimal tripotent of V_+. Then $L(c, x)V_1(c) = L(x, c)V_1(c) = P(x, c)V_1(c) = 0$. Hence

$$V_- \cap V_2(c) \subset I_2(c) = \{y \in V_2(c);\, L(y, c)V_1(c) = L(c, y)V_1(c) = 0\}.$$

But $I_2(c)$ is an ideal of $V_2(c)$ ([**16**] Corollary 1.8) and $V_2(c)$ is simple by [**13**]. If $I_2(c) = V_2(c)$, then $c \in I_2(c)$ implies $V_1(c) = 0$. Therefore, by our assumption, $V_- \cap V_2(c) \subset I_2(c) = 0$ and, by (5.5), $V_- \subset V_0(c)$. From the Peirce multiplication rules now follows $L(c, V_-) = 0$. Since this holds for every minimal tripotent of V_+ and since every element of V_+ has a minimal decomposition as mentioned above, we conclude $L(V_+, V_-) = 0 = P(V_+, V_-)$. This implies that $V = V_+ \oplus V_-$ is a direct sum of two non-trivial ideals, a contradiction to (5.3).

Therefore, there exists a minimal tripotent c of V_+ such that $V_1(c) = 0$. Since V is simple, this forces $V_0(c)$ to vanish and (5.6) follows.

We denote by X_+ the underlying vector space of the Jordan triple system V_+. Then

(5.7)   There exists a scalar product $\langle \cdot, \cdot \rangle$ on X_+ such that V_+ is the Jordan triple system associated to $\langle \cdot, \cdot \rangle$. Moreover, $\{x_+ y_+ y_+ z_-\} = 2\langle x_+, y_+ \rangle z_-$.

Proof. By (5.6) we know $V_+ = (V_+)_2(c)$ where c is a minimal tripotent of V_+. Therefore $(V_+)_2^+(c) = \mathbf{R}c$ and Lemma 2.1 says that V_+ is the Jordan triple associated to a bilinear form $\langle \cdot, \cdot \rangle$ on X_+. It is easy to compute that the trace form σ of V_+ is

$$\sigma(x, y) = \tfrac{1}{2}\operatorname{trace}(L(x, y) + L(y, x)) = 2\langle x, y \rangle \dim X_+$$

which implies that $\langle \cdot, \cdot \rangle$ is positive-definite.

Let $d \in V_+$ and assume $\langle d, d \rangle = 1$. Then by (2.4) d is a tripotent of $V_+ = (V_+)_2(d)$. In particular, $c \in V_2(d)$ and $L(c, c)(V_- \cap V_0(d)) = 0$. But $V_- \subset V_2(c)$ and therefore $L(c, c)|(V_- \cap V_0(d)) = 2\,\mathrm{Id}$ which shows $V_- \cap V_0(d) = 0$ and, by (5.5), $V_- \subset V_2^+(d)$. Hence, for $x_+ \in V_+$ arbitrary, $\{x_+ x_+ z_-\} = 2\langle x_+, x_+ \rangle z_-$. Since $\{x_+ y_+ z_-\} = \{y_+ x_+ z_-\}$, we get by linearization $\{x_+ y_+ z_-\} = 2\langle x_+, y_+ \rangle z_-$.

We denote by X_- the underlying vector space of the Jordan triple V_-. Then

(5.8) There exists a scalar product $\langle \cdot, \cdot \rangle$ on X_- such that V_- is the Jordan triple associated to $-\langle \cdot, \cdot \rangle$. Moreover,
$$\{x_- y_- z_+\} = -2\langle x_-, y_- \rangle z_+.$$

Proof. Let $\{ \cdots \}$ denote the Jordan triple product of V and V^- the Jordan triple system with triple product $-\{ \cdots \}$. Then V^- is simple, $\mathrm{Aut}\, V^- = \mathrm{Aut}\, V$ is compact and $-\alpha$ is a Cartan involution of V, hence $(V^-)_+ = V_-$, $(V^-)_- = V_+$. We therefore can apply (5.7) to V_- and get (5.8).

Let X be the underlying vector space of V. Then X has a scalar product $\langle \cdot, \cdot \rangle$, given by $\langle x_+ + x_-, y_+ + y_- \rangle = \langle x_+, y_+ \rangle + \langle x_-, y_- \rangle$. We define $\bar{x} = \overline{x_+ + x_-} = x_+ - x_-$. Then $^-$ is a nontrivial orthogonal reflection. Moreover, we have for $x, y, z \in V$

(5.9) $$\{xyz\} = 2[\langle x, \bar{y} \rangle z + \langle z, \bar{y} \rangle x - \langle x, z \rangle \bar{y}].$$

Proof. From (5.7) resp. (5.8) and (2.1) we derive

$$\{x_\varepsilon y_\varepsilon z_\varepsilon\} = 2\varepsilon[\langle x_\varepsilon, y_\varepsilon \rangle z_\varepsilon + \langle z_\varepsilon, y_\varepsilon \rangle x_\varepsilon - \langle x_\varepsilon, z_\varepsilon \rangle y_\varepsilon].$$

Moreover, by (5.4), we know $\{x_+ y_+ z_-\} = \{x_+ z_- y_+\} = \{z_- x_+ y_+\}$ and $\{x_- y_- z_+\} = \{x_- z_+ y_-\} = \{z_+ x_- y_-\}$. Hence, by (5.7) resp. (5.8), we get for $\varepsilon = \pm$:

$$\{x_\varepsilon y_\varepsilon z_{-\varepsilon}\} = 2\varepsilon \langle x_\varepsilon, y_\varepsilon \rangle z_{-\varepsilon},$$
$$\{x_\varepsilon y_{-\varepsilon} z_\varepsilon\} = 2\varepsilon \langle x_\varepsilon, z_\varepsilon \rangle y_{-\varepsilon} \quad \text{and}$$
$$\{x_{-\varepsilon} y_\varepsilon z_\varepsilon\} = 2\varepsilon \langle y_\varepsilon, z_\varepsilon \rangle x_{-\varepsilon}.$$

It is now a straightforward verification, that in all cases the triple product $\{x_\varepsilon y_\mu z_\delta\}$, $\varepsilon, \mu, \delta = \pm$, can be expressed by the right-hand side of (5.9). This proves (5.9).

Since $P(x)y = \frac{1}{2}\{xyx\} = 2\langle x, \bar{y} \rangle x - \langle x, x \rangle \bar{y}$ by (5.9) we have finally proved one direction of Theorem 5.1. We consider now the other direction: Let $V = V^1 \oplus \cdots \oplus V^q$ be the decomposition of V into its simple ideals and $G = (\mathrm{Aut}\, V^1) \times \cdots \times (\mathrm{Aut}\, V^q)$. Then $\mathrm{Aut}\, V/G$ is finite, since it is represented by automorphisms which map isomorphic ideals onto each other. It is therefore enough to show that $\mathrm{Aut}\, V^j$ is compact for $1 \leq j \leq q$. This is clear if V^j is compact or of non-compact type since

every automorphism is orthogonal relative to the trace form. Hence it remains to prove that Aut V is compact for a Jordan triple V given by (5.9).

Let $\varphi \in \text{Aut } V$. Then φ is orthogonal relative to the trace form σ of V which is $\sigma(x, y) = 2\langle x, \bar{y} \rangle \dim V$. Hence $\langle x, \bar{y} \rangle = \langle \varphi x, \overline{\varphi y} \rangle$ for $x, y \in V$. Moreover

$$2\langle x, \bar{y} \rangle \varphi x - \langle x, x \rangle \varphi(\bar{y}) = \varphi(P(x)y)$$
$$= P(\varphi x)\varphi y = 2\langle \varphi x, \overline{\varphi y} \rangle \varphi x - \langle \varphi x, \varphi x \rangle \overline{\varphi y}$$

implies $\langle x, x \rangle \varphi(\bar{y}) = \langle \varphi x, \varphi x \rangle \overline{\varphi y}$ for all $x, y \in V$, i.e. $\varphi(\bar{y}) = \alpha \overline{\varphi y}$ with $\alpha \in \mathbf{R}$, $\alpha > 0$. Taking the determinant shows $\alpha = 1$. Thus φ is also orthogonal relative to $\langle \cdot, \cdot \rangle$. Conversely, it is obvious that a map with $\varphi(\bar{y}) = \overline{\varphi(y)}$ and $\langle \varphi x, \varphi y \rangle = \langle x, y \rangle$ for every $x, y \in V$ is an automorphism of V, i.e. $\text{Aut } V = \{\varphi \in 0(V, \langle \cdot, \cdot \rangle); \ \varphi(\bar{y}) = \overline{\varphi(y)} \text{ for all } y \in V\}$ which clearly is a compact group.

6. Jordan triple systems with completely reducible structure algebras. As a further application of Theorem 4.1 we will classify Jordan triple systems with completely reducible structure algebras, even in the case where the ground field is not algebraically closed. The link to Theorem 4.1 is provided by the following

LEMMA 6.1. *If $\mathfrak{T}(V)$ is completely reducible, then $\text{Der } V$ is completely reducible..*

Proof. Since $\mathfrak{T}(V)$ is completely reducible there is a decomposition $V = W^1 \oplus \cdots \oplus W^q$ where the W^i are $\mathfrak{T}(V)$-irreducible subspaces. This yields a decomposition

$$(V, V) = (W^1, 0) \oplus \cdots \oplus (W^q, 0) \oplus (0, W^1) \oplus \cdots \oplus (0, W^q)$$

into $\text{Der}(V, V)$-irreducible subspaces. Hence $\text{Der}(V, V)$ is completely reducible.

The map $\theta: \text{Der}(V, V) \to \text{Der}(V, V): (A, B) \to (B, A)$ is an involutive automorphism of $\text{Der}(V, V)$ inducing a decomposition

$$\text{Der}(V, V) = \mathfrak{K} \oplus \mathfrak{N}$$

where $\mathfrak{K} = \{(A, A) \in \text{Der}(V, V)\}$ and $\mathfrak{N} = \{(A, -A) \in \text{Der}(V, V)\}$. Since $\text{Der}(V, V)$ is completely reducible, every nilpotent element can be imbedded in a three-dimensional split simple subalgebra of $\text{Der}(V, V)$ ([8] III Theorem 17.1). Because $[\mathfrak{K}, \mathfrak{K}] \subset \mathfrak{K}$ and $[\mathfrak{K}, \mathfrak{N}] \subset \mathfrak{N}$ the same

property holds for $\mathfrak{K}$ by [8] III Lemma 8. Moreover $\mathfrak{K}$ is almost algebraic and thus the center of $\mathfrak{K}$ is almost algebraic, too. Thus [8] III Theorem 17.2 implies that $\mathfrak{K}$ is completely reducible.

Now $\mathfrak{K} \to \mathrm{Der}\, V: (A, A) \to A$ is an isomorphism, in particular $\mathrm{Der}\, V$ is reductive. If D is in the center of $\mathrm{Der}\, V$, (D, D) is in the center of $\mathfrak{K}$ and thus semisimple, which shows that D is semisimple. By (1.1) $\mathrm{Der}\, V$ is completely reducible.

THEOREM 6.2. *For a Jordan triple system V the following are equivalent*:
(a) $\mathfrak{T}(V)$ *is completely reducible*,
(b) *V is the direct sum of a semisimple and a trivial ideal.*

Proof. Since (b) $\Rightarrow$ (a) was already proved in Theorem 2.7 it remains to show (a) $\Rightarrow$ (b).

We first consider the case where the ground field k is algebraically closed. Since $\mathrm{Der}\, V$ is completely reducible we can apply Theorem 4.1 which in conjunction with Theorem 2.7(b) shows (b), i.e.

$$V = \{VVV\} \oplus \mathrm{Rad}\, V$$

where $\{VVV\}$ is a semisimple ideal and $\mathrm{Rad}\, V$ is a trivial ideal.

In general, let $\bar{k}$ be an algebraic closure of k. Since $\bar{k} \otimes \{VVV\} = \{\bar{k} \otimes V, \bar{k} \otimes V, \bar{k} \otimes V\}$ we see that $\{VVV\}$ is a semisimple deal, similarly $\mathrm{Rad}\, V$ is a trivial ideal. Because $\bar{k} \otimes V = (\bar{k} \otimes \{VVV\}) \oplus (\bar{k} \otimes \mathrm{Rad}\, V)$ we get $V = \{VVV\} \oplus \mathrm{Rad}\, V$.

REMARK 6.3. If $V = I^1 \oplus I^2$ is a direct sum of ideals where I^1 is semisimple and I^2 is trivial, then $I^1 = \{VVV\}$ and $I^2 = \mathrm{Rad}\, V$, i.e. the decomposition in Theorem 6.2(b) is unique.

It seems to be worthwhile to mention the following

COROLLARY 6.4. *Let $\mathfrak{J}$ be a unital Jordan algebra. Then $\mathfrak{J}$ is semisimple iff the structure algebra of $\mathfrak{J}$ is completely reducible.*

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MATH. INSTITUT DER UNIVERSITÄT
EINSTEINSTR. 64/IV
4400 MÜNSTER
WEST-GERMANY

Current address: Department of Mathematics
                   University of Ottawa
                   Ottawa, Ontario K1N 9B4
                   Canada

Pacific Journal of Mathematics

Vol. 113, No. 1   March, 1984