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Dedicated to Professor F. Burton Jones

In answer to a question of T. Mackowiak and E. D. Tymchatyn [20] we prove that every atriodic homogeneous continuum is 1-dimensional. This is accomplished by showing that every atriodic homogeneous continuum that is not a solenoid and has a decomposable subcontinuum admits a continuous decomposition to a solenoid and that all elements of this decomposition are homeomorphic tree-like hereditarily indecomposable homogeneous continua. It follows from this decomposition theorem that every tree-like atriodic homogeneous continuum is hereditarily indecomposable. This decomposition theorem also provides another proof of the author's theorem [11] that every indecomposable homogeneous plane continuum is hereditarily indecomposable.

1. Introduction. A space is *homogeneous* if for each pair p, q of its points there is a homeomorphism of the space onto itself that takes p to q. A *continuum* is a compact connected nondegenerate metric space. A continuum M is a *triod* if M has a subcontinuum H such that $M \setminus H$ is the union of three nonempty disjoint opens sets. When a continuum does not contain a triod it is *atriodic*. A continuum M is *tree-like* if for each positive number ε there is an open covering of M with mesh less than ε whose nerve is a tree. A continuum is *decomposable* if it is the union of two of its proper subcontinua; otherwise, it is *indecomposable*. When a continuum does not have a decomposable subcontinuum it is *hereditarily indecomposable*. Note that every hereditarily indecomposable continuum is atriodic.

In 1951 R. H. Bing [2] proved that every finite-dimensional hereditarily indecomposable homogeneous continuum is 1-dimensional. Recently J. T. Rogers, Jr. [25] proved that every hereditarily indecomposable homogeneous continuum is tree-like and, therefore, 1-dimensional. Mackowiak and Tymchatyn [20, Theorem 13.4] proved that every finite-dimensional atriodic homogeneous continuum is 1-dimensional. In §13 of [20], Mackowiak and Tymchatyn asked if every atriodic homogeneous continuum is 1-dimensional. Corollary 1 of §4 (below) answers this question in the affirmative.

Our arguments involve a decomposition theory for homogeneous continua that was originated in 1951 by F. B. Jones [15]. Recently Rogers [24] surveyed this area and presented a general decomposition theory for homogeneous spaces. Theorem 2 of §4 (below) solves a problem of Jones' that Rogers [**24**, page 142] called the outstanding problem in decompositions of homogeneous continua.

Mackowiak and Tymchatyn [**20**, Theorem 14.7] proved that every decomposable atriodic homogeneous continuum that is not a simple closed curve has a continuous decomposition to a circle and that the elements of this decomposition are homeomorphic indecomposable homogeneous continua. In §14 of [**20**] they asked if every atriodic homogeneous continuum that is not a solenoid and has a decomposable subcontinuum admits a continuous decomposition to a solenoid such that all elements of the decomposition are homeomorphic tree-like hereditarily indecomposable homogeneous continuum. Theorem 2 of §4 (below) answers this question in the affirmative.

Bing [**4**, Theorem 10] proved that no tree-like atriodic homogeneous continuum contains an arc. Mackowiak and Tymchatyn [**20**, Theorem 14.8] generalized Bing's theorem by proving that no tree-like atriodic homogeneous continuum has a hereditarily decomposable subcontinuum. According to Corollary 2 of §4 (below), no tree-like atriodic homogeneous continuum has a decomposable subcontinuum.

The known examples of atriodic homogeneous continua are the solenoids [**12**], the pseudo-arc [**1**], and the solenoids of pseudo-arcs [**23**] [**13**]. By Rogers' theorem [**25**] and Theorem 2 of §4 (below), if every tree-like homogeneous continuum is a pseudo-arc, then there are no other examples of atriodic homogeneous continua. Unfortunately, it is not known whether the pseudo-arc is the only tree-like continuum that is homogeneous. For additional information and unsolved problems involving 1-dimensional homogeneous continua see C. E. Burgess' expository article [**7**].

2. More definitions and related results. A *chain* is a finite collection $\{L_i\colon 1 \leq i \leq n\}$ of open sets such that $L_i \cap L_j \neq \varnothing$ if and only if $|i-j| \leq 1$. If L_1 also intersects L_n the collection is called a *circular chain*. Each L_i is called a *link*. A chain (circular chain) is called an *ε-chain* (*ε-circular chain*) if each of its links has diameter less than ε. A continuum is *arc-like* (*circle-like*) if for each positive number ε, it can be covered by an ε-chain (ε-circluar chain). Bing [**1**] [**3**] proved that a continuum is a pseudo-arc if and only if it is homogeneous and arc-like.

A continuum is a *solenoid* if it is homeomorphic to an inverse limit of circles with covering maps as the bonding maps. Note that simple closed curves are solenoids. The author [**12**] proved that a continuum M is a solenoid if and only if M is homogeneous and every proper subcontinuum of M is an arc. Rogers [**24**, Theorem 3] proved that every atriodic homogeneous 1-dimensional continuum that contains an arc is a solenoid. In [**20**, Theorem 14.8], Mackowiak and Tymchatyn generalized these results by showing that every atriodic homogeneous continuum that contains a hereditarily decomposable continuum is a solenoid.

A continuum M is a *solenoid of pseudo-arcs* if M is circle-like and there exists a continuous decomposition $\mathfrak{D}$ of M to a solenoid such that each element of $\mathfrak{D}$ is a pseudo-arc. In 1959 Bing and Jones [**5**] constructed the circle of pseudo-arcs. Rogers [**23**] used this continuum to construct a solenoid of pseudo-arcs for each solenoid. The author and Rogers [**13**] proved that every circle-like homogeneous continuum is either a solenoid, a pseudo-arc, or a solenoid of pseudo-arcs.

Following K. Kuratowski [**18**] we define a continuum M to be of *type* λ if M is irreducible and every indecomposable continuum in M is a continuum of condensation. Type λ continua are studied by E. S. Thomas in [**26**]. There they are called continua of type A'.

If a continuum M is of type λ, then M admits a unique monotone upper semi-continuous decomposition $\mathfrak{D}$ such that $M/\mathfrak{D}$ is an arc and each element of $\mathfrak{D}$ has a void interior relative to M [**19**, Theorem 3, page 216] [**26**, Theorem 10, page 15]. We shall refer to $\mathfrak{D}$ as simply the decomposition of M.

3. Preliminaries. Throughout this section M is an atriodic homogeneous continuum with metric ρ.

Let ε be a positive number. A homeomorphism h of M onto M is called an *ε-homeomorphism* if $\rho(x, h(x)) < \varepsilon$ for each point x of M.

NOTATION. Let x be a point of M. We denote $\{y \in M$: an ε-homeomorphism of M onto M takes x to $y\}$ by $W(x, \varepsilon)$. Let X be a subset of M. We denote $\bigcup\{W(x, \varepsilon): x \in X\}$ by $W(X, \varepsilon)$.

LEMMA 1. *For every positive number ε and every point x of M, the set $W(x, \varepsilon)$ is open in M.*

Proof. Lemma 1 follows from a short argument [**10**, Lemma 4, proof] involving E. G. Effros' topological transformation group theorem [**8**, Theorem 2.1].

A continuum is *unicoherent* provided that if it is the union of two subcontinuum H and K, then $H \cap K$ is connected.

LEMMA 2. *Every proper subcontinuum of M is unicoherent* [**20**, *Theorem* 13.8].

In the remainder of this section we assume there is a continuum E of type λ in M. Let k: $E \to [0, 1]$ be a quotient map associated with the decomposition of E. We call $k^{-1}(0)$ and $k^{-1}(1)$ the *end sets* of E.

LEMMA 3. *Let Y be an element of the decomposition of E distinct from $k^{-1}(0)$ and $k^{-1}(1)$. Let F be a type λ subcontinuum of M with ends T and V, and let U be an element of the decomposition of F distinct from T and V. Suppose h is a homeomorphism of M onto M such that $U \cap h[Y] \neq \varnothing$ and $U \cap h[k^{-1}(0) \cup k^{-1}(1)] = \varnothing = h[Y] \cap (T \cup V)$. Then $h[Y] = U$.*

Proof. Lemma 3 follows from the argument given in paragraphs 9 through 11 in the proof of Theorem 1 of [**10**].

A subcontinuum F of M is called an *extension* of E away from $k^{-1}(0)$ if F is a continuum of type λ that contains E and has $k^{-1}(0)$ as an end set.

NOTATION. We denote the set of all extensions of E away from $k^{-1}(0)$ by $\mathfrak{S}(k^{-1}(0), E)$.

LEMMA 4. *The set $\mathfrak{S}(k^{-1}(0), E)$ is linearly ordered by inclusion and does not have a maximal element* [**11**, *Lemma* 4].

LEMMA 5. *The decomposition of each continuum of type λ in M is continuous* [**11**, *Lemma* 5].

A continuum H in E is an *essential subcontinuum* of E if H intersects more than one element of the decomposition of E.

LEMMA 6. *If H is an essential subcontinuum of E, then H is a continuum of type λ and every element of the decomposition of H is an element of the decomposition of E.*

Proof. Lemma 6 follows from Lemma 5 and the irreducibility properties of E [**26**, Theorem 8, page 14].

LEMMA 7. *Suppose F is a continuum of type λ in M such that $E \setminus F \neq \varnothing \neq F \setminus E$. Suppose $F \cap k^{-1}(r) \neq \varnothing$ for some number r $(0 \leq r \leq 1)$. Then $k^{-1}(r)$ is an element of the decomposition of F.*

Proof. By Lemma 4, there is a continuum H of type λ in M such that

(1) E is an essential subcontinuum of H that misses both end sets of H.

Let I be a continuum of type λ in M such that F is an essential subcontinuum of I that misses both end sets of I.

Observe that

(2) $k^{-1}(r) \cap (I \setminus F) = \varnothing$.

To see this note that since $F \setminus E \neq \varnothing$, the continuum F is not in $k^{-1}(r)$. Since M is atriodic, it follows from (1) that there is a point x of F in $H \setminus k^{-1}(r)$. Let J be the continuum of type λ in H such that x belongs to one end set of J and the other end set of J is $k^{-1}(r)$. By Lemma 2, $F \cap J$ is a subcontinuum of J.

Suppose that (2) is false. Then $F \cap J$ does not contain $k^{-1}(r)$. Hence $F \cap J$ is a proper subcontinuum of J. Since $x \in F \cap J$ and $F \cap J$ intersects $k^{-1}(r)$, this contradicts the fact that J is irreducible between x and $k^{-1}(r)$. Hence (2) is true.

Since M is atriodic and $F \cap k^{-1}(r) \neq \varnothing$, it follows from (2) that $k^{-1}(r) \subset F$. Therefore, since M is atriodic and $E \setminus F \neq \varnothing$, there is a point y of $E \setminus k^{-1}(r)$ in $I \setminus F$.

Let K be the continuum of type λ in E such that y belongs to one end set of K and the other end set of K is $k^{-1}(r)$. Let L be a subcontinuum of I that is irreducible between y and $k^{-1}(r)$. It follows from Lemma 2 and the irreducibility of K and L that $K = K \cap L = L$. Hence L is a continuum of type λ and $k^{-1}(r)$ is an element of the decomposition of L. Since $k^{-1}(r) \subset F$ and $y \in I \setminus F$, the continuum L is an essential subcontinuum of I. By Lemma 6, $k^{-1}(r)$ is an element of the decomposition of I. Since F is an essential subcontinuum of I and $k^{-1}(r) \subset F$, it follows from Lemma 6 that $k^{-1}(r)$ is an element of the decomposition of F.

LEMMA 8. *Suppose F is a continuum of type λ in M that intersects E and misses either $k^{-1}(0)$ or $k^{-1}(1)$. Then $\bigcup(\mathfrak{S}(k^{-1}(0), E) \cup \mathfrak{S}(k^{-1}(1), E))$ contains F.*

Proof. The conclusion follows immediately if E contains F. Therefore we assume that $F \not\subset E$. Assume without loss of generality that F misses $k^{-1}(0)$. By Lemma 7, for each number r $(0 < r \leq 1)$ if $k^{-1}(r)$ intersects an element Y of the decomposition of F, then $k^{-1}(r) = Y$. Hence the union $\mathfrak{D}$ of the decompositions of E and F is a monotone continuous (Lemma 5) decomposition of the continuum $E \cup F$. Each element of $\mathfrak{D}$ has void interior relative to $E \cup F$. The quotient space $(E \cup F)/\mathfrak{D}$ is the union of two arcs. Since M is atriodic, $(E \cup F)/\mathfrak{D}$ is atriodic. Moreover $(E \cup F)/\mathfrak{D}$ is not a simple closed curve since F misses $k^{-1}(0)$. It follows from Lemma 2 that $(E \cup F)/\mathfrak{D}$ does not contain a simple closed curve. Thus $(E \cup F)/\mathfrak{D}$ is an arc and $E \cup F$ is a continuum of type λ. Furthermore $k^{-1}(0)$ is an end set of $E \cup F$. Hence $E \cup F$ belongs to $\mathfrak{S}(k^{-1}(0), E)$. This completes the proof of Lemma 8.

LEMMA 9. *Every element of the decomposition of E is homogeneous.*

Proof. Lemma 9 follows from paragraphs 5 through 12 in the proof of Theorem 1 of [**10**].

LEMMA 10. *Suppose N is an indecomposable subcontinuum of M that contains E. Then N contains* $\bigcup \mathfrak{S}(k^{-1}(0), E)$.

Proof. Assume N does not contain $\bigcup \mathfrak{S}(k^{-1}(0), E)$. Let F be an element of $\mathfrak{S}(k^{-1}(0), E)$ that intersects $M \setminus N$. Let A be the composant of N that contains E. Let Y be the end set of F opposite $k^{-1}(0)$. It follows from Lemma 2 and the irreducibility of F that $N \cap Y = \varnothing$. Let $\varepsilon = \rho(N, Y)$. By Lemma 1, there exist two ε-homomorphisms f and g of M onto M and two composants B and C of N distinct from A such that $f[F] \cap B \neq \varnothing \neq g[F] \cap C$. By Lemma 2, $F, f[F]$, and $g[F]$ are disjoint. Since f and g are ε-homeomorphisms, $M \setminus N$ contains $f[Y] \cup g[Y]$. Hence $N \cup F \cup f[F] \cup g[F]$ is a triod. This contradicts the assumption that M is atriodic. Therefore N contains $\bigcup \mathfrak{S}(k^{-1}(0), E)$.

NOTATION. Let X be a subset of M. We denote the boundary of X and the closure of X in M by Bd X and Cl X, respectively.

LEMMA 11. *If N is an indecomposable subcontinuum of M that contains E, then* Cl $\bigcup \mathfrak{S}(k^{-1}(0), E)$ *is an indecomposable subcontinuum of N.*

Proof. By Lemma 10, Cl $\bigcup \mathfrak{S}(k^{-1}(0), E)$ is a subcontinuum of N. The argument given in paragraphs 2 through 11 in the proof of Lemma 6 of [**11**] proves that Cl $\bigcup \mathfrak{S}(k^{-1}(0), E)$ is indecomposable.

Suppose $\mathfrak{L} = \{L_i\colon 1 \leq i \leq 5\}$ is a 5-linked chain in M.

NOTATION. We denote the 3-linked subchain $\{L_i\colon 2 \leq i \leq 4\}$ of $\mathfrak{L}$ by $\mathfrak{L}'$.

The chain $\mathfrak{L}$ is *free* if $\text{Bd}(L_1 \cup L_5) \setminus \text{Cl} \bigcup \mathfrak{L}'$ contains $\text{Bd} \bigcup \mathfrak{L}$.

The chain $\mathfrak{L}$ is *normal* if $\text{Cl}\, L_i \cap \text{Cl}\, L_j = \varnothing$ whenever $|i - j| > 1$.

The continuum E *runs straight through* $\mathfrak{L}$ provided

(1) $E \subset \bigcup \mathfrak{L}$,

(2) $k^{-1}(0) \subset L_1 \setminus L_2$,

(3) $k^{-1}(1) \subset L_5 \backslash L_4$,
(4) if $0 \leq r \leq 1$ and $k^{-1}(r) \cap \operatorname{Bd} L_i \neq \varnothing$, then $k^{-1}(r) \subset \operatorname{Bd} L_i$, and
(5) if $0 \leq r < t \leq 1$ and $k^{-1}(r) \cup k^{-1}(t) \subset L_i$, then $k^{-1}[[r, t]] \subset L_i$.

The chain $\mathfrak{L}$ is *regular* if for each component K of $\bigcup \mathfrak{L}'$, the set $\operatorname{Cl} K$ is a continuum of type λ that runs straight through $\mathfrak{L}$.

A chain $\{P_i\colon 1 \leq i \leq 5\}$ is an *ordered refinement* of $\mathfrak{L}$ if for each i, the link L_i contains P_i.

LEMMA 12. *Suppose E runs straight through a normal regular chain $\mathfrak{L} = \{L_i\colon 1 \leq i \leq 5\}$. Suppose $k^{-1}(0) \subset L_1 \backslash \operatorname{Cl} L_2$ and $k^{-1}(1) \subset L_5 \backslash \operatorname{Cl} L_4$. Then E runs straight through a free normal regular ordered refinement of $\mathfrak{L}$.*

Proof. Let A and B denote the open sets $L_1 \backslash \operatorname{Cl} L_2$ and $L_5 \backslash \operatorname{Cl} L_4$, respectively. Since $\mathfrak{L}$ is regular, $E \backslash (A \cup B)$ is an essential subcontinuum of E. Since M is atriodic, $E \backslash (A \cup B)$ is a component of $M \backslash (A \cup B)$. Since no component of $M \backslash (A \cup B)$ intersects both $E \backslash (A \cup B)$ and $M \backslash \bigcup \mathfrak{L}$, there exist disjoint open sets C and D in M such that (1) C contains $E \backslash (A \cup B)$, (2) D contains $M \backslash \bigcup \mathfrak{L}$, and (3) $C \cup D$ contains $M \backslash (A \cup B)$ [**22**, Theorem 44, page 15].

Let $P_1 = L_1$ and $P_5 = L_5$. For $i = 2$, 3, and 4, let $P_i = C \cap L_i$. The chain $\mathfrak{P} = \{P_i\colon 1 \leq i \leq 5\}$ is a free normal regular ordered refinement of $\mathfrak{L}$. Note that E runs straight through $\mathfrak{P}$.

LEMMA 13. *Suppose ε is a positive number, A is a closed set that misses E, and B is an open set that contains $k^{-1}(0) \cup k^{-1}(1)$. Then E runs straight through a free normal regular chain $\mathfrak{P} = \{P_i\colon 1 \leq i \leq 5\}$ in $M \backslash A$ with the property that B contains $\operatorname{Cl}(P_1 \cup P_5)$ and for each component K of $\bigcup \mathfrak{P}'$ there is an ε-homeomorphism h of M onto M such that $\operatorname{Cl} K$ is an essential subcontinuum of $h[E]$.*

Proof. Let $\{x_i\colon 0 \leq i \leq 15\}$ be a set of numbers such that $x_0 = 0$, $x_{15} = 1$, $x_i < x_{i+1}$ for each i $(0 \leq i \leq 14)$, and $k^{-1}[[0, x_3] \cup [x_{12}, 1]] \subset B$. Let δ be a positive number less than ε, $\rho(A, E)$, $\rho(M \backslash B, k^{-1}[[0, x_3] \cup [x_{12}, 1]])$, and the minimum of $\{\frac{1}{2}\rho(k^{-1}[[0, x_i]], k^{-1}[[x_{i+1}, 1]])\colon 1 \leq i \leq 13\}$.

For each integer i $(1 \leq i \leq 5)$, let L_i be the open set $W(k^{-1}[[x_{3i-3}, x_{3i}]], \delta)$ (Lemma 1). Let $\mathfrak{L} = \{L_i\colon 1 \leq i \leq 5\}$. Note that $\mathfrak{L}$ is a normal chain in $M \backslash A$.

Next we prove that $\mathfrak{L}$ is regular. To accomplish this let h be a δ-homeomorphism of M onto M.

Note that

(1) $h[E] \subset \bigcup \mathfrak{L}$, and

(2) $h[k^{-1}(0)] \subset L_1 \setminus \mathrm{Cl}\, L_2$ and $h[k^{-1}(1)] \subset L_5 \setminus \mathrm{Cl}\, L_4$.

For each number r $(0 \leq r \leq 1)$ and each integer i $(1 \leq i \leq 5)$

(3) if $h[k^{-1}(r)] \cap L_i \neq \varnothing$, then $h[k^{-1}(r)] \subset L_i$.

To prove (3) assume for some numbers r and i, $h[k^{-1}(r)]$ intersects both L_i and $M \setminus L_i$. It follows from the definition of δ that $r \in [x_2, x_{13}] \setminus [x_{3i-3}, x_{3i}]$. There exist a number s in $[x_{3i-3}, x_{3i}] \cap [x_2, x_{13}]$ and a δ-homeomorphism g of M onto M such that $g[k^{-1}(s)] \cap h[k^{-1}(r)] \neq \varnothing$. By Lemma 3, $g[k^{-1}(s)] = h[k^{-1}(r)]$. This contradicts the fact that L_i contains $g[k^{-1}(s)]$. Hence (3) holds.

For each number r $(0 \leq r \leq 1)$

(4) if $h[k^{-1}(r)] \cap \mathrm{Bd}\, L_i \neq \varnothing$, then $h[k^{-1}(r)] \subset \mathrm{Bd}\, L_i$.

To see this let p be a point of $h[k^{-1}(r)] \cap \mathrm{Bd}\, L_i$ and assume that $\mathrm{Bd}\, L_i$ misses a point q of $h[k^{-1}(r)]$. By (3), $q \notin L_i$. Let $\mu = \rho(q, L_i)$. By Lemma 1, there exists a μ-homeomorphism f of M onto M such that fh is a δ-homeomorphism and $f(p) \in L_i$. It follows from the argument for (3) that $fh[k^{-1}(r)] \subset L_i$. This contradicts the fact that $f(q) \notin L_i$. Hence (4) holds.

Note that

(5) if $0 \leq r < t \leq 1$ and $k^{-1}(r) \cup k^{-1}(t) \subset L_i$, then $k^{-1}[[r, t]] \subset L_i$.

To see this assume the contrary. By (1), (2), and (3), there exist numbers r, s, and t $(0 \leq r < s < t \leq 1)$ and an integer i $(1 \leq i \leq 4)$ such that $h[k^{-1}(s)] \subset L_i$ and $h[k^{-1}(r) \cup k^{-1}(t)] \subset L_{i+1} \setminus L_i$. Let u be a number in $[x_{3i-3}, x_{3i}]$ and g be a δ-homeomorphism of M onto M such that $g[k^{-1}(u)] \cap h[k^{-1}(s)] \neq \varnothing$. Since M is atriodic and $g[k^{-1}(0, x_{3i})]]$ misses $L_{i+1} \setminus L_i$, it follows that $g[k^{-1}[0, x_{3i}]] \subset h[k^{-1}[[r, t]]$. This contradicts the definition of δ. Hence (5) holds.

It follows from (1), (2), (4), and (5) that

(6) $h[E]$ runs straight through $\mathfrak{L}$.

Let K be a component of $\bigcup \mathfrak{L}'$. Let h be a δ-homeomorphism of M onto M such that $K \cap h[E] \neq \varnothing$. Since M is atriodic, it follows from (6) and Lemma 6 that $\mathrm{Cl}\, K$ is an essential subcontinuum of $h[E]$ that runs straight through $\mathfrak{L}$. Hence $\mathfrak{L}$ is regular.

Since h in (2) and (6) can be the identity, $k^{-1}(0) \subset L_1 \setminus \mathrm{Cl}\, L_2$, $k^{-1}(1) \subset L_5 \setminus \mathrm{Cl}\, L_4$, and E runs straight through $\mathfrak{L}$. By Lemma 12, E runs straight through a free normal regular ordered refinement $\mathcal{P} = \{P_i\colon 1 \leq i \leq 5\}$ of $\mathfrak{L}$.

Since $\bigcup \mathfrak{L} \subset M \setminus A$ and $\mathrm{Cl}(L_1 \cup L_5) \subset B$, it follows that $\bigcup \mathcal{P} \subset M \setminus A$ and $\mathrm{Cl}(P_1 \cup P_5) \subset B$. Let K be a component of $\bigcup \mathcal{P}'$. Since $\delta < \varepsilon$,

M is atriodic, and $\mathcal{P}$ is a regular ordered refinement of $\mathcal{L}$, there is an ε-homeomorphism h of M onto M such that $\mathrm{Cl}\, K$ is an essential subcontinuum of $h[E]$. This completes the proof of Lemma 13.

NOTATION. Suppose $\mathcal{P} = \{P_i\colon 1 \leq i \leq 5\}$ is a regular chain. Let $\Omega(\mathcal{P})$ denote the collection $\{Z\colon Z$ is an element of the decomposition of the closure of a component of $\bigcup \mathcal{P}'\}$. Let $\Delta(\mathcal{P})$ be $\{Z\colon Z \in \Omega(\mathcal{P})$ and $Z \cap \mathrm{Cl}(P_1 \cup P_5) = \varnothing\}$. Note that since $\mathcal{P}$ is regular, $\Delta(\mathcal{P})$ is a decomposition of $(\bigcup \mathcal{P}) \setminus \mathrm{Cl}(P_1 \cup P_5)$.

LEMMA 14. *Suppose $\mathcal{P} = \{P_i\colon 1 \leq i \leq 5\}$ is a free regular chain. Then the decomposition $\Delta(\mathcal{P})$ of $(\bigcup \mathcal{P}) \setminus \mathrm{Cl}(P_1 \cup P_5)$ is continuous.*

Proof. Suppose $\{Z_i\}$ and $\{z_i\}$ are sequences such that (1) $Z_i \in \Delta(\mathcal{P})$ and $z_i \in Z_i$ for each positive integer i, and (2) $\{z_i\}$ converges to a point z of $\bigcup \Delta(\mathcal{P})$. Let Z be the element of $\Delta(\mathcal{P})$ that contains z. It suffices to show that $\{Z_i\}$ converges to Z.

Let ε be any positive number less than $\frac{1}{2}\rho(\bigcup \Delta(\mathcal{P}), M \setminus \bigcup \mathcal{P}')$. Let K_0 be the z-component of $\bigcup \mathcal{P}'$. For each positive integer i, let K_i be the z_i-component of $\bigcup \mathcal{P}'$. Since $\mathcal{P}$ is regular, for each non-negative integer i, the end sets of $\mathrm{Cl}\, K_i$ are in $M \setminus \bigcup \mathcal{P}'$. By Lemma 3, if z_i belongs to the open set $W(z, \varepsilon)$ (Lemma 1), then there is an ε-homeomorphism h of M onto M such that $h[Z] = Z_i$. Since ε is arbitrarily small, $\{Z_i\}$ converges to Z. Therefore $\Delta(\mathcal{P})$ is continuous.

LEMMA 15. *Suppose $\mathcal{P} = \{P_i\colon 1 \leq i \leq 5\}$ is a free regular chain. Then there is a positive number ε such that if $Z \in \Delta(\mathcal{P})$ and h is an ε-homeomorphism of M onto M, then $h[Z] \in \Omega(\mathcal{P})$.*

Proof. Let $\varepsilon = \frac{1}{2}\rho(\bigcup \Delta(\mathcal{P}), M \setminus \bigcup \mathcal{P}')$. By Lemma 3, if $Z \in \Delta(\mathcal{P})$ and h is an ε-homeomorphism of M onto M, then $h[Z] \in \Omega(\mathcal{P})$.

LEMMA 16. *Suppose $\mathcal{P} = \{P_i\colon 1 \leq i \leq 5\}$ is a free regular chain. Suppose K is a component of $\bigcup \mathcal{P}'$ and X is an end set of* $\mathrm{Cl}\, K$. *Suppose F is an extension of* $\mathrm{Cl}\, K$ *away from X and $\mathcal{D}$ is the decomposition of F. Then no element of $\mathcal{D}$ intersects three consecutive links of $\mathcal{P}$.*

Proof. Assume the contrary. Let $d\colon F \to [0, 1]$ be a quotient map associated with $\mathcal{D}$ such that $d^{-1}(0) = X$. Let s be the greatest lower bound of $S = \{r \in [0, 1]\colon d^{-1}(r)$ intersects three consecutive links of $\mathcal{P}\}$. Since no element of the decomposition of Cl K intersects three consecutive links of $\mathcal{P}$, it follows that $s > 0$. Since $\mathcal{D}$ is continuous (Lemma 5), $s \notin S$ and $s \neq 1$.

Since $\mathcal{P}$ is free, there exist a 3-linked subchain $\mathcal{Q}$ of $\mathcal{P}$ and numbers t, w greater than s such that

(1) for each number u ($t \leq u \leq w$), $d^{-1}(u)$ intersects each link of $\mathcal{Q}$ and no element of $\mathcal{P} \setminus \mathcal{Q}$ that intersects a link of $\mathcal{Q}$.

Let v be a number between t and w.

Since $\mathcal{P}$ is regular, there exists a component H of $\bigcup \mathcal{P}'$ such that $H \cap d^{-1}(v) \neq \varnothing$ and

(2) Cl H runs straight through $\mathcal{P}$.

Since Cl H intersects each link of $\mathcal{P}$, the continuum $d^{-1}[[t, w]]$ does not contain Cl H. Therefore, since M is atriodic, Cl H intersects either $d^{-1}(t)$ or $d^{-1}(w)$. Assume without loss of generality that Cl $H \cap d^{-1}(t) \neq \varnothing$. By (1) and (2), there exist distinct elements Y and Z of the decomposition of Cl H such that $Y \cap d^{-1}(t) \neq \varnothing$ and $Z \cap d^{-1}(v) \neq \varnothing$. Let I be the essential subcontinuum of Cl H that is irreducible between Y and Z. It follows from Lemma 2 and the irreducibility of I and $d^{-1}[[t, v]]$ that $I = I \cap d^{-1}[[t, v]] = d^{-1}[[t, v]]$. Since no element of the decomposition of I intersects all three links of $\mathcal{Q}$, each element of the decomposition of I is properly contained in an element of the decomposition of $d^{-1}[[t, v]]$. This contradicts the fact that the decomposition of $d^{-1}[[t, v]]$ is unique. Hence Lemma 16 is true.

4. Principal results.

THEOREM 1. *Suppose M is an atriodic homogeneous continuum. Suppose N is an indecomposable subcontinuum of M that contains a decomposable continuum. Then $M = N$. Furthermore M admits a continuous decomposition $\mathcal{D}$ such that $M/\mathcal{D}$ is a solenoid and the elements of $\mathcal{D}$ are homeomorphic. Moreover if the elements of $\mathcal{D}$ are not points, then they are tree-like hereditarily indecomposable homogeneous continua.*

Proof. By the argument in paragraphs 1 and 2 in the proof of Theorem 1 of [10], N has a subcontinuum E of type λ. Let $k: E \to [0, 1]$ be a quotient map associated with the decomposition of E.

Let ε be a positive number.

By Lemma 13, E runs straight through a free normal regular chain $\mathcal{P} = \{P_i \colon 1 \leq i \leq 5\}$ with the property that

(1) for each component K of $\bigcup \mathcal{P}'$ there is an ε-homeomorphism h of M onto M such that Cl K is an essential subcontinuum of $h[E]$.

For each element Z of $\Omega(\mathcal{P})\setminus\Delta(\mathcal{P})$, let $K(Z)$ be the component of $\bigcup\mathcal{P}'$ that contains Z. Let $X(Z)$ be the end set of $\mathrm{Cl}\, K(Z)$ that is separated from Z in $\mathrm{Cl}\, K(Z)$ by P_3. By Lemma 11, $\mathrm{Cl} \bigcup \mathfrak{S}(X(Z), \mathrm{Cl}\, K(Z))$ is an indecomposable subcontinuum of N. Thus $\bigcup \mathfrak{S}(X(Z), \mathrm{Cl}\, K(Z))$ intersects $P_3\setminus K(Z)$. Hence there is an element $F(Z)$ of $\mathfrak{S}(X(Z), \mathrm{Cl}\, K(Z))$ that intersects $P_3\setminus K(Z)$. By Lemmas 7 and 16, each element of the decomposition of $F(Z)$ that intersects $\bigcup\mathcal{P}'$ is an element of the decomposition of the closure of a component of $\bigcup\mathcal{P}'$. Hence there is an essential subcontinuum $J(Z)$ of $F(Z)$ that contains Z, misses $\mathrm{Cl}\, P_3$, and has one end set in $K(Z) \cap (\bigcup\Delta(\mathcal{P}))$ and the other end set in $(\bigcup\Delta(\mathcal{P}))\setminus K(Z)$.

By Lemma 13, $J(Z)$ runs straight through a free normal regular chain $\mathcal{P}(Z) = \{P(Z)_i\colon 1 \leq i \leq 5\}$ such that

(2) $\bigcup\mathcal{P}(Z)$ misses $\mathrm{Cl}\, P_3$, and

(3) $\bigcup\Delta(\mathcal{P})$ contains $\mathrm{Cl}(P(Z)_1 \cup P(Z)_5)$.

(The sets A and B in Lemma 13 are $\mathrm{Cl}\, P_3$ and $\bigcup\Delta(\mathcal{P})$, respectively.)

Let $\mathfrak{B}$ be the collection of open sets $\{\bigcup\Delta(\mathcal{P})\} \cup \{\bigcup\Delta(\mathcal{P}(Z))\colon Z \in \Omega(\mathcal{P})\setminus\Delta(\mathcal{P})\}$. Since $\mathcal{P}$ is free and since $\mathcal{P}(Z)$ is free and (3) holds for each element Z of $\Omega(\mathcal{P})\setminus\Delta(\mathcal{P})$, it follows that $\bigcup\mathfrak{B}$ is a closed open subset of M. Hence $\mathfrak{B}$ covers M.

Let S be $\bigcup(\mathfrak{S}(k^{-1}(0), E) \cup \mathfrak{S}(k^{-1}(1), E))$.

Next we prove that

(4) $\rho(p, S) < \varepsilon$ for every point p of M.

If $p \in \bigcup\Delta(\mathcal{P})$, then (4) follows from (1). Therefore we assume that $p \notin \bigcup\Delta(\mathcal{P})$. Let Z be an element of $\Omega(\mathcal{P})\setminus\Delta(\mathcal{P})$ such that $p \in \bigcup\Delta(\mathcal{P}(Z))$. Let H be the p-component of $\bigcup\Delta(\mathcal{P}(Z))$. Let X be an end set of $\mathrm{Cl}\, H$. Let K be the component of $\bigcup\mathcal{P}'$ that contains X. By (1), there is an ε-homeomorphism h of M onto M such that $\mathrm{Cl}\, K$ is an essential subcontinuum of $h[E]$. By (2) and Lemma 2, $\mathrm{Cl}\, H$ does not intersect both $h[k^{-1}(0)]$ and $h[k^{-1}(1)]$. Hence, by Lemma 8, H is a subset of $T = \bigcup(\mathfrak{S}(h[k^{-1}(0)], h[E]) \cup \mathfrak{S}(h[k^{-1}(1)], h[E]))$. Note that $T = h[S]$. Since $p \in H \subset h[S]$ and h is an ε-homeomorphism, (4) holds.

Since ε may be arbitrarily small, it follows from (4) that S is dense in M. Hence, by Lemma 10, $M = N$.

For convenience we define $\mathcal{P}(Z_0)$ to be $\mathcal{P}$.

Since M is compact, there exists a finite subcollection $\mathcal{C} = \{\bigcup\Delta(\mathcal{P}(Z_i))\colon 0 \leq i \leq n\}$ of $\mathfrak{B}$ that covers M.

Let $\mathfrak{D}$ be $\bigcup\{\Delta(\mathcal{P}(Z_i))\colon 0 \leq i \leq n\}$.

We must show that $\mathfrak{D}$ is a decomposition of M. Since $\mathcal{C}$ covers M, it suffices to show that the elements of $\mathfrak{D}$ are disjoint. Let A and B be intersecting elements of $\mathcal{D}$. If $A \cup B$ intersects $\bigcup \Delta(\mathcal{P})$, then, by (2), (3), and Lemma 7, $A = B$. Hence we assume that $A \cup B$ misses $\bigcup \Delta(\mathcal{P})$. Let i and j be positive integers such that $A \in \Delta(\mathcal{P}(Z_i))$ and $B \in \Delta(\mathcal{P}(Z_j))$. Let I be the component of $\bigcup \mathcal{P}(Z_i)'$ that contains A. Let J be the component of $\bigcup \mathcal{P}(Z_j)'$ that contains B. Since $A \cup B$ misses $\bigcup \Delta(\mathcal{P})$, it follows from (3) that $A \cup B$ misses both end sets of Cl I and both end sets of Cl J. Thus, by Lemma 3, $A = B$. Hence $\mathcal{D}$ is a decomposition of M.

It follows from Lemma 14 that $\mathcal{D}$ is continuous. According to Lemma 9, each element of $\mathcal{D}$ is either a point or a homogeneous continuum. Since M is atriodic, it follows that the quotient space $M/\mathcal{D}$ is an atriodic continuum.

The quotient space $M/\mathcal{D}$ is homogeneous and the elements of $\mathcal{D}$ are homeomorphic. To see this first note that for each integer i $(1 \leq i \leq n)$, since $\mathcal{P}(Z_i)$ is a free regular chain, it follows from (2) and (3) that each component of $\bigcup \Delta\mathcal{P}(Z_i)$ misses P_3 and intersects $M \setminus \bigcup \mathcal{P}'$.

Since $\mathcal{P}$ is a free regular chain, it follows from (3) and Lemma 7 that

(5) for each integer i $(0 \leq i \leq n)$, $\Omega(\mathcal{P}(Z_i)) \subset \mathcal{D}$.

By Lemma 15, for each integer i $(0 \leq i \leq n)$ there is a positive number ε_i such that if $Z \in \Delta(\mathcal{P}(Z_i))$ and h is an ε_i-homeomorphism of M onto M, then $h[Z] \in \Omega(\mathcal{P}(Z_i))$. Let δ be the minimum of $\{\varepsilon_i\colon 0 \leq i \leq n\}$.

It follows from (5) that

(6) if $Z \in \mathcal{D}$ and h is a δ-homeomorphism of M onto M, then $h[Z] \in \mathcal{D}$.

Let X and Y be elements of $\mathcal{D}$. Since M is a continuum, there is a finite subset $\{x_i\colon 1 \leq i \leq m\}$ of M such that $Y \cap W(x_1, \delta/2) \neq \varnothing$, $X \cap W(x_m, \delta/2) \neq \varnothing$, and $W(x_i, \delta/2) \cap W(x_{i+1}, \delta/2) \neq \varnothing$ for each integer i $(1 \leq i < m)$.

Define a set $\{v_i\colon 0 \leq i \leq m\}$ such that $v_0 \in Y \cap W(x_1, \delta/2)$, $v_m \in X \cap W(x_m, \delta/2)$, and $v_i \in W(x_i, \delta/2) \cap W(x_{i+1}, \delta/2)$ for each integer i $(1 \leq i < m)$. For each i $(1 \leq i \leq m)$, let h_i be a δ-homeomorphism of M onto M such that $h_i(v_i) = v_{i-1}$. By (6), each h_i maps each element of $\mathcal{D}$ onto an element of $\mathcal{D}$. Therefore $h_1 h_2 \cdots h_m$ induces a homeomorphism of $M/\mathcal{D}$ onto itself that takes X to Y. Hence $M/\mathcal{D}$ is homogeneous. Since $h_1 h_2 \cdots h_m[X] = Y$, it follows that the elements of $\mathcal{D}$ are homeomorphic.

Since $M/\mathcal{D}$ is an atriodic homogeneous continuum that contains an arc, $M/\mathcal{D}$ is a solenoid [**20**, Theorem 14.4].

Suppose $\mathcal{D}$ has a nondegenerate element Z. By Lemma 2, Z is hereditarily unicoherent. Since Z is homogeneous, it follows that Z is indecomposable [**15**] [**9**]. In fact, Z is hereditarily indecomposable; for if Z

has a decomposable subcontinuum, then, by the above argument, $M = Z$ and this is impossible. Hence Z is tree-like [**25**]. Therefore, if the elements of $\mathfrak{D}$ are not points, they are homeomorphic tree-like hereditarily indecomposable homogeneous continua.

THEOREM 2. *Suppose M is an atriodic homogeneous continuum that is not a solenoid and has a decomposable subcontinuum. Then M admits a continuous decomposition $\mathfrak{D}$ such that $M/\mathfrak{D}$ is a solenoid and the elements of $\mathfrak{D}$ are homeomorphic tree-like hereditarily indecomposable homogeneous continua.*

Proof. If M is indecomposable, the conclusion follows immediately from Theorem 1. Therefore we assume that M is decomposable. According to Theorem 14.7 of [**20**], M admits a continuous decomposition $\mathfrak{D}$ such that $M/\mathfrak{D}$ is a simple closed curve and the elements of $\mathfrak{D}$ are homeomorphic indecomposable homogeneous continua. Let Z be an element of $\mathfrak{D}$. The indecomposable continuum Z is hereditarily indecomposable; for if Z has a decomposable subcontinuum, it follows from Theorem 1 that $Z = M$ and this is impossible. Hence Z is tree-like [**25**] and the proof is complete.

The following corollary to Theorem 2 answers in the affirmative Mackowiak and Tymchatyn's question [**20**, § 13].

COROLLARY 1. *If M is an atriodic homogeneous continuum, then M is 1-dimensional.*

Proof. If M is hereditarily indecomposable, then M is tree-like [**25**], and, therefore, 1-dimensional. Furthermore, if M is a solenoid, then M is 1-dimensional. Hence we assume that M is not a solenoid and has a decomposable subcontinuum. By Theorem 2, M admits a continuous decomposition $\mathfrak{D}$ such that $M/\mathfrak{D}$ is 1-dimensional and the elements of $\mathfrak{D}$ are 1-dimensional continua. According to the second inequality in the proof of Theorem VI 7 on page 92 of [**14**], the dimension of M is either 1 or 2. Hence, by Theorem 13.4 of [**20**], M is 1-dimensional.

COROLLARY 2. *If M is a tree-like atriodic homogeneous continuum, then M is hereditarily indecomposable.*

Proof. Suppose M has a decomposable subcontinuum. By Theorem 2, M admits a monotone continuous decomposition $\mathfrak{D}$ such that $M/\mathfrak{D}$ is a solenoid. Since M is indecomposable [**15**] [**9**], $M/\mathfrak{D}$ is not a simple closed curve. This contradicts the fact that no tree-like continuum can be mapped onto a solenoid that is not a simple closed curve [6] [17]. Therefore M is hereditarily indecomposable.

In 1968 F. B. Jones [16] suggested the following method for proving that every indecomposable homogeneous plane continuum M is hereditarily indecomposable. Assume that M has a decomposable subcontinuum. Find a monotone decomposition $\mathfrak{D}$ of M such that $M/\mathfrak{D}$ is a homogeneous plane continuum that contains an arc. It follows from a theorem of Bing [4] that $M/\mathfrak{D}$ is a simple closed curve and this contradicts the fact that M is indecomposable.

Every indecomposable homogeneous plane continuum is atriodic [10, Lemma 1]. In Theorem 2 (above), since the elements of $\mathfrak{D}$ are tree-like, if M is planar, then $M/\mathfrak{D}$ is planar [21]. Hence Theorem 2 provides the decomposition that Jones requested and we obtain the following:

COROLLARY 3. *Every indecomposable homogeneous plane continuum is hereditarily indecomposable* [11].

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REFERENCES

[1] R. H. Bing, *A homogeneous indecomposable plane continuum*, Duke Math. J., **15** (1948), 729–742.

[2] ______, *Higher-dimensional hereditarily indecomposable continua*, Trans. Amer. Math. Soc., **71** (1951), 267–273.

[3] ______, *Each homogeneous nondegenerate chainable continuum is a pseudo-arc*, Proc. Amer. Math. Soc., **10** (1959), 345–346.

[4] ______, *A simple closed curve is the only homogeneous bounded plane continuum that contains an arc*, Canad. J. Math., **12** (1960), 209–230.

[5] R. H. Bing and F. B. Jones, *Another homogeneous plane continuum*, Trans. Amer. Math. Soc., **90** (1959), 171–192.

[6] K. Borsuk, *On movable compacta*, Fund. Math., **66** (1969), 137–146.

[7] C. E. Burgess, *Homogeneous 1-dimensional continua*, General Topology and Modern Analysis, Academic Press, New York, 1981, 169–175.

[8] E. G. Effros, *Transformation groups and C*-algebras*, Ann. of Math., **81** (1965), 38–55.

[9] G. R. Gordh, Jr., *On homogeneous hereditarily unicoherent continua*, Proc. Amer. Math. Soc., **51** (1975), 198–202.

[10] C. L. Hagopian, *Homogeneous plane continua*, Houston J. Math., **1** (1975), 35–41.

[11] ______, *Indecomposable homogeneous plane continua are hereditarily indecomposable*, Trans. Amer. Math. Soc., **224** (1976), 339–350.

[12] ______, *A characterization of solenoids*, Pacific J. Math., **68** (1977), 425–435.

[13] C. L. Hagopian and J. T. Rogers, Jr., *A classification of homogeneous circle-like continua*, Houston J. Math., **3** (1977), 471–474.

[14] W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton University Press, Princeton, N. J., 1948.

[15] F. B. Jones, *Certain homogeneous unicoherent indecomposable continua*, Proc. Amer. Math. Soc., **2** (1951), 855–859.

[16] _____, *Homogeneous plane continua*, Proc. Auburn Topology Conf., Auburn University, Auburn, Ala., (1969), 46–56.

[17] J. Krasinkiewicz, *Curves which are continuous images of tree-like continua are movable*, Fund. Math., **89** (1975), 233–260.

[18] K. Kuratowski, *Theorie des continus irreductible entre deux points II*, Fund. Math., **10** (1927), 225–276.

[19] _____, *Topology*, Volume 2, 3rd ed., Monografie Mat., TOM 21, PWN, Warsaw, 1961; English translation, Academic Press, New York; PWN, Warsaw, 1968.

[20] T. Mackowiak and E. D. Tymchatyn, *Continuous mappings on continua II*, Dissertationes Math., **225** (1983).

[21] R. L. Moore, *Concerning upper semi-continuous collections of continua*, Trans. Amer. Math. Soc., **27** (1925), 416–428.

[22] _____, *Foundations of point set theory*, rev. ed., Amer. Math. Soc. Colloq. Publ., vol. 13, Amer. Math. Soc., Providence, R. I., 1962.

[23] J. T. Rogers, Jr., *Solenoids of pseudo-arcs*, Houston J. Math., **3** (1977), 531–537.

[24] _____, *Decompositions of homogeneous continua*, Pacific J. Math., **99** (1982), 137–144.

[25] _____, *Homogeneous, hereditarily indecomposable continua are tree-like*, Houston J. Math., **8** (1982), 421–428.

[26] E. S. Thomas, Jr., *Monotone decompositions of irreducible continua*, Rozprawy Matematyczne, **50** (1966), 1–74.

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