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In this paper, we introduce the notion of D-complexes which are defined by replacing metric spaces with Nagami's D-spaces in the definition of Hyman's M-spaces, and prove a main theorem that every D-complex is a space with a σ-almost locally finite base (this notion was introduced by Itō and Tamano). This theorem sharpens a theorem of Nagata. Furthermore, we deal with the adjunction spaces of two spaces with a σ-almost locally finite base.

1. Introduction. In [**8**], M. Itō and K. Tamano introduced the notion of almost local finiteness and the class of all spaces with a σ-almost locally finite base. This class is countably productive, hereditary and the closed image of a space in the class is M_1 (see [**8**]). Furthermore, this class is an intermediate class between that of free L-spaces and that of M_1-spaces. Indeed, there exists a space with a σ-almost locally finite base which is not a free L-space (see [**8**]). But it is not known whether there exists an M_1-space which is not a space with a σ-almost locally finite base. If M_1-spaces are spaces with a σ-almost locally finite base, Ceder's long-standing unsolved question will be affirmatively answered; that is, every stratifiable space is M_1.

In §2, we introduce the notion of D-complexes which generalizes that of Hyman's M-spaces ([**6**]). Note that, in [**1**], C. J. R. Borges used the words paracomplex or n-paracomplex instead of Hyman's M-space or his M_n-space, respectively. Furthermore, we give some results for D-complexes which obtained in [**10**]. In §3, we give some preliminary lemmas. In §4, we prove main results.

Throughout this paper, all spaces are assumed to be regular T_1 and all maps to be continuous. N denotes the set of all natural numbers. For the definitions of uniformly approaching anti-cover and D-space, see K. Nagami [**12**]. For M_1-spaces and free L-space, see J. G. Ceder [**2**] and K. Nagami [**13**], respectively. In each monotonically normal space X, we assume that X has a monotone normality operator G satisfying the properties [5, Lemma 2.2].

2. D-complexes and some results. In this section, we define D-complexes, and study some properties of D-complexes.

DEFINITION 2.1. A $D(0)$-complex is a D-space. Assume that $D(n-1)$-complexes have been defined for an $n \in N$. Then a space Z is a $D(n)$-complex if it is homeomorphic to the adjunction space $X \cup_f Y$, where X is a D-space, A a closed set of X, Y a $D(n-1)$-complex and f a map from A into Y. Let $X = \cup\{X_i: i \in N\}$, where $\{X_i: i \in N\}$ is a closed cover of the space X such that $X_i \subset X_{i+1}$ and each X_i is a $D(n_i)$-complex for some $n_i \in N \cup \{0\}$. If X is dominated by $\{X_i: i \in N\}$ (namely, $F \subset X$ is closed in X if and only if $F \cap X_i$ is closed in X_i for every $i \in N$), then X is said to be a D-complex.

REMARK 2.2. Since a metric space is a D-space and the closed image of a D-space is a D-space by [**12**, Remark 4.5], each Lašnev space is a D-space. Furthermore there exist a D-space which is not a Lašnev space (see [**12**, Example 2.1]), and a Lašnev space which is not a paracomplex (see [**3**, Example 2]). Therefore the class of all D-complexes properly contains those of all Lašnev spaces and all paracomplexes.

The following two theorems was established in [**10**] and those are generalizations of Theorems 1 and 2 in [**16**].

THEOREM 2.3. *Every D-complex is an M_1-space.*

THEOREM 2.4. *Let X be a D-complex. Then* $\dim X \leq n$ *if and only if X has a σ-closure preserving base* $\mathfrak{U}$ *such that* $\dim B(U) \leq n-1$ *for every* $U \in \mathfrak{U}$, *where* $\dim X$ *is the covering dimension of X and* $B(U)$ *is the boundary of U.*

Outline of proofs of Theorems 2.3 and 2.4. The property ECP was defined in [**16**]. We consider ECP in monotonically normal spaces. Then, first, we prove that every D-space X has ECP. Outline of this proof is the following: Let X' be a monotonically normal space and $X' = F \cup X$, where F and X are closed in X', and G a monotone normality operator in X'. Suppose $\mathfrak{U} = \{U_\alpha: \alpha \in A\}$ is a closure preserving open family in F, and $\mathfrak{V} = \{V_\lambda: \lambda \in \Lambda\}$ a uniformly approaching anti-cover of $X \cap F$ in X such that $\mathfrak{V}$ is locally finite in $X - F$. For each $U_\alpha \in \mathfrak{U}$, let $U'_\alpha = \cup\{G(x, F - U_\alpha): x \in U_\alpha\}$. Then U'_α is open in X'. For the fixed element $\alpha \in A$, let $B_\alpha = \{\gamma(\alpha) \subset \Lambda: U'_{\gamma(\alpha)}$ is open in $U'_\alpha\}$, where $U'_{\gamma(\alpha)} = U_\alpha \cup (\cup\{V_\lambda: \lambda \in \gamma(\alpha)\})$. Let $B = \cup\{B_\alpha: \alpha \in A\}$, $\mathfrak{U}' = \{U'_\beta: \beta \in B\}$. Then $\mathfrak{U}'$ satisfies the conditions (1), (2), (3) of Definition 2 in [**16**]. Next, by the methods of the above proof and [**16**, Lemma 2] we can prove that every $D(n)$-complex has ECP. Last, Theorem 2.3 is proved by the same way as proof of [**16**, Theorem 1]. If we use the results of K. Nagami [**12**], [**13**], [**14**] and the method of the above proof, Theorem 2.4 can be shown by the same way as proof of [**16**, Theorem 2].

For adjunction spaces, we proved the following theorem in [**10**]. Since a D-space is a free L-space, the subsequent corollary is a direct consequence.

THEOREM 2.5. *Let X and Y be free L-spaces, A a closed set of X which has a uniformly approaching anti-cover, and f a map from A into Y. Then the adjunction space $X \cup_f Y$ is a free L-space.*

Proof. In [7], M. Itō proved that weak L-spaces are free L-spaces. Therefore this theorem can be proved by some slight modifications of the proof in [**9**, Theorem 3.1].

COROLLARY 2.6 (*cf. Theorem* 2.3). *Every $D(n)$-complex is a free L-space.*

3. Preliminary lemmas. In this section, we define a property EP-ALF — this is an abbreviation of "extension property of an almost locally finite family" —, and give some preliminary lemmas. We begin with the definition of almost local finiteness.

DEFINITION 3.1 ([**8**]). Let X be a space, x a point of X and $\mathfrak{U}$ a family of subsets of X. $\mathfrak{U}$ is said to be almost locally finite at x if there exists a neighborhood V of x and a finite subset $\{K_1,\ldots,K_n\}$ of X such that

$$\mathfrak{U}|V = \{U \cap V\colon U \in \mathfrak{U}\} \subset \{K_i \cap W\colon i = 1,\ldots,n \text{ and } W \text{ is a neighborhood of } x\}.$$

$\mathfrak{U}$ is said to be almost locally finite in X if $\mathfrak{U}$ is almost locally finite at every point of X.

DEFINITION 3.2. By EP-ALF we mean the following property of a monotonically normal space X: If X is a closed set of a monotonically normal space X' such that $X' = F \cup X$, F and X closed in X', and if $\mathfrak{U} = \{U_\alpha\colon \alpha \in A\}$ is an almost locally finite open family in F, then for each $\alpha \in A$ there is a family $\{U'_\beta\colon \beta \in B_\alpha\}$ of open sets in X' satisfying

(C1) $\mathfrak{U}' = \{U'_\beta\colon \beta \in B_\alpha, \alpha \in A\}$ is almost locally finite in X',

(C2) for each $\beta \in B_\alpha$, $U'_\beta \cap F = U_\alpha$, and for every open set V in X' with $V \cap F = U_\alpha$ there is $\beta \in B_\alpha$ such that $U_\alpha \subset U'_\beta \subset V$, and

(C3) for every open set W in F, there is an open set W' of X' such that $W' \cap F = W$ and such that $W' \cap U'_\beta = \varnothing$ whenever $\beta \in B_\alpha$ and $W \cap U_\alpha = \varnothing$.

LEMMA 3.3. *Every D-space has EP-ALF.*

Proof. Let X be a D-space, X' a monotonically normal space and $X' = F \cup X$, where F and X are closed in X'. Furthermore let G be a monotone normality operator of X'. Suppose $\mathcal{U} = \{U_\alpha\colon \alpha \in A\}$ is an almost locally finite open family of F. Let $\mathcal{V} = \{V_\lambda\colon \lambda \in \Lambda\}$ be a uniformly approaching anti-cover of $X \cap F$ in X. In particular, since X is hereditarily paracompact, we may assume that $\mathcal{V}$ is locally finite in $X - F$. For each $U_\alpha \in \mathcal{U}$, let $U'_\alpha = \bigcup\{G(x, F - U_\alpha)\colon x \in U_\alpha\}$. Then U'_α is obviously open in X'. For the fixed element $\alpha \in A$, let $B_\alpha = \{\gamma(\alpha) \subset \Lambda\colon U'_{\gamma(\alpha)}$ is open in $U'_\alpha\}$, where $U'_{\gamma(\alpha)} = U_\alpha \cup (\bigcup\{V_\lambda\colon \lambda \in \gamma(\alpha)\})$. Let $B = \bigcup\{B_\alpha\colon \alpha \in A\}$, $\mathcal{U}' = \{U'_\beta\colon \beta \in B\}$. Then condition (C2) of Definition 3.2 is obviously satisfied by $\mathcal{U}'$, because for each open set V with $V \cap F = U_\alpha$ there is a set $U'_\beta = U_\alpha \cup (\bigcup\{V_\lambda \in \mathcal{V}\colon V_\lambda \subset V \cap U'_\alpha\})$ for some $\beta \in B_\alpha$ such that $U_\alpha \subset U'_\beta \subset V$. To prove (C3), let W be open in F. Then it is easy to see that $W' = \bigcup\{G(x, F - W)\colon x \in W\}$ is an open set in X' satisfying (C3).

Finally to prove (C1), first we consider the case $x \in F$. There exist an open neighborhood V of x in F and open finite subsets $\{H_1,\dots,H_n\}$ of F such that

$$\mathcal{U}|V \subset \{H_i \cap W\colon i = 1,\dots,n \text{ and } W \text{ is a neighborhood of } x \text{ in } F\}.$$

Without loss of generality, we assume that

$$H_i \supset \bigcup \{U_\alpha \in \mathcal{U}\colon U_\alpha \cap V = H_i \cap W \text{ for some neighborhood } W \text{ of } x\}.$$

Let $V' = \bigcup\{G(y, F - V)\colon y \in V\}$ and $H'_i = \bigcup\{G(y, F - H_i)\colon y \in H_i\}$ for each $i \in \{1,\dots,n\}$. Then it is easy to see that

$$\mathcal{U}'|V' \subset \{H'_i \cap W\colon i = 1,\dots,n \text{ and } W \text{ is a neighborhood of } x \text{ in } X\},$$

and V' is a neighborhood of x in X'. Thus $\mathcal{U}'$ is almost locally finite at x. Next, we consider the case $x \in X - F$. Since $\mathcal{V}$ is locally finite in $X - F$, there is a neighborhood V of x such that

$$\{\lambda \in \Lambda\colon V \cap V_\lambda \neq \varnothing, x \in V_\lambda, V_\lambda \in \mathcal{V}\} = \{\lambda_1,\dots,\lambda_n\}.$$

Let

$$\left\{ \bigcup \{V_{\lambda_i}\colon \lambda_i \in \gamma\}\colon \gamma \text{ is a non-empty subset of } \{\lambda_1,\dots,\lambda_n\}\right\} = \{K_1,\dots,K_m\}.$$

Then it is clear that

$$\mathcal{U}'|V \subset \{K_i \cap W: i = 1,\ldots,m \text{ and } W \text{ is a neighborhood of } x \text{ in } X'\}.$$

Thus $\mathcal{U}'$ is almost locally finite at x. This completes the proof.

LEMMA 3.4. *Every $D(n)$-complex has EP-ALF.*

Proof. We use induction on n. Since by Lemma 3.3 the present assertion is true for $n = 0$, we assume that every $D(n-1)$-complex has EP-ALF. Let X_0 be a D-space, Y_0 a $D(n-1)$-complex and f a map from a closed set E of X_0 into Y_0. Then it suffices to prove that the adjunction space $Z = X_0 \cup_f Y_0$ has EP-ALF. Let p be the projection from the free union $X_0 \cup Y_0$ onto Z. Note that p is a topological map from Y_0 onto a closed subset Y of Z. Now, let $Z' = F \cup Z$, where Z' is monotonically normal and F and Z are closed in Z'. Suppose $\mathcal{U} = \{U_\alpha: \alpha \in A\}$ is an almost locally finite open family in F. Let $Y' = Y \cup F$. Then F and Y are obviously closed in the monotonically normal space Y'. Since by the induction hypothesis Y has EP-ALF, each U_α can be extended to open sets $\{U'_\beta: \beta \in B_\alpha\}$ in Y' satisfying (C1), (C2), (C3). Let us denote by q the restriction of p to X_0. Define a closed set K of X_0 by $K = q^{-1}(Y')$. Since X_0 is a D-space, X_0 has a monotone normality operator G. Let $\mathcal{V} = \{V_\lambda: \lambda \in \Lambda\}$ be a uniformly approaching anti-cover of K in X_0 and locally finite in $X_0 - K$. For each $\beta \in B_\alpha$ ($\alpha \in A$) and each $\gamma \subset \Lambda$, let

$$V_\beta = \bigcup \{G(x, K - q^{-1}(U'_\beta)): x \in q^{-1}(U'_\beta)\},$$

$$V'_{\beta\gamma} = q^{-1}(U'_\beta) \bigcup \left(\bigcup \{V_\lambda \in \mathcal{V}: \lambda \in \gamma\}\right).$$

For the fixed element $\alpha \in A$ and $\beta \in B_\alpha$, let

$$C_\alpha(\beta) = \{\gamma \subset \Lambda: V'_{\beta\gamma} \text{ is open in } V_\beta\}, \; C_\alpha = \bigcup \{C_\alpha(\beta): \beta \in B_\alpha\}.$$

Let $U''_\gamma = p(V'_{\beta\gamma}) \cup U'_\beta$ and $\mathcal{U}''_\alpha = \{U''_\gamma: \gamma \in C_\alpha\}$. Then $\mathcal{U}''_\alpha$ are extensions of U_α into Z' satisfying (C1), (C2), (C3).

First, we can easily show that each $U''_\gamma \in \mathcal{U}''_\alpha$ is open in Z'. (C2) is obviously satisfied by $\mathcal{U}''_\alpha$ ($\alpha \in A$), because $\{U'_\beta: \beta \in B_\alpha\}$ satisfies (C2). Next, to prove (C3), let W be an open set in F. Since $\{U'_\beta: \beta \in B_\alpha, \alpha \in A\}$ satisfies (C3), there exists an open set W' in Y' such that $W' \cap F = W$ and such that $U_\alpha \cap W = \varnothing$ implies $W' \cap U'_\beta = \varnothing$ for all $\beta \in B_\alpha$. Since $q^{-1}(W')$ is open in K, let

$$W'' = W' \cup p\left(\bigcup \{G(x, K - q^{-1}(W')): x \in q^{-1}(W')\}\right).$$

Then W'' is obviously open in Z'. Furthermore, $W \cap U_\alpha = \varnothing$ implies that $W' \cap U'_\beta = \varnothing$ for every $\beta \in B_\alpha$, so that $W'' \cap U''_\gamma = \varnothing$ for every $\gamma \in C_\alpha(\beta)$. This proves (C3).

Finally, we shall prove that $\mathfrak{U}'' = \bigcup\{\mathfrak{U}''_\alpha: \alpha \in A\}$ is almost locally finite in Z'. Let $x \in Y'$. Since $\mathfrak{U}' = \{U'_\beta: \beta \in B_\alpha, \alpha \in A\}$ is almost locally finite in Y', there exist an open neighborhood V of x in Y' and open finite subsets $\{H_1, \dots, H_m\}$ of Y' such that

$$\mathfrak{U}'|V \subset \{H_i \cap W: i = 1, \dots, n \text{ and } W \text{ is a neighborhood of } x \text{ in } Y'\}.$$

Without loss of generality, we assume that for each i

$$H_i \supset \bigcup \{U'_\beta \in \mathfrak{U}': U'_\beta \cap V = H_i \cap W \text{ for some neighborhood } W \text{ of } x \text{ in } Y'\}.$$

Let $V' = V \cup p(\bigcup\{G(y, K - q^{-1}(V)): y \in q^{-1}(V)\})$ and for each i

$$H'_i = H_i \cup p\left(\bigcup \{G(y, K - q^{-1}(H_i)): y \in q^{-1}(H_i)\}\right).$$

Then it is easy to see that

$$\mathfrak{U}''|V' \subset \{H'_i \cap W: i = 1, \dots, m \text{ and } W \text{ is a neighborhood of } x \text{ in } Z'\},$$

and V' is a neighborhood of x in Z'. Thus $\mathfrak{U}''$ is almost locally finite at x. Let $x \in Z' - Y'$. Then by the same method as last part in the proof of Lemma 3.3, it is easily seen that $\mathfrak{U}''$ is almost locally finite at x. This completes the proof.

4. Main theorems. We begin with the proof of the following main theorem which sharpens Theorem 2.3 in this paper (therefore Nagata's Theorem [**16**, Theorem 1]).

THEOREM 4.1. *Every D-complex is a space with a σ-almost locally finite base.*

Proof. Suppose that $X = \bigcup\{X_i: i \in N\}$, $X_i \subset X_{i+1}$, where each X_i is a $D(n_i)$-complex and closed in X, and X is dominated by $\{X_i: i \in N\}$. By Corollary 2.6 and [**8**, Theorem 3.3], each X_i has a σ-almost locally finite base $\{\mathfrak{U}_{ij}: j \in N\}$. For each $j \in N$, let $\mathfrak{U}_{1j} = \{U(\alpha_1): \alpha_1 \in A\}$. Since X_2 is a $D(n_2)$-complex, $X_1 \subset X_2$ and X_1 is closed in X (therefore in X_2), by Lemma 3.4 X_2 has EP-ALF. Therefore every $U(\alpha_1)$ can be extended to open sets $\{U(\alpha_1, \alpha_2): \alpha_2 \in A(\alpha_1)\}$ in X_2 in such a way that the family $\{U(\alpha_1, \alpha_2): \alpha_1 \in A, \alpha_2 \in A(\alpha_1)\}$ satisfies (C1), (C2), (C3). (In particular, we assume that the method of extensions is the same one of Lemma 3.4.)

Repeating this process we get for each k an almost locally finite open family

$$\{U(\alpha_1,\ldots,\alpha_k)\colon \alpha_1 \in A, \alpha_2 \in A(\alpha_1),\ldots,\alpha_k \in A(\alpha_1,\ldots,\alpha_{k-1})\}$$

in X_k. Let

$$\Sigma = \{(\alpha_1, \alpha_2, \alpha_3,\ldots)\colon \alpha_1 \in A, \alpha_2 \in A(\alpha_1), \alpha_3 \in A(\alpha_1, \alpha_2),\ldots\}.$$

For each $(\alpha_1, \alpha_2,\ldots) \in \Sigma$, let

$$U(\alpha_1, \alpha_2,\ldots) = \bigcup \{U(\alpha_1,\ldots,\alpha_k)\colon k \in N\}.$$

Then $U(\alpha_1, \alpha_2,\ldots)$ is an open set of X, because for each $k \in N$, $U(\alpha_1, \alpha_2,\ldots) \cap X_k = U(\alpha_1,\ldots,\alpha_k)$ is open in X_k. Let

$$\mathcal{U}'_{1j} = \{U(\alpha_1, \alpha_2,\ldots)\colon (\alpha_1, \alpha_2,\ldots) \in \Sigma\}.$$

Now we claim that $\{\mathcal{U}'_{1j}\colon j \in N\}$ is a σ-almost locally finite local base at each point $x \in X_1$. First, it is easily seen by (C2) that $\{\mathcal{U}'_{1j}\colon j \in N\}$ is a local base at x. Next, to prove that each $\mathcal{U}'_{1j}$ is almost locally finite, let $y \in X_1$. Since $\mathcal{U}_{1j}$ is almost locally finite at y in X_1, there exist an open neighborhood $V(1)$ of y in X_1 and finite open subsets $\{H_1(1),\ldots,H_n(1)\}$ of X_1 such that

$$\mathcal{U}_{1j}|V(1) \subset \{H_i(1) \cap W\colon i = 1,\ldots,n \text{ and } W \text{ is a neighborhood of } y \text{ in } X_1\}.$$

Since the extension $\{U(\alpha_1, \alpha_2)\colon \alpha_1 \in A,\ \alpha_2 \in A(\alpha_1)\}$ of $\mathcal{U}_{1j}$ is the same one of Lemma 3.4, there exist an open neighborhood $V(1,2)$ of y in X_2 and finite open subsets $\{H_1(1,2),\ldots,H_n(1,2)\}$ of X_2 such that

$$\{U(\alpha_1, \alpha_2)\colon \alpha_1 \in A, \alpha_2 \in A(\alpha_1)\}|V(1,2)$$
$$\subset \{H_i(1,2) \cap W\colon i = 1,\ldots,n \text{ and } W \text{ is a neighborhood of } y \text{ in } X_2\},$$

and $V(1,2) \cap X_1 = V(1)$, $H_i(1,2) \cap X_1 = H_i(1)$ for each i. Repeating this process we get for each $k \in N$ an open neighborhood $V(1,\ldots,k)$ of y in X_k and finite open subsets $\{H_1(1,\ldots,k),\ldots,H_n(1,\ldots,k)\}$ of X_k such that

$$\{U(\alpha_1,\ldots,\alpha_k)\colon \alpha_1 \in A,\ldots,\alpha_k \in A(\alpha_1,\ldots,\alpha_{k-1})\}|V(1,\ldots,k)$$
$$\subset \{H_i(1,\ldots,k) \cap W\colon i = 1,\ldots,n \text{ and } W \text{ is a neighborhood of } y \text{ in } X_k\},$$

and $V(1,\ldots,k) \cap X_{k-1} = V(1,\ldots,k-1)$, for each i, $H_i(1,\ldots,k) \cap X_{k-1} = H_i(1,\ldots,k-1)$. Let $V = \bigcup\{V(1,\ldots,k)\colon k \in N\}$ and $H_i = \bigcup\{H_i(1,\ldots,k)\colon k \in N\}$ for each i. Then it is easily verified that V is an open neighborhood of y in X and, for each i, H_i is open in X such that

$$\mathcal{U}'_{1j}|V \subset \{H_i \cap W\colon i = 1,\ldots,n \text{ and } W \text{ is a neighborhood of } y \text{ in } X\}.$$

Thus $\mathcal{U}'_{1j}$ is almost locally finite at y in X. Furthermore, we can prove the same results even if $y \in X_k$ for $k \neq 1$. Therefore $\mathcal{U}'_{1j}$ is almost locally finite in X.

Finally, we can prove the same results even if $i \neq 1$, namely for $\mathcal{U}_{ij}$ $(i \neq 1)$ we can construct $\mathcal{U}'_{ij}$ such that $\bigcup\{\mathcal{U}'_{ij}\colon j \in N\}$ is a σ-almost locally finite local base at each point $x \in X_i$. Thus $\bigcup\{\mathcal{U}'_{ij}\colon i, j \in N\}$ is a σ-almost locally finite base of X. This completes the proof.

EXAMPLE 4.2. By this theorem, we can give a space with a σ-almost locally finite base which is not a free L-space. In **[15]**, K. Nagami and K. Tsuda proved that an infinite dimensional full complex with weak topology of Whitehead is not free L. This example is a different one from [**8**, Example 3.9].

COROLLARY 4.3. *Every paracomplex has a σ-almost locally finite base.*

COROLLARY 4.4. *Every CW-complex has a σ-almost locally finite base.*

In [**16**, Problem 1], J. Nagata proposed whether every closed image of a paracomplex is an M_1-space or not. This problem was affirmatively solved by G. Gruenhage **[4]** and T. Mizokami **[11]**, independently. Now we can this problem as a corollary of Theorem 4.1 in a slightly generalized form.

COROLLARY 4.5. *Every closed image of a D-complex is M_1.*

Proof. This follows immediately by Theorem 4.1 and [**8**, Theorem 3.6].

Finally, we consider the adjunction space of two spaces with a σ-almost locally finite base. We begin with the following theorem.

THEOREM 4.6. *Every D-complex has EP-ALF.*

Proof. Let X be a D-complex. Suppose that $X = \bigcup\{X_i\colon i \in N\}$, $X_i \subset X_{i+1}$, where each X_i is a $D(n_i)$-complex and closed in X, and X is dominated by $\{X_i\colon i \in N\}$. Let $X' = F \cup X$ be a monotonically normal space, where F and X are closed sets of X'. Suppose $\mathcal{U} = \{U(\alpha_0)\colon \alpha_0 \in A\}$ is an almost locally finite open family in F. Let $X'_1 = F \cup X_1$.

Since X_1' is monotonically normal, F and X_1 closed in X_1' and X_1 a $D(n_1)$-complex, by Lemma 3.4 every $U(\alpha_0)$ can be extend to open sets $\{U(\alpha_0, \alpha_1): \alpha_1 \in A(\alpha_0)\}$ in $F \cup X_1$ satisfying (C1), (C2), (C3). (In particular, we assume that the method of extensions is the same one of Lemma 3.4.) Repeating this process we get for each k an almost locally finite open family

$$\{U(\alpha_0, \alpha_1, \ldots, \alpha_k): \alpha_0 \in A, \alpha_1 \in A(\alpha_0), \ldots, \alpha_k \in A(\alpha_0, \alpha_1, \ldots, \alpha_{k-1})\}$$

in $F \cup X_k$. Let

$$\Sigma = \{(\alpha_0, \alpha_1, \alpha_2, \ldots): \alpha_0 \in A, \alpha_1 \in A(\alpha_0), \alpha_2 \in A(\alpha_0, \alpha_1), \ldots\}.$$

For each $(\alpha_0, \alpha_1, \alpha_2, \ldots) \in \Sigma$, let

$$U(\alpha_0, \alpha_1, \alpha_2, \ldots) = \bigcup \{U(\alpha_0, \alpha_1, \ldots, \alpha_k): k \in N\}.$$

Then it is easily verified by the same method of Theorem 4.1 that

$$\mathcal{U}' = \{U(\alpha_0, \alpha_1, \alpha_2, \ldots): (\alpha_0, \alpha_1, \alpha_2, \ldots) \in \Sigma\}$$

is an almost locally finite open family satisfying (C1), (C2), (C3). Thus X has EP-ALF.

THEOREM 4.7. *Let X be a D-complex, Y a space with a σ-almost locally finite base, F a closed set of X and f a map from F into Y. Then the adjunction space $X \cup_f Y$ has a σ-almost locally finite base.*

Proof. Let $Z = X \cup_f Y$, p the projection from the free union $X \cup Y$ onto Z and q the restriction of p to X. Suppose $\{\mathcal{U}_i: i \in N\}$ is a σ-almost locally finite base of $p(Y)$. Now, for the fixed element $i \in N$, let $\mathcal{U}_i = \{U_\alpha: \alpha \in A\}$. Since $q^{-1}(\mathcal{U}_i) = \{q^{-1}(U): U \in \mathcal{U}_i\}$ is obviously an almost locally finite open family in F, by Theorem 4.6 there exists an almost locally finite open family $\mathcal{V}_i = \{V_\beta: \beta \in B = \bigcup\{B_\alpha: \alpha \in A\}\}$ in X satisfying (C1), (C2), (C3). For $\beta \in B_\alpha$, let $U_\beta' = U_\alpha \cup p(V_\beta)$ and $\mathcal{U}_i' = \{U_\beta': \beta \in B\}$. Then it can be easily verified that U_i' is an almost locally finite open family in Z and $\bigcup\{\mathcal{U}_i': i \in N\}$ is a σ-almost locally finite local base at each point $z \in p(Y)$. Let $\{\mathcal{W}_i: i \in N\}$ be a σ-almost locally finite base in $X - F$ and $\mathcal{W}_i' = \{p(W): W \in \mathcal{W}_i\}$. Then $\{\mathcal{U}_i', \mathcal{W}_i': i \in N\}$ is obviously a σ-almost locally finite base of Z. This completes the proof.

COROLLARY 4.8. *The adjunction space of two D-complexes has a σ-almost locally finite base.*

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