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A FOLK THEOREM IN THE SPECTRAL THEORY OF C_0-SEMIGROUPS

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If A is the infinitesimal generator of a C_0-semigroup $T(t)$, a classical theorem of Hille and Phillips relates the point spectrum of A and that of $T(\xi)$ for $\xi > 0$. Specifically, if μ is in the point spectrum of $T(\xi)$ and $\mu \neq 0$, then there exists α_0 in the point spectrum of A with $\exp(\xi\alpha_0) = \mu$ and the null space of $\mu - T(\xi)$ is the closed linear span of the null spaces of $\alpha_n - A$ for $\alpha_n = \alpha_0 + 2\pi i n\xi^{-1}$ and n ranging over the integers. In this note we shall extend the Hille-Phillips theorem by proving that the null space of $(\mu - T(\xi))^k$ is the closed linear span of the null spaces of $(\alpha_n - A)^k$ as n ranges over the integers. Such a result is useful in relating the order of poles of the resolvent of A and the order of poles of the resolvent of $T(\xi)$, and as an example we shall give an application to the theory of positive (in the sense of cone-preserving) linear operators.

The generalization which we describe above has been known for many years. Jack Hale states it in his book on functional differential equations [3, Lemma 4.1, p. 180], where the generalization is left as an exercise to the reader. This seems an unwarranted burden on the reader. In our proof of the theorem for general k we shall encounter several nontrivial complications which are not present when $k = 1$. Partly because of these difficulties and partly because the extension provides useful additional information (Theorem 3 below gives an application to the theory of positivity-preserving C_0-semigroups), it seems worthwhile to provide a detailed proof.

Before stating our theorem formally, we establish some notation. If B is a closed, densely defined linear operator on a Banach space X, $N(B)$ will denote the null space of B,

$$N(B) = \{x \in X: Bx = 0\}.$$

If X is complex, $\sigma_p(B)$ will denote the point spectrum of B, so $\sigma_P(B)$ is the collection of complex λ with $N(\lambda - B) \neq \{0\}$. If $\{F_j: j \in J\}$ is a collection of linear subspaces F_j of X, we shall denote F, the smallest closed linear subspace of X such that $F_j \subset F$ for all $j \in J$, by

$$F = \bigvee_{j \in J} F_j$$

or by

$$F = \bigvee_{j=-\infty}^{\infty} F_j$$

if J is the set of integers.

The following result is a generalization of Theorem 16.7.2 in [**4**] (although it should be noted that the Hille-Phillips theorem allows semigroups more general than C_0-semigroups).

THEOREM 1. (*Compare* [**4**], *Theorem* 16.7.2, p. 467 *and* [**3**, p. 180].) *Let* $T(t)$, $t \geq 0$, *be a* C_0*-semigroup with infinitesimal generator* A. *Then for any* $\xi > 0$ *one has*

$$\sigma_P(T(\xi)) - \{0\} = \{\exp(\xi\alpha) \colon \alpha \in \sigma_P(A)\}. \tag{1}$$

If $\mu \in \sigma_P(T(\xi)) - \{0\}$, $\mu = \exp(\xi\alpha_0)$ *for some* $\alpha_0 \in \sigma_P(A)$, *and if* $\alpha_n \equiv \alpha_0 + 2\pi i n \xi^{-1}$ *for* n *an integer, then for any integer* $k \geq 1$ *one has*

$$N\big((\mu - T(\xi))^k\big) = \bigvee_{n=-\infty}^{\infty} N\big((\alpha_n - A)^k\big). \tag{2}$$

As we have said, the novelty of Theorem 1 is that we allow $k > 1$.

The main tool in proving Theorem 1, as in the proof of the original Hille-Phillips theorem, is the theory of Fourier series for Banach space valued functions. Specifically, suppose $g\colon \mathbf{R} \to X$ (X a complex Banach space) is a piecewise continuous, periodic function of period ξ. For each integer n define the nth Fourier coefficient $x_n \in X$ of $g(t)$ by

$$x_n \equiv \xi^{-1} \int_0^{\xi} \exp(-2\pi i n t \xi^{-1}) g(t)\, dt. \tag{3}$$

For any elements $z_n \in X$ (n ranging over all integers) denote the Cesàro sum of the z_n by

$$(C,1) \sum_n z_n$$

if the Cesàro sum exists. The definition of the Cesàro sum is the same as for $z_n \in \mathbf{R}$, i.e., if

$$T_j \equiv \sum_{|k| \leq j} z_k$$

then

$$(C,1)\sum z_n = \lim_{m\to\infty}\left(\frac{1}{m}\sum_{j=1}^{m} T_j\right) \tag{4}$$

if the limit on the right exists in the norm topology. Just as for real-valued functions one has

$$\frac{1}{2}g(t^-) + \frac{1}{2}g(t^+) = (C,1)\sum_n x_n \exp(2\pi int\xi^{-1}) \tag{5}$$

where $g(t^-) = \lim_{s\to t; s<t} g(s)$ and the Cesàro sum on the right in (5) converges.

We shall prove Theorem 1 in a series of technical lemmas. For notational convenience we fix $\xi > 0$ and $\mu \in \sigma_P(T(\xi))$, $\mu \neq 0$. Select $\alpha_0 \in \mathbf{C}$ such that

$$\exp(\xi\alpha_0) = \mu \tag{6}$$

and define $\alpha_n = \alpha_0 + 2\pi in\xi^{-1}$ for integers n. Define a C_0-semigroup $S(t)$ by

$$S(t) = \exp(-\alpha_0 t)T(t) \tag{7}$$

and for each integer n we define (as in [4]) a bounded linear operator J_n: $X \to X$ by

$$J_n(x) = \xi^{-1}\int_0^{\xi} \exp(-2\pi in\xi^{-1}r)S(r)x\,dr. \tag{8}$$

If, for $x \in X$, we define $g(t) = S(t)x$ for $0 \leq t < \xi$, and then extend g to be periodic of period ξ, then $J_n(x)$ is the nth Fourier coefficient of the piecewise continuous function $g(t)$. It will also be convenient to define Q: $X \to X$ by

$$Q = I - \mu^{-1}T(\xi) \tag{9}$$

and closed linear subspaces M_k, $k \geq 1$, by

$$M_k = N\big((\mu - T(\xi))^k\big) = N(Q^k). \tag{10}$$

LEMMA 1. *Let $T(t)$, $t \geq 0$, be a C_0-semigroup and let notation be as above. Then J_n maps X into the domain of A and*

$$A(J_n x) = \alpha_n J_n(x) - \xi^{-1}Q(x). \tag{11}$$

Proof. If $s > 0$ and $x \in X$, some simple manipulations give

$$(12)\quad \begin{aligned} &s^{-1}[T(s)(J_n x) - J_n x] \\ &= (s\xi)^{-1}\left[\int_{\xi}^{s+\xi} \exp(-2\pi int\xi^{-1}) \exp\left(s\left(2\pi in\xi^{-1} + \alpha_0\right)\right)S(t)x\,dt\right] \\ &\quad -(s\xi)^{-1}\left[\int_0^s \exp(-2\pi int\xi^{-1})S(t)x\,dt\right] \\ &\quad +(\xi)^{-1}\int_0^{\xi} \exp(-2\pi int\xi^{-1})s^{-1} \\ &\qquad \times\left[\exp\left((2\pi in\xi^{-1} + \alpha_0)s\right) - 1\right]S(t)x\,dt. \end{aligned}$$

Using the continuity of $t \to S(t)x$, one obtains from (12) that

$$(13)\quad \lim_{s\to 0^+} s^{-1}[T(s)(J_n x) - J_n x] = A(J_n x) = \alpha_n J_n x - \xi^{-1}Qx$$

which completes the proof. □

LEMMA 2. *Let notation and assumptions be as in Lemma* 1. *Then one has for* $x \in X$,

$$(14)\quad x = (C, 1)\sum_n J_n(x) + \frac{1}{2}Qx$$

where the summation is over all integers.

Proof. Define $g(t) = S(t)x$ for $0 \le t < \xi$ and extend $g(t)$ to be periodic of period ξ. Then $J_n x$ is the nth Fourier coefficient of $g(t)$ and as already remarked

$$(15)\quad \begin{aligned} \frac{1}{2}\left(\lim_{t\to\xi^+} g(t) + \lim_{t\to\xi^-} g(t)\right) &= \frac{1}{2}(x + \mu^{-1}T(\xi)x) \\ &= x - \frac{1}{2}Qx = (C, 1)\sum_n J_n x \end{aligned}$$

which completes the proof. □

LEMMA 3. *Let notation and assumptions be as in Lemma* 1. *If* $z \in M_k$ *one has*

$$(16)\quad z = \sum_{j=0}^{k-1} (2^{-j})\left((C, 1)\sum_m J_m(Q^j z)\right).$$

Proof. Apply equation (14) to z, then to Qz, then to Q^2z and so on and use the fact that $Q^k z = 0$ for $z \in M_k$. □

Our next three lemmas provide the tools to prove Theorem 1.

LEMMA 4. *Let notation and assumptions be as in Lemma* 1. *If* $u \in M_k$ *and* n *is a fixed integer, there exists* $v \in M_k \cap D(A)$ ($D(A)$ *denotes the domain of* A) *such that*

(17) $$(A - \alpha_n)v - u \in J_n(M_k).$$

Proof. It suffices to prove that if $w \in M_k$ there exists $v \in M_k \cap D(A)$ such that

(18) $$(A - \alpha_n)v - (C, 1)\sum_m J_m(w) \in J_n(M_k).$$

If we know (18) is true and $u \in M_k$, apply (18) to $w = Q^j(u)$ to obtain $v_j \in M_k \cap D(A)$ satisfying (18). Lemma 3 then implies that if

$$v = \sum_{j=0}^{k-1} 2^{-j} v_j$$

one has

$$(A - \alpha_n)v - u \in J_n(M_k).$$

Thus we assume $w \in J_n(M_k)$ and try to find $v \in M_k \cap D(A)$ satisfying (18). Define v by

(19) $$v = (C, 1) \sum_{m \neq n} (\alpha_m - \alpha_n)^{-1} J_m(w).$$

First, we must show that the expression in (19) is Cesàro summable. Define $h(t)$ by

$$h(t) = \exp(2\pi int\xi^{-1}) \int_0^t [\exp(-2\pi ins\xi^{-1})S(s)w - J_n w]\, ds.$$

One can easily check that $h(0) = h(\xi) = 0$. For $m \neq n$, a simple integration by parts gives

(20) $$\xi^{-1} \int_0^{\xi} \exp\left(-\frac{2\pi imt}{\xi}\right) h(t)\, dt = (\alpha_m - \alpha_n)^{-1} J_m w$$

and of course for $m = n$ one obtains

$$\text{(21)} \qquad \begin{aligned} &\xi^{-1}\int_0^\xi \exp\left(-\frac{2\pi int}{\xi}\right)h(t)\,dt \\ &\quad = \xi^{-1}\int_0^\xi\left(\int_0^t \exp\left(-\frac{2\pi ins}{\xi}\right)S(s)w - J_n w\,ds\right)dt. \end{aligned}$$

Since the Cesàro sums of the Fourier series for $h(t)$ converge to $h(t)$, by using (20) and (21) and recalling that $h(0) = 0$ one obtains

$$\text{(22)} \qquad \begin{aligned} &(C,1)\sum_{m\neq n}(\alpha_m - \alpha_n)^{-1}J_m(w) \\ &\quad = -\xi^{-1}\int_0^\xi\left(\int_0^t \exp\left(-\frac{2\pi ins}{\xi}\right)S(s)w - J_n w\,ds\right)dt. \end{aligned}$$

If v denotes the left-hand side of (22), it remains to show that $v \in D(A - \alpha_n) \cap M_k$ and that (18) is satisfied. Let v_N be a Cesàro approximating sum for v, so

$$v_N = \sum_{|m|\leq N} c_{mN}(\alpha_m - \alpha_n)^{-1}J_m(w)$$

where the c_{mN} are the constants for Cesàro summability and $v_N \to v$. One has that

$$\text{(23)} \qquad \begin{aligned} (A - \alpha_n)v_N &= \sum_{\substack{|m|\leq N \\ m\neq n}} c_{mN}(A - \alpha_m + \alpha_m - \alpha_n)(\alpha_m - \alpha_n)^{-1}J_m w \\ &= \sum_{\substack{|m|\leq N \\ m\neq n}} c_{mN}J_m w - \sum_{\substack{|m|\leq N \\ m\neq n}} c_{mN}(\alpha_m - \alpha_n)^{-1}(\xi^{-1}Qw). \end{aligned}$$

We have used Lemma 1 in obtaining (23). We know that

$$\text{(24)} \qquad \lim_{N\to\infty}\sum_{\substack{|m|\leq N \\ m\neq n}} c_{mN}J_m w = (C,1)\sum_{m\neq n} J_m w.$$

Also one has that

$$\sum_{|m|\leq N} c_{mN}(\alpha_m - \alpha_n)^{-1}(\xi^{-1}Qu) = \frac{1}{2\pi i}\sum_{|m|\leq N} c_{mN}(m - n)^{-1}Qu$$

are just the Cesàro approximating sums for

$$(2\pi i)^{-1}(C,1)\sum_{m\neq n}(m - n)^{-1}Qu = 0.$$

It follows that

$$\text{(25)} \qquad \lim_{N\to\infty}(A - \alpha_N)v_N = (C,1)\sum_{m\neq n} J_m w.$$

Since A is closed, one concludes from (25) that $v \in D(A)$ and

$$(A - \alpha_n)v = (C, 1) \sum_{m \neq n} J_m w. \tag{26}$$

Equation (26) immediately implies that (18) is satisfied.

To complete the proof it suffices to prove that $v \in M_k$. However, it is easy to see that J_m commutes with $T(t)$ for all $t \geq 0$ and all integers m, so one finds $J_m(M_k) \subset M_k$ for all m. It follows that $v_N \in M_k$ and since v_N approaches v as $N \to \infty$ and M_k is closed, $v \in M_k$. □

We shall also need a slight refinement of Lemma 4.

LEMMA 5. *If* $u \in M_k$ $(k \geq 1)$ *and* $j \geq 1$ *is an integer, there exists* $v \in M_k \cap D((A - \alpha_n)^j)$ *such that*

$$(A - \alpha_n)^j v - u \in J_n(M_k).$$

Proof. We proceed by induction on k. First we claim the lemma is true for $k = 1$. Select $u \in M_1$ and (using Lemma 4) select $w_1 \in M_1 \cap D(A - \alpha_n)$ such that

$$(A - \alpha_n)w_1 - u \in J_n(M_1). \tag{27}$$

Using Lemma 4 select $w_2 \in M_1 \cap D(A - \alpha_n)$ such that

$$(A - \alpha_n)w_2 - w_1 \in J_n(M_1). \tag{28}$$

Lemma 1 implies that $J_n(M_1) \subset D(A - \alpha_n)$ and that $A - \alpha_n$ vanishes on $J_n(M_1)$, so we find that

$$(A - \alpha_n)^2 w_2 - u = (A - \alpha_n)w_1 - u \in J_n(M_1). \tag{29}$$

Continuing in this way we eventually find $w_j \in M_1 \cap D((A - \alpha_n)^j)$ such that

$$(A - \alpha_n)^j w_j - u \in J_n(M_1). \tag{30}$$

Thus the lemma is true for $k = 1$.

Now we assume the lemma is true for all integers less than or equal to a fixed integer k; we have to prove the lemma for $k + 1$.

First, we make a technical observation: If $u \in M_{k+1} \cap D((A - \alpha_n)^j)$ and p is an integer such that $1 \leq p \leq j + 1$, we claim that there exist vectors $w_p \in M_{k+1} \cap D((A - \alpha_n)^p)$ and $z_p \in J_n(M_{k+2-p})$ such that

$$(A - \alpha_n)^{p-1} u = (A - \alpha_n)^p w_p + z_p. \tag{31}$$

(Here we adopt the convention that $M_{k+2-p} = \{0\}$ if $k + 2 - p \leq 0$.) If $p = 1$, equation (31) follows directly from Lemma 4, so we assume that w_p and z_p have been found for a fixed $p < j + 1$ and try to find w_{p+1} and z_{p+1}. Lemma 1 insures that $z_p \in D(A - \alpha_n)$ and that

$$(A - \alpha_n)z_p \in M_{k+1-p}. \tag{32}$$

Since $p < j + 1$ we also know that $(A - \alpha_n)^{p-1}u \in D(A - \alpha_n)$, so we can apply $(A - \alpha_n)$ to (31) and obtain

$$(A - \alpha_n)^p u = (A - \alpha_n)^{p+1} w_p + (A - \alpha_n)z_p. \tag{33}$$

However $(A - \alpha_n)z_p \in M_{k+1-p}$ and $k + 1 - p \leq k$, so our inductive hypothesis applies and there exist $z_{p+1} \in J_n(M_{k+1-p})$ and $v_p \in D((A - \alpha_n)^{p+1}) \cap M_{k+1-p}$ such that

$$(A - \alpha_n)z_p = (A - \alpha_n)^{p+1} v_p + z_{p+1}. \tag{34}$$

Thus if we define $w_{p+1} = w_p + v_p$ we have

$$w_{p+1} \in M_{k+1} \cap D\big((A - \alpha_n)^{p+1}\big), \qquad z_{p+1} \in J_n\big(M_{k+2-(p+1)}\big)$$

and

$$(A - \alpha_n)^p u = (A - \alpha_n)^{p+1} w_{p+1} + z_{p+1}.$$

Continuing in this way we obtain equation (31) for $1 \leq p \leq j + 1$.

We now return to the induction in k. If $u \in M_{k+1}$ we must show that for every $j \geq 1$ there exist

$$v_j \in D\big((A - \alpha_n)^j\big) \cap M_{k+1} \quad \text{and} \quad \zeta_j \in J_n\big(M_{k+1}\big)$$

such that

$$u = (A - \alpha_n)^j v_j + \zeta_j. \tag{35}$$

If $j = 1$, this is simply Lemma 4. Assuming v_j and ζ_j in equation (34) have been determined for a fixed $j \geq 1$, the preceding technical observation implies that there exist

$$v_{j+1} \in D\big((A - \alpha_n)^{j+1}\big) \cap M_{k+1} \quad \text{and} \quad z_{j+1} \in J_n\big(M_{k+1-j}\big)$$

such that

$$(A - \alpha_n)^j v_j = (A - \alpha_n)^{j+1}(v_{j+1}) + z_{j+1}. \tag{36}$$

Substituting (36) in (35) and writing $\zeta_{j+1} = z_{j+1} + \zeta_j$, one sees that $\zeta_{j+1} \in J_n(M_{k+1})$ and that

$$u = (A - \alpha_n)^{j+1}(v_{j+1}) + \zeta_{j+1}. \tag{37}$$

This shows that (35) can be satisfied for all j, which completes the induction in k. □

LEMMA 6. *If $x \in M_k$, there exists $w \in M_{k-1}$ such that $J_n(x) + w \in D((A - \alpha_n)^k)$ and*

$$(A - \alpha_n)^k(J_n(x) + w) = 0. \tag{38}$$

Proof. Lemma 1 implies that $(A - \alpha_n)(J_n x) \in M_{k-1}$, so according to Lemma 5 there exists $w_1 \in D((A - \alpha_n)) \cap M_{k-1}$ such that

$$(A - \alpha_n)(J_n x + w_1) \in J_n(M_{k-1}). \tag{39}$$

Equation (39) implies that

$$(A - \alpha_n)^2(J_n x + w_1) \in M_{k-2} \tag{40}$$

so Lemma 5 gives $w_2 \in D((A - \alpha_n)^2) \cap M_{k-2}$ such that

$$(A - \alpha_n)^2(J_n x + w_1 + w_2) \in J_n(M_{k-2}). \tag{41}$$

Assume that we have found $w_j \in M_{k-j}$ for $1 \le j \le p < k$ such that $J_n x + \sum_{j=1}^{p} w_j$ is an element of $D(A - \alpha_n)^p$ and

$$(A - \alpha_n)^p\left(J_n x + \sum_{j=1}^{p} w_j\right) \in J_n(M_{k-p}). \tag{42}$$

Equation (42) implies that we can apply $(A - \alpha_n)$ again to obtain

$$(A - \alpha_n)^{p+1}\left(J_n x + \sum_{j=1}^{p} w_j\right) \in M_{k-(p+1)}. \tag{43}$$

It follows from Lemma 5 that there exists

$$w_{p+1} \in M_{k-(p+1)} \cap D\left((A - \alpha_n)^{p+1}\right)$$

such that

$$(A - \alpha_n)^{p+1}\left(J_n x + \sum_{j=1}^{p} w_j + w_{p+1}\right) \in J_n\left(M_{k-(p+1)}\right). \tag{44}$$

Continuing in this way we eventually obtain

$$(45)\qquad (A - \alpha_n)^{k-1}\left(J_n x + \sum_{j=1}^{k-1} w_j\right) \in J_n(M_1)$$

so that, defining $w = \sum_{j=1}^{k-1} w_j$, gives

$$(A - \alpha_n)^k (J_n x + w) = 0$$

which is the desired result. □

Proof of Theorem 1. We proceed by induction on k. The case $k = 1$ is the previously mentioned theorem of Hille and Phillips (and also follows directly from Lemmas 1 and 2). Assume that we have proved the theorem for an integer k; we need to prove the proposition for $k + 1$.

First, we shall prove the difficult part:

$$(46)\qquad M_{k+1} \subset \bigvee_{j=-\infty}^{\infty} N\left((\alpha_j - A)^{k+1}\right).$$

Lemma 3 shows (if we recall that $Q^p z \in M_{k+1}$ whenever $z \in M_{k+1}$) that it suffices to prove that if $x \in M_{k+1}$, then

$$J_n(x) \in \bigvee_{j=-\infty}^{\infty} N\left((\alpha_j - A)^{k+1}\right).$$

Lemma 6 shows that there exists $w \in M_k$ such that

$$(47)\qquad J_n(x) + w \in N\left((\alpha_n - A)^{k+1}\right)$$

and the inductive hypothesis implies that

$$(48)\qquad -w \in M_k \subset \bigvee_{j=-\infty}^{\infty} N\left((\alpha_j - A)^k\right) \subset \bigvee_{j=-\infty}^{\infty} N\left((\alpha_j - A)^{k+1}\right).$$

We obtain (46) by combining (47) and (48).

To prove that

$$M_{k+1} \supset \bigvee_{j=-\infty}^{\infty} N\left((\alpha_j - A)^{k+1}\right)$$

it suffices to prove that

$$(49)\qquad M_{k+1} \supset N\left((\alpha_n - A)^{k+1}\right)$$

whenever α_n is in the point spectrum of A. This is rather easy, so we shall only sketch the proof. Define a new C_0-semigroup $T_1(t)$ by

$$T_1(t) = e^{-\alpha t} T(t)$$

where we have written α for α_n. It is easy to show that $T_1(t)$ has infinitesimal generator $B \equiv -\alpha I + A$. We have to show that if $x \in N(B^{k+1})$, then $x \in N((I - T_1(\xi))^{k+1})$. Notice that if we define $y(t)$, for $t \geq 0$ by

$$y(t) = T_1(t)x$$

then $y(t) \in C^{k+1}[0, \infty)$ (because $x \in D(B^{k+1})$) and $y(t)$ satisfies the ordinary differential equation

$$\text{(50)} \qquad \begin{aligned} \left(\frac{d}{dt}\right)^{k+1} y(t) &= 0, \qquad t \geq 0, \\ \left(\frac{d}{dt}\right)^{j} y(t)\Big|_{t=0} &= B^j(x), \qquad 0 \leq j \leq k. \end{aligned}$$

The equations (50) are also satisfied by

$$\text{(51)} \qquad z(t) = \sum_{j=0}^{\infty} \left(\frac{t^j B^j x}{j!}\right)$$

where the summation in (51) is actually from $j = 0$ to k. Uniqueness for solutions of the initial value problem (50) implies that

$$\text{(52)} \qquad y(t) = T_1(t)x = \sum_{j=0}^{\infty} \left(\frac{t^j B^j x}{j!}\right).$$

Equation (52) implies that

$$(I - T_1(\xi))x = \sum_{j=1}^{\infty} \left(\frac{1}{j!}\right) \xi^j B^j x.$$

Assume for $p \geq 1$ that there are constants $c_{j,p}$, $j \geq p$, such that

$$\text{(53)} \qquad (I - T_1(\xi))^p x = \sum_{j=p}^{\infty} c_{j,p} \xi^j B^j x.$$

Substituting $B^p x$ for x in (52) gives

$$\text{(54)} \qquad T_1(\xi) B^p x = \sum_{j=0}^{\infty} \left(\frac{1}{j!}\right) \xi^j B^{j+p} x$$

and using (53) and (54) one finds for constants $c_{j,p+1}$ that

$$\text{(55)} \qquad (I - T_1(\xi))^{p+1} x = \sum_{j=p+1}^{\infty} c_{j,p+1} \xi^j B^j x.$$

Taking $p = k$ in (55) shows that

$$(I - T_1(\xi))^{k+1}x = 0$$

since $B^j x = 0$ for $j \geq k + 1$, and this completes the proof. □

We wish now to obtain some consequences of Theorem 1. Recall that if A is a closed, densely defined linear operator and α_0 is an isolated point of the spectrum $\sigma(A)$ of A, then $(\lambda - A)^{-1}$ has a convergent Laurent series expansion for $|\lambda - \alpha_0|$ small:

$$(\lambda - A)^{-1} = \sum_{j=-\infty}^{\infty} (\lambda - \alpha_0)^j B_j \tag{56}$$

where the B_j are bounded linear operators. If there exists $-m < 0$ such that $B_j = 0$ for $j < -m$ and $B_{-m} \neq 0$, then α_0 is called a pole of the resolvent $(\lambda - L)^{-1}$ of order m. Standard results (see, for example, [**5**], pp. 178–181 or [**11**]) imply that if α_0 is a pole of A, then α_0 is in $\sigma_P(A)$, the point spectrum of A; and m, the order of the pole, is the first integer j such that

$$N\big((\alpha_0 - A)^j\big) = N\big((\alpha_0 - A)^{j+1}\big). \tag{57}$$

In particular, there must be such a j if α_0 is a pole of the resolvent. If α_0 is a pole of the resolvent of A and the dimension of the range of the spectral projection associated to α_0 is a finite number d, α_0 is called a pole of finite rank d; d is also called the algebraic multiplicity of the eigenvalue α_0.

We shall also need the idea of the essential spectrum of A, $\operatorname{ess}(A)$. There are several inequivalent definitions of the essential spectrum; we shall use a definition given by F. E. Browder in [**1**]. A complex number α is defined to be in $\operatorname{ess}(A)$ if at least one of the following conditions holds: (1) the range of $\alpha - A$ is not closed, (2) α is a limit point of $\sigma(A)$ or (3) $\bigcup_{k\geq 1} N((\alpha - A)^k)$ is not finite dimensional. Browder proves that $\beta \notin \operatorname{ess}(A)$ if and only if $\beta \notin \sigma(A)$ or β is a pole of finite rank of the resolvent of A. Other useful characterizations have been given by D. C. Lay [**6**] and A. E. Taylor [**11**, §9].

If B is a bounded linear operator on a complex Banach space X, the radius of the essential spectrum of B, $r_e(B)$, is defined by

$$r_e(B) = \sup\{|\lambda| : \lambda \in \operatorname{ess}(B)\}. \tag{58}$$

If, for any bounded linear operator B, a seminorm $p(B)$ is defined by

$$p(B) = \inf\{\|B + C\|: C \text{ a compact linear operator}\} \tag{59}$$

it is proved in [7] that

$$r_e(B) = \lim_{n\to\infty} (p(B^n))^{(1/n)}. \tag{60}$$

The formula (60) is valid for all the definitions of the essential spectrum. As usual, the radius of the spectrum of B, $r(B)$ is given by

$$r(B) = \sup\{|\lambda| : \lambda \in \sigma(B)\} = \lim_{n\to\infty} \|B^n\|^{(1/n)}. \tag{61}$$

If L is a closed, densely defined linear operator, it will also be useful to define $t(L)$ by

$$t(L) = \inf\{\mathrm{Re}(\alpha) : \alpha \in \sigma(L)\}. \tag{62}$$

With these preliminaries we have

THEOREM 2. *Assume that $-L$ is the generator of a C_0-seminorm $T(s)$, $s \geq 0$, and that for some $\xi > 0$*

$$r_e(T(\xi)) < r(T(\xi)). \tag{63}$$

Then $t(L)$ satisfies

$$-\xi t(L) = \log(r(T(\xi))). \tag{64}$$

If $F_1 = \{\lambda \in \sigma(T(\xi)) : |\lambda| = r(T(\xi))\}$, then F_1 is a finite set and every element of F_1 is a pole of finite rank of the resolvent of $T(\xi)$. If F_2 is defined by

$$F_2 = \{\alpha \in \sigma_P(L) : \mathrm{Re}(\alpha) = t(L)\}$$

then F_2 is also a finite, nonempty set consisting of isolated points of $\sigma(A)$ and for every element α of F_2 one has that

$$\bigcup_{k=1}^{\infty} N(\alpha - L)^k \tag{65}$$

is finite dimensional. If, for every $\lambda \in F_1$, $d(\lambda)$ is defined to be the dimension of the null space of $\lambda - T(\xi)$, then the cardinality of F_2 satisfies

$$\mathrm{card}(F_2) \leq \sum_{\lambda \in F_1} d(\lambda) \tag{66}$$

If m_1 is the maximum of the orders of the poles of $(\lambda - T(\xi))^{-1}$ for $\lambda \in F_1$ and m_2 the maximum of the order of the poles of $(\alpha - L)^{-1}$ for $\alpha \in F_2$ and α a pole of the resolvent of L, then

$$m_2 \leq m_1. \tag{67}$$

Proof. The statements about F_1 follow from the properties of the essential spectrum and equation (63). Since it is known (see Corollary 2, p. 457 in [**4**]) that

$$\exp(-\xi\sigma(L)) \subset \sigma(T(\xi)) \tag{68}$$

it follows that $\alpha \in \sigma(L)$ implies

$$-\xi \operatorname{Re}(\alpha) \leq \log(r(T(\xi))) \equiv \tau$$

and that there exists $\varepsilon > 0$ such that $-\xi \operatorname{Re}(\alpha)$ is not in the interval $(\tau - \varepsilon, \tau)$ for $\alpha \in \sigma(L)$. Theorem 1 states that if $\lambda \in F_1$, there exists $\alpha_0 \in F_2$ such that $\exp(-\xi\alpha_0) = \lambda$ and if $\alpha_m = \alpha_0 + (2\pi i m)\xi$ and $k \geq 1$,

$$N\left((\lambda - T(\xi))^k\right) = \bigvee_{m=-\infty}^{\infty} N\left((\alpha_m - L)^k\right). \tag{69}$$

This immediately gives (65). If one takes $k = 1$ in (69) and uses the fact that the linear subspaces $N(\alpha_m - L)$ are linearly independent for different m's, one also obtains (66). Equation (68) and the fact that F_1 consists of isolated points of $\sigma(T(\xi))$ implies that F_2 consists of isolated points of $\sigma(L)$.

It remains to prove that $m_2 \leq m_1$. Equation (69) shows that it suffices to prove that if $\lambda \in F_1$ and

$$N\left((\lambda - T(\xi))^{k+1}\right) = N\left((\lambda - T(\xi))^k\right)$$

then

$$N\left((\alpha_m - L)^{k+1}\right) = N\left((\alpha_m - L)^k\right)$$

for all integers m. By way of contradiction, suppose not, so assume

$$w \in N\left((\alpha_j - L)^{k+1}\right), \qquad w \notin N\left((\alpha_j - L)^k\right) \tag{70}$$

for some j. Define $W_m = N((\alpha_m - L)^k)$; since W_m is finite dimensional and only finitely many W_m are nonzero, equation (69) implies that

$$w = \sum_m w_m \tag{71}$$

where $w_m \in W_m$, $w_n \neq 0$ for some $n \neq j$ and the sum is finite. It is easy to see that L is defined on all of W_m and that $\sigma(L \mid W) = \alpha_m$; thus $(L - \alpha_m)^{k+1}$ is a one-one map of W_n onto W_n for $m \neq n$. It follows that if we apply $(L - \alpha_j)^{k+1}$ to (71) and then $(L - \alpha_m)^{k+1}$ for all $m \neq n$ such that $w_m \neq 0$ we obtain a contradiction: a nonzero element of W_n will be zero. □

REMARK 1. If, for a given $\alpha \in F_2$, we know that the range of $\alpha - L$ is closed, it follows that α is a pole of finite rank for the resolvent of A (see [**11**]).

Finally, we wish to give an application of Theorems 1 and 2 to the theory of positive linear operators. Recall that if X is a real Banach space and K a closed, convex subset of X, K is called a cone if $x \in K - \{0\}$ implies $-x \notin K$ and $x \in K$ and $t \geq 0$ implies $tx \in K$. The cone K is called total if X is the norm closure of $\{u - v\colon u, v \in K\}$. A bounded linear operator B on X is called positive (with respect to K) if $B(K) \subset K$. By the spectrum of B we shall mean the spectrum of the natural extension of B to the complexification of X.

We need to recall an extension of the Krein-Rutman theorem which is proved in the appendix of [**10**]; a different proof of a closely-related result (which would be adequate for the application below) is given in [**9**].

THEOREM (*See appendix of* [**10**]). *Let X be an ordered real Banach space with total positive cone K and assume that B is a continuous, positive linear map of X to X whose resolvent has a pole on the spectral circle $|\lambda| = r(B)$. Then $r(B)$ is in the spectrum of B, and if $r(B)$ is a pole of the resolvent, its order is greater than or equal to the order of any other pole on $|\lambda| = r(B)$.*

Our next theorem is very close to a theorem of Greiner, Vogt and Wolff [**2**]; our assumptions on the cone K are weaker, but we must assume that $r_e(T(\xi)) < r(T(\xi))$ for some $\xi > 0$.

THEOREM 3 (Compare [**2**]). *Assume that K is a total cone in a real Banach space X. Let $T(t)$, $t \geq 0$, be a C_0-semigroup in X with infinitesimal generator $-L$, and suppose $T(t)$ and L are extended naturally to the complexification of X. Assume that $T(t)$ is positive for each $t \geq 0$ and that for some $\xi > 0$ the essential spectral radius of $T(\xi)$ is strictly less than the spectral radius of $T(\xi)$. Then if $t(L)$ is given by* (62)

$$t(L) \in \sigma_P(L).$$

If $t(L)$ is a pole of the resolvent of L (see Remark 1), *its order is greater than or equal to the order of any other pole of the resolvent on the line* $\{\alpha\colon \operatorname{Re}(\alpha) = t(L)\}$.

Proof. Let notation be as in Theorem 2. Since F_2 is finite, select d such that $d > |\alpha|$ for all α in F_2. Select a positive integer k such that

$$2\pi k \xi^{-1} > 2d \tag{72}$$

and define $\eta = \xi k^{-1}$. The spectral mapping theorem and the spectral mapping theorem for Browder's essential spectrum imply that

$$[r_e(T(\eta))]^k = r_e(T(\xi)) \tag{73}$$
$$[r(T(\eta))]^k = r(T(\xi))$$

so

$$r_e(T(\eta)) < r(T(\eta)). \tag{74}$$

The definition of the essential spectrum and the Krein-Rutman theorem imply that $r(T(\eta))$ is a pole of finite rank for the resolvent of $T(\eta)$. Theorem 1 implies that there exists $\alpha \in F_2$ such that

$$\exp(-\eta\alpha) = r(T(\eta)). \tag{75}$$

However η has been chosen so that the only possible solution $\alpha \in F_2$ of (75) is real, and using (64) (for η instead of ξ) we conclude that

$$t(L) \in F_2. \tag{76}$$

Finally, suppose that $\alpha = t(L)$ is a pole of the resolvent of A and that $\beta \in F_2$ is another pole. We have chosen η so that the map $x \to \exp(-\eta x)$ is one-one for $x \in F_2$, so Theorem 1 gives (writing $\lambda_1 = r(T(\eta))$ and $\lambda_2 = \exp(-\eta\beta)$)

$$N((\beta - A)^k) = N((\lambda_2 - T(\eta))^k) \tag{77}$$
$$N((\alpha - A)^k) = N((\lambda_1 - T(\eta))^k).$$

Equation (77) implies that the order of the pole β of the resolvent of A is the same as the order of the pole λ_2 of the resolvent of $T(\eta)$, and similarly for α and λ_1. However, the Krein-Rutman theorem implies that the order of the pole λ_2 is less than or equal to that of λ_1. □

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