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Let X be a compact normal complex space on which $\mathbf{C}^*$ acts 'in a nice manner'. We describe all invariant open subsets U of X such that the holomorphic map $U \to U/\mathbf{C}^*$ of U onto the categorical quotient for the category of compact complex spaces, $U/\mathbf{C}^*$, is locally Stein. The description depends on a partial ordering of the fixed point components which arises from the Bialynicki-Birula decompositions of X.

Introduction. Let $\rho\colon T \times X \to X$ be a meromorphic action, (cf. §1), of $T = \mathbf{C}^*$ on an irreducible compact normal complex analytic space X. Such an action is said to be locally linearizable if and only if given any $x \in X$ there is a T-invariant neighborhood V of x and a proper T-equivariant holomorphic embedding of V into $\mathbf{C}^n$ with T acting linearly on $\mathbf{C}^n$.

In this paper we solve the following problem:

Describe all T-invariant Zariski open subsets U of X, such that U/T is a compact complex analytic space and $U \to U/T$ is a semi-geometric quotient (i.e. a categorical quotient which is locally Stein cf. (1.8)).

This problem has been solved by A. Bialynicki-Birula and A. Sommese, [**B** − **B** + **S**], under the above setting when U contains no fixed points and by A. Bialynicki-Birula and J. Swiecieka, [**B** − **B** + **Sw**], when the action is algebraic and X is a compact algebraic variety.

As in [**B** − **B** + **S**], our description of semi-geometric quotients $U \to U/T$ is intimately linked to a certain partial ordering of the fixed point components $F_1,\ldots,F_r$. So that we can state our results precisely we shall introduce the following notation. We assume that all analytic spaces are Hausdorff, reduced and have countable topology.

Let $\{F_1,\ldots,F_r\}$ be the connected components of the fixed point set of T, X^T. Define ϕ^+, ϕ^-: $X \to X^T$ by $\phi^+(x) = \lim_{t\to 0} tx$ and $\phi^-(x) = \lim_{t\to\infty} tx$, respectively.

Let $X_i^+ = \{x \in X \mid \phi^+(x) \in F_i\}$, $i = 1,\ldots,r$, and $X_i^- = \{x \in X \mid \phi^-(x) \in F_i\}$, $i = 1,\ldots,r$.

An index i is said to be *directly less than* an index j if $C_{ij} = (X_i^+ - F_i) \cap (X_j^- - F_j) \neq \varnothing$. We say that i is *less than* j, denoted $i < j$, if there exists a sequence $i = i_0,\ldots,i_k = j$ such that i_l is directly less than i_{l+1} for $l = 0,\ldots,k-1$. This relation forms an ordering of the indices $\{1,\ldots,r\}$.

A *cross section* of $\{1,\ldots,r\}$ is a division of $\{1,\ldots,r\}$ into two non-empty disjoint subsets A^- and A^+ satisfying the condition that $i \in A^-$ and $j < i$ implies that $j \in A^-$.

A *semi-cross section* of $\{1,\ldots,r\}$ is a division of $\{1,\ldots,r\}$ into three disjoint subsets, A^-, A^0, A^+, at least two of which are nonempty, which satisfy the following two conditions:

(a) if $i < j$ and $j \in A^0$ then $i \notin A^0$

(b) if $A^+ \neq \varnothing$ then $(A^- \cup A^0, A^+)$ is a cross section and if $A^- \neq \varnothing$ then $(A^-, A^0 \cup A^+)$ is a cross section.

A subset B of X is a *semi-sectional set* if $B = X - \bigcup_{i \in A^+} X_i^+ - \bigcup_{j \in A^-} X_j^-$ for some semi-cross section (A^-, A^0, A^+).

MAIN THEOREM. *Let ρ: $T \times X \to X$ be as above and let U be a T-invariant Zariski open subset of X. Then U/T is a compact complex analytic space and $U \to U/T$ is a semi-geometric quotient if and only if U is a semi-sectional set with respect to some semi-cross section* (A^-, A^0, A^+). □

Our proof uses the techniques of [**B** − **B** + **S**].

We conclude this paper with a simple illustration of the Theorem for the case of a diagonal action of $\mathbf{C}^*$ on $\mathbf{P}^1 \times \mathbf{P}^1$.

I would like to express my sincere thanks to Professor Andrew J. Sommese for his generous support and encouragement in completing this project.

1. Notation and background material. In this section we establish the pertinent notation and background material needed for the proof of the Theorem. The principal reference for this material is [**B** − **B** + **S**].

Let T denote $\mathbf{C}^*$, the multiplicative group of non-zero complex numbers. A holomorphic action ρ: $T \times X \to X$ of T on a normal compact analytic space X is said to be a *meromorphic action* if ρ extends to a meromorphic map $\tilde{\rho}$: $\mathbf{P}^1 \times X \to X$, where $\mathbf{P}^1$ in one-dimensional complex projective space. This condition is satisfied if X is a Kaehler manifold and X^T has non-empty intersection with every connected component of X, [**So**].

The maps ϕ^+, ϕ^-: $X \to X^T$ as defined in the introduction always exist for meromorphic actions, [**Kor$_1$**]. The collections of subsets $\{X_i^+ \mid i = 1,\ldots,r\}$ and $\{X_i^- \mid i = 1,\ldots,r\}$ form two decompositions of the space X, called respectively the plus and the minus Bialynicki-Birula decompositions. They satisfy the following properties:

(1.1) (a) $X = \bigcup X_i^+ = \bigcup X_i^-$ is a disjoint union of T-invariant sets.

(b) There are two special components of X^T, F_1 called the *source* and F_r called the *sink* (renumbering if necessary), such that X_1^+ and X_r^- are Zariski open in X.

(c) Each X_i^+ and X_j^- is a constructible set, i.e., the finite union of locally closed sets.

These properties were proven in the algebraic category by Bialynicki-Birula, [**B** − **B**], and in the Kaehler category by Carrell and Sommese, [**C** + **S**], and Fujiki [$\mathbf{Fu_2}$].

We will now state a result found in [**B** − **B** + **S**] which is modeled on a result of Fujiki [$\mathbf{Fu_2}$]. It provides the basis for the proof of the Main Theorem.

THEOREM (1.2). *Let* $\rho\colon T \times X \to X$ *be a meromorphic action of* T *on an irreducible compact complex analytic space* X. *There is a diagram*:

$$\begin{array}{ccc} Z & \xrightarrow{\mu} & X \\ f\downarrow & & \\ Q & & \end{array}$$

with the following properties:

(a) f *is a flat morphism of irreducible compact complex spaces* Z *and* Q.

(b) μ *is a bimeromorphic holomorphic map of* Z *onto* X *such that the restriction of* μ *to each fiber* $Z_q = f^{-1}(q)$ *is an embedding.*

(c) *There is a natural holomorphic action of* T *on* Z *making* f *and* μ *T-equivariant with respect to the trivial action on* Q *and* ρ *on* X *respectively.*

(d) *There is a dense Zariski open subset* $\mathfrak{U}$ *of* Q *such that for every* $q \in \mathfrak{U}$, Z_q *is reduced and* $\mu(Z_q)$ *is the closure of a T-orbit from* $X_1^+ \cap X_r^-$.

(e) *Every fiber* Z_q *of* f *is one-dimensional and for fibers* Z_q, $Z_{q'}$ *that are reduced*, $\mu(Z_q) = \mu(Z_{q'})$ *if and only if* $q = q'$.

(f) $\mu(Z_q)$ *is connected and meets* F_1, *the source, and* F_r, *the sink, for all* $q \in Q$.

(g) *For all* $q \in Q$, $Z_q \cap Z^T$ *is finite.*

(h) *Any continuous map* $\tau\colon A \to Y$ *of an open subset* A *of* Q *to a complex analytic space* Y *which is holomorphic on a Zariski open dense subset of* A *is holomorphic on all of* A. □

Let K be a compact complex space and let Comp(K) be the set of all compact subsets of K. The *Hausdorff metric* on Comp(K) is defined by:

$$\underline{\operatorname{dist}}(A, B) = \max_{a\in A}\left\{\min_{b\in B} \operatorname{dist}(a, b)\right\} + \max_{b\in B}\left\{\min_{a\in A} \operatorname{dist}(b, a)\right\}$$

where $\operatorname{dist}(a, b)$ is the metric on K. Let $A, A_i, i \in I$, be elements of $\operatorname{Comp}(K)$. When we say that the A_i's converge to A we mean they converge in the Hausdorff metric.

We have the following Corollary to (1.2).

COROLLARY (1.3). *Let ρ, X, Q, Z, F and μ be as in* (1.2). *Let $\{q_n\}$ be a sequence in Q. If q_n converges to q in Q then $\mu(Z_{q_n})$ converges to $\mu(Z_q)$ in X.*

Proof. We claim that q_n converges to q in Q implies that Z_{q_n} converges to Z_q in the Hausdorff metric in Z, where $Z_{q_n} = f^{-1}(q_n)$, $Z_q = f^{-1}(q)$. Let z be an arbitrary point of Z_q, then any open neighborhood V of z must intersect Z_{q_n} for $n \gg 0$. Suppose not, since $f\colon Z \to Q$ is flat it is an open map and hence $f(V)$ is an open neighborhood of q. If Z_{q_n} does not intersect V then q_n would not be an element of $f(V)$ and therefore q_n would not converge to q. Thus we have that Z_{q_n} converges to Z_q and by the continuity of $\mu\colon Z \to X$ that $\mu(Z_{q_n})$ converges to $\mu(Z_q)$. □

DEFINITION (1.4). Let $\rho\colon T \times X \to X$ be a meromorphic action of T on a normal compact analytic space. We say that ρ is a *locally linearizable action* if given any $x \in X$ there is a T-invariant neighborhood V of x and a proper T-equivariant holomorphic embedding of V into $\mathbf{C}^N$ with T acting linearly on $\mathbf{C}^N$.

PROPOSITION (1.5). *A holomorphic action $\rho\colon T \times X \to X$ on a normal irreducible compact complex space X is locally linearizable if either of the following is true*:

(a) *X is an algebraic variety and ρ is an algebraic action or*

(b) *$X^T \neq \varnothing$ and X can be equivariantly embedded in a compact Kaehler manifold Y with a holomorphic action $\tilde{\rho}\colon T \times Y \to Y$.*

Proof. (a) is due to Sumihiro [**Su**] and b) is due to Koras [**Kor$_2$**]. □

We shall also use extensively the following (cf. Corollary (0.2.4) of [**B − B + S**]).

PROPOSITION (1.6). *Let $\rho\colon T \times X \to X$ be a locally linearizable action of T on a compact analytic space X. Given any $q \in Q$ we can choose $\{x_1, \ldots, x_k\}$ in $\mu(Z_q) - \mu(Z_q)^T$ with*:

(a) $\phi^+(x_1) \in F_1$ *and* $\phi^-(x_k) \in F_r$

(b) $\phi^-(x_j) = \phi^+(x_{j+1})$ *for* $j = 1, \ldots, k-1$

(c) *if* $\phi^-(x_j) = \phi^+(x_i)$, *then* $i = j + 1$
(d) $\overline{T\{x_1,\ldots,x_k\}} = \mu(Z_q)$

Moreover, if X is normal, then $\mu(Z_q) \cap F_1 = \{x_1\}$, $\mu(Z_q) \cap F_r = \{x_k\}$. □

We note that the last statement of Proposition (1.6) may not hold if X is not normal, i.e., it is possible in such a case that $F_1 = F_r$, for example simply identify a point of F_1 with a point of F_r.

COROLLARY (1.7). *Let* X *and* ρ *be as in* (1.6). *For any connected component* F_i *of* X^T, $F_1 < F_i < F_r$. □

Let $\bar{\rho}$: $G \times Z \to Z$ be an action of a reductive group G on complex space Z. We can definie an equivalence relation on the points of Z by $x \sim y$ if and only if there is a sequence of points $x = x_0, x_1, \ldots, x_n = y$ in Z such that $\overline{Gx_i} \cap \overline{Gx_{i+1}} \neq \varnothing$, $i = 0,\ldots,n-1$. We define Z/G to be the set of equivalence classes under the above relation and define a map π: $Z \to Z/G$ by $\pi(x) = [x]$, where $[x]$ denotes the equivalence class containing x. Z/G is given the quotient topology, i.e. V is an open subset of Z/T if and only if $\pi^{-1}(V)$ is an open subset of Z. We call π: $Z \to Z/G$ the *categorical quotient* of Z by G.

We note that in general our definition of a categorical quotient does not coincide with the usual definition, in which the equivalence relation is defined by the invariant holomorphic functions. Our definition implies that fibers of π are connected and thus the quotient, X/G, need not be Hausdorff. When the quotient is assumed to be Hausdorff either definition will suffice.

DEFINITION (1.8). A categorical quotient π: $Z \to Z/G$ is a *semi-geometric quotient* if it is locally Stein, i.e. given any point $y \in Z/G$ there is a neighborhood W of y such that $\pi^{-1}(W)$ is Stein.

LEMMA (1.9). *Let* ρ: $T \times X \to X$ *be a meromorphic action of* T *on* X *a compact complex analytic space. Let* U *be a* T*-invariant open subset of* X. *If* π: $U \to U/T$ *is a semi-geometric quotient then each fiber contains at most one fixed point.*

Proof. Let $x, y \in U^T$ and suppose $x \sim y$. Then we can find a sequence of fixed points in U, $x = z_0, z_1, \ldots, z_n = y$ such that $z_i \in [x]$ and z_i is directly related to z_{i+1}, i.e. there is a point $z \in U$ with $\phi^+(z) = z_i$ and $\phi^-(z) = z_{i+1}$ or $\phi^+(z) = z_{i+1}$ and $\phi^-(z) = z_i$. Thus, if $x \neq y$ then

$\pi^{-1}([x])$ contains $\phi^+(z) \cup Tz \cup \phi^-(z)$ which is homeomorphic to $\mathbf{P}^1$ contradicting the assumption that π is locally Stein. □

COROLLARY (1.10). *Let ρ, U, and π be as in* (1.9). *Then π restricted to U^T is one to one onto $\pi(U^T)$.* □

The above allows us to identify U^T with a subset of U/T, namely $\pi(U^T)$.

LEMMA (1.11). *Let ρ, U, and π be as in* (1.9). *Then fibers of π are either orbits or $x^+ \cup x^-$ for some $x \in U^T$, where $x^+ = \{z \in X \mid \phi^+(z) = x\}$ and $x^- = \{z \in X \mid \phi^-(z) = x\}$.*

Proof. By (1.9) each fiber contains at most one fixed point. If the fiber does not contain a fixed point then it must consist of a single orbit since the intersection of the closures of two distinct orbits is either empty or contained in the set of fixed points. If the fiber contains a fixed point then since U is open and T-invariant, it follows that the fiber contains $x^+ \cup x^-$. For the fiber to contain anything else it must contain a second fixed point which is impossible. Thus the fibers are as stated. □

LEMMA (1.12). *Let ρ: $T \times X \to X$ be a meromorphic action of T on X a normal compact complex analytic space X. Let U be a T-invariant open subset of X such that ρ: $U \to U/T$ is a semi-geometric quotient. If U/T is Hausdorff it possess the structure of a complex analytic space and π is a holomorphic map.*

Proof. The definition of a semi-geometric quotient implies that we may cover U with π-saturated Stein sets, A_i. Each A_i/T is a complex Stein space such that π: $A_i \to A_i/T$ is holomorphic, **[Sn]**. Since the structure on A_i/T is induced by the invariant holomorphic functions on A_i it follows easily that the structure on the A_i/T's are compatable. Thus, if U/T is Hausdorff it is a complex analytic space and π is holomorphic. □

2. Semi-geometric quotients. Throughout this section we shall assume that ρ: $T \times X \to X$ is a locally linearizable action of $T = \mathbf{C}^*$ on an irreducible normal compact complex analytic space X with fixed point components $F_1, \ldots, F_r$.

We want to describe all T-invariant Zariski open subsets U of X whose quotient U/T is semi-geometric and a compact complex space. The following propositions enable us to partition all such U into these three disjoint classes:

Class I. U contains no fixed point components, i.e. $U \subset X - X^T$.

Class II. The only fixed point component U contains is either the source F_1 or the sink F_r.

Class III. U contains fixed point components F_i, $i \neq 1, r$, and if U contains F_i and F_j they are not directly related to each other.

PROPOSITION (2.1). *Let U be a T-invariant Zariski open subset of X whose quotient U/T is semi-geometric and a compact complex space. If $F_i \cap U \neq \varnothing$, then $X_i^+ \cup X_i^- \subset U$.*

Proof. Let $x \in F_i \cap U$. Since U is open and T-invariant the sets x^+ and x^- must both be contained in U. Since $X_i^+ = \bigcup_{x \in F_i} x^+$ and $X_i^- = \bigcup_{x \in F_i} x^-$ we must only show that if $F_i \cap U \neq \varnothing$, then $F_i \subset U$. Furthermore, since $F_i \cap U$ is open in F_i, this reduces to showing that $F_i \cap U$ is closed.

Let $x \in \overline{F_i \cap U} \subset F_i$ and let $\{x_n\}$ be a sequence of distinct points contained in F_i converging to x. Since $U \to U/T$ is a semi-geometric quotient each distinct x_n must have a distinct image in U/T, and so we may consider $\{x_n\}$ as a sequence in U/T. Now U/T is assumed to be compact and so, passing to a subsequence and renumbering if necessary, we have that x_n converges to some $y \in U/T$. The locally Stein condition of semi-geometric quotients implies that we can find a neighborhood $W_T \subset U/T$ of y and a Stein set $W = \pi^{-1}(W_T) \subset U$.

We can assume that $\{x_n\}$ is contained in $F_i \cap W$, which is a closed T-invariant subset of W. By Corollary 3.6 of [**Sn**] since W is Stein we have that $\pi(F_i \cap W) = F_i \cap W_T$ is closed in W_T (we note that in this case our definition of categorical quotient coincides with that of [**Sn**]). This implies that $y \in \pi(F_i \cap W)$ and thus by identification, cf (1.10), $y \in F_i \cap W \subset U$. The convergence of the $\{x_n\}$ yields $y = x$. □

PROPOSITION (2.2). *Let U be as in* (2.1). *Let F_i and F_j be two fixed point components and suppose that $F_i \subset U$. If F_j is directly related to F_i then $F_j \not\subset U$.*

Proof. Assume $F_i < F_j$. Suppose $F_j \subset U$ then we can find an $x \in U$ such that $\phi^+(x) \in F_i$ and $\phi^-(x) \in F_j$. (2.1) implies that $U \supset \phi^+(x) \cup Tx \cup \phi^-(x)$ which is biholomorphic to $\mathbf{P}^1$. This contradicts the local Steinness of the quotient since there can be no neighborhood of $\pi(x)$ in U/T whose inverse image in U is Stein. □

PROPOSITION (2.3). *Let U be as in* (2.1). *If U contains the source, F_1, or the sink, F_r, then U does not contain any other fixed point component.*

Proof. Assume $U \supset F_1$. Since F_r is directly related to F_1, $F_r \not\subset U$. Let $F_i \subset U$, $i \neq 1, r$. Let $x \in F_i$ and choose $q \in Q$ such that $x \in \mu(Z_q)$, where Q, Z_q and μ are as in (1.2). Let $\mathfrak{U}$ also be as in (1.2) and choose a sequence $\{q_n\} \subset \mathfrak{U}$ converging to q. $\mu(Z_{q_n})$ converges to $\mu(Z_q)$ by (1.3). Thus we can find a sequence of points $\{x_n\} \subset U$ such that $x_n \in \mu(Z_{q_n})$ and x_n converges to x. Let $\{y_n\}$, y be the image of $\{x_n\}$, x respectively in U/T. $\{y_n\} \subset$ image of F_1, which is identified with F_1, but since $x \notin F_1$, $y \notin F_1$. But every open neighborhood of y meets $\{y_n\}$, so y is in the closure of F_1 in U/T. Since F_1 is closed this contradiction proves the proposition. □

We now show our Main Theorem holds for each of the three Classes separately.

Assume U is of Class I, i.e. $U \subset X - X^T$. This case was done in [**B** − **B** + **S**]. Their description is given in terms of cross sections. However, by considering a cross section (A^-, A^+) as a semi-cross section (A^-, A^0, A^+) with $A^0 = \varnothing$, their result coincides with our Main Theorem.

We next assume U is of Class II, i.e. the only fixed point component U contains is either F_1 or F_r. To simplify things we assume $F_1 \subset U$.

PROPOSITION (2.4). *Let U be as in the preceding paragraph. If the quotient U/T is semi-geometric and a compact complex space then there exists a semi-cross section $A = (A^-, A^0, A^+)$ such that U is a semi-sectional set with respect to A.*

Proof. Since $F_1 \subset U$ we have by (2.1) that $X_1^+ \subset U$. The proof of Proposition (2.3) can in fact be used to show that U contains only X_1^+, i.e. $U = X_1^+$. Hence:

$$U = X_1^+ = X - \bigcup_{i=2}^{r} X_i^+ = X - \bigcup_{i \in A^+} X_i^+ - \bigcup_{j \in A^-} X_j^-$$

where $A^+ = \{2,\ldots,r\}$, $A^- = \varnothing$. Taking $A^0 = \{1\}$ we have the desired semi-cross section. □

PROPOSITION (2.5). *Suppose U is the semi-sectional set associated to the semi-cross section (A^-, A^0, A^+) where $A^- = \varnothing$, $A^0 = \{1\}$, $A^+\{2,\ldots,r\}$. Then U is a T-invariant Zariski open subset of X whose quotient U/T is semi-geometric and a compact complex space.*

Proof. By definition we have:

$$U = X - \bigcup_{i \in A^+} X_i^+ - \bigcup_{j \in A^-} X_j^-.$$

Using the facts that $A^- = \varnothing$ and that $X = \bigcup_{i=1}^r X_i^+$ which is a disjoint union we have that $U = X_1^+$. Thus we have that U is a T-invariant Zariski open subset of X.

Since $U/T = X_1^+/T = F_1$ we have that U/T is a compact complex analytic space. For each $x \in F_1$ let V_x be a T-invariant Stein neighborhood of x in U given by definition of the action being locally linearizable. Then we have covered U/T by sets whose inverse images in U are Stein. U/T is obviously a categorical quotient and so it is a semi-geometrical quotient. The proposition is proven. □

Combining (2.4) and (2.5) gives the Main Theorem for Class II sets.

From now on unless stated otherwise, we assume that U is of Class III, i.e. U contains fixed point components F_i, $i \neq 1$, r and any two are not directly related.

LEMMA (2.6). *Let U be a T-invariant Zariski open subset of X whose quotient U/T is semi-geometric and a compact complex space. Then $X_1^+ \cap X_r^- \subset U$.*

Proof. Let $C = X_1^+ \cap X_r^-$, then C is a Zariski open subset of X. Since X is assumed to be irreducible C must also be irreducible. By Zariski openness and denseness C must intersect U. The same proof as that of Lemma (1.1.1) in [**B** − **B** + **S**] yields that C is contained in U. □

Let $\mathfrak{U}$ be the subset of Q from Theorem (1.2). The above Lemma allows us to identify $\mathfrak{U}$ with a dense open subset of U/T.

We have need of the following fact. Let A be a complex space and let B be a dense subset of A. Let $\{x_n\}$ be a sequence of points of A, then we can find a sequence of points contained in B, $\{y_n\}$, such that $\mathrm{dist}(x_n, y_n) < 1/n$ where dist is the metric on A. If $\{x_n\}$ diverges then $\{y_n\}$ diverges and if $\{x_n\}$ converges then $\{y_n\}$ converges to the same point. Thus if we have a sequence in U/T we can assume it is contained in $\mathfrak{U}$.

We shall make the following convention. Let $y \in U/T$, when we choose a point $x \in \pi^{-1}(y)$ we assume x is the unique fixed point if $\pi^{-1}(y)$ contains one, otherwise x may be any point of $\pi^{-1}(y)$.

LEMMA (2.7). *Let U be a T-invariant Zariski open subset of X. Then U/T is a semi-geometric quotient and a compact complex space if and only if given $q \in 0$, either*:

(a) *There exists a $y \in X - X^T$ such that $\mu(Z_q) \cap U = Ty$ or*

(b) *There exist $y_1, y_2 \in X - X^T$ with $\phi^-(y_1) = \phi^+(y_2)$ such that $\mu(Z_q) \cap U = Ty_1 \cup \phi^-(y_1) \cup Ty_2$, where Q, Z_q and μ are as given in* (1.2).

Proof. To prove the necessity of (a) or (b) we first shall show that $\mu(Z_q) \cap U \neq \varnothing$. Suppose not, we can find a sequence $\{q_n\} \subset \mathfrak{U} \subset Q$, such that q_n converges to q and thus $\mu(Z_{q_n})$ converges to $\mu(Z_q)$ in the Hausdorff metric. We note this implies that any open neighborhood of $\mu(Z_q)$ contaiins $\mu(Z_{q_n})$, $n \gg 0$. By (2.6) we can consider $\{q_n\} \subset U/T$. By assumption U/T is compact and therefore, after passing to a subsequence and renumbering if necessary, q_n converges to an element y of U/T. Let $x \in \pi^{-1}(y)$. Let V_1 and V_2 be disjoint open subsets of X which contain $\mu(Z_q)$ and x respectively. We can assume that $V_2 \subset U$. $\pi(V)$ contains a dense open subset consisting of elements of $\mathfrak{U}$. Since q_n converges to y and $y \in \pi(V)$ we can replace elements of $\{q_n\}$ for $n \gg 0$, with elements of $\pi(V) \cap \mathfrak{U}$ without affecting convergence, so we may consider $q_n \in \pi(V)$ for $n \gg 0$. This implies $\pi^{-1}(q_n) \cap V_2 \neq \varnothing$. But $\pi^{-1}(q_n) = \mu(Z_{q_n}) \cap U \subset V_1$. This contradiction implies that $\mu(Z_q) \cap U \neq \varnothing$.

We now claim that $\mu(Z_q) \cap U$ is connected. Obviously this is true if $q \in \mathfrak{U}$. Suppose q is not an element of $\mathfrak{U}$ and that $\mu(Z_q) \cap U$ is not connected. Then we can find two disjoint closed invariant sets, S_1 and S_2, with $S_1 \cup S_2 = \mu(Z_{q_n}) \cap U$. Note $\pi(S_1) \neq \pi(S_2)$. As before we can find a sequence $\{q_n\}$ contained in $\mathfrak{U}$, such that $\mu(Z_{q_n})$ converges to $\mu(Z_q)$. Let $x_n = \pi(\mu(Z_{q_n}))$, then by continuity we have that x_n converges to both $\pi(S_1)$ and $\pi(S_2)$ in U/T. This contradicts U/T being Hausdorff.

$\mu(Z_q) \cap U$ can contain at most one fixed point since if it contained two, connectivity would imply that it contains $\mathbf{P}^1$ and then U/T could not be a semi-geometric quotient. If $\mu(Z_q) \cap U$ contains no fixed point it has the form of (a), if it has a fixed point, x, since $x^+ \cup x^- \subset U$ by (2.1) it has the form of (b).

Suppose $\mu(Z_q) \cap U$ is of the form (a) or (b) for any $q \in Q$. We will first show that π: $U \to U/T$ is a semi-geometric quotient. The fiber over any point in U/T must either be a single orbit or $x^+ \cup x^-$ for some fixed point x. This can be seen by considering $\mu(Z_q) \cap U$. If it is just an orbit then it goes to a point in U/T and is the fiber over that point. If it contains a fixed point then every $\mu(Z_{q'}) \cap U$ which contains the x will go to the same point in U/T. Thus the fibers are as stated above and it is easily seen that this implies that U/T is a categorical quotient. For each $y \in U/T$ choose $x \in \pi^{-1}(y)$. For each x let V_x be the T-invariant Stein neighborhood of x in X given by the action being locally linearizable. Since U is T-invariant we can consider that $V_x \subset U$ and by the description of the fibers that the V_x are π-saturated. Thus we can cover U/T with sets whose inverse images are Stein and, so the quotient U/T is semi-geometric.

Assume U/T is not Hausdorff, then we can find $\{y_n\} \subset U/T$ with y_n converging to two distinct points z_1 and z_2. We may assume $\{y_n\} \subset \mathcal{U}$ and so $\{y_n\} \subset Q$ which is compact. We may assume, after passing to a subsequence and renumbering if necessary that y_n converges to $q \in Q$. Let $x_i \in \pi^{-1}(z_i)$ and V_i be an open neighborhood of z_i in U/T. $V_i \supset y_n$ for $n \gg 0$ and so we can find a sequence of points $\{x_{i,n}\} \subset U$ with $x_{i,n} \in \mu(Z_{y_n}) \cap \pi^{-1}(V_i)$ such that $x_{i,n}$ converges to x_i. Since $\mu\colon Z \to X$ is continuous the above implies that $\mu(Z_q)$ contains both x_1 and x_2. But since x_1 and x_2 are both contained in U this implies that their image in U/T must be the same, i.e. $z_1 = z_2$. This contradiction implies U/T is Hausdorff. Applying Proposition (1.10) gives us that U/T is a complex analyltic space.

It remains to show that U/T is compact. Let $\{x_n\}$ be a sequence in U/T. We can assume it is contained in $\mathcal{U}$ and therefore in Q which is compact and so we can find a convergent subsequence $\{x'_m\}$ with x'_m converging to $q \in Q$. Let $x \in \mu(Z_q) \cap U$. Since x'_m converges to q we have that $\mu(Z_{x'_m})$ contained in U with $z_m \in \mu(Z_{x'_m}) \cap U$ and such that the z_m converges to x. By the continuity of $U \to U/T$ we have that x'_m converges to y where y is the image of x in U/T. Thus we have shown that every sequence in U/T has a convergent subsequence which converges to a point in U/T. Therefore, U/T is compact.

This completes the proof of Lemma (2.7). □

THEOREM (2.8). *Let $A = (A^-, A^0, A^+)$ be a semi-cross section. If U is the semi-sectional set which corresponds to the semi-cross section then U is a T-invariant Zariski open subset of X whose quotient U/T is semi-geometric and a compact complex space.*

Proof. Recall

$$U = X - \bigcup_{i \in A^+} X_i^+ - \bigcup_{j \in A^-} X_j^-.$$

It is obvious that U is T-invariant. The proof that U is Zariski open is the same as that given in Theorem (1.3) of [**B** − **B** + **S**].

Let $q \in Q$. We claim that $\mu(Z_q) \cap U \neq \varnothing$. Suppose not, then $\{h \in \{1,\dots,r\} \mid F_h \cap \mu(Z_q) \neq \varnothing\}$ would all lie in A^- or all in A^+. This follows from Proposition (1.6) since otherwise we could find an $x \in \mu(Z_q)$ and an i and j with $j \in A^-$ and $i \in A^+$, such that $\phi^+(x) \in F_j$ and $\phi^-(x) \in F_i$. By the above description of U we see that this implies that $x \in U$ and thus that $\mu(Z_q) \cap U \neq \varnothing$. Therefore the set of h with $F_h \cap \mu(Z_q) \neq \varnothing$ lies totally in either A^- or A^+. By (1.6) we would have that either $r \in A^-$ or $1 \in A^+$. The former implies that $A^- = \{1,\dots,r\}$ and the latter that $A^+ = \{1,\dots,r\}$. In either case this would imply that $U = \varnothing$. Thus for all $q \in 0$ we must have that $\mu(Z_q) \cap U \neq \varnothing$.

Let $q \in Q$ and let y_1 and y_2 be two points in $X - X^T$ such that $Ty_1 \cup Ty_2$ is contained in $\mu(Z_q) \cap U$. We claim that either $Ty_1 = Ty_2$ or either $\phi^+(y_1) = \phi^-(y_2)$ or $\phi^-(y_1) = \phi^+(y_2)$. Suppose $Ty_1 \neq Ty_2$. Under this condition assume also that $\phi^+(y_1) \neq \phi^-(y_2)$ and $\phi^-(y_1) \neq \phi^+(y_2)$. Then again applying (1.6) we can find a and b such that either $\phi^+(y_2) \in F_b$ and $\phi^-(y_1) \in F_a$ and $a < b$ or $\phi^+(y_1) \in F_b$ and $\phi^-(y_2) \in F_a$ and $a < b$. Either way we get a contradiction. In the former case if $a \in A^-$ then $y_1 \in F_a^-$ and is not in U, if $a \in A^0 \cup A^+$ then $b \in A^+$, (since (A^-, A^0, A^+) is a semi-cross section), and therefore $y_2 \in X_b^+$ and thus not in U. Likewise the latter case also implies that either y_1 or y_2 is not an element of U.

Assume that in fact $\phi^-(y_1) = \phi^+(y_2) = x$. Let $x \in F_k$, then $k \in A^0$, since otherwise if $k \in A^-$ this would mean y_1 is not an element of U and if $k \in A^+$ this would mean y_2 is not an element of U. Hence $x \in U$.

Therefore, we have shown that for every $q \in 0$ either $\mu(Z_q) \cap U = Ty$ for some $y \in X - X^T$ or $\mu(Z_q) \cap U = Ty_1 \cup \phi^-(y_1) \cup Ty_2$ for some y_1 and y_2 in $X - X^T$. Applying Lemma (2.7) finishes the proof. □

LEMMA (2.9). *Let U be a T-invariant Zariski open subset of X whose quotient U/T is semi-geometric and a compact complex space. Let $\{F_k\}$ be the set of fixed point components contained in U. Let $U' = U - \bigcup X_k^-$. Then U' is a Class* I *T-invariant Zariski open subset of X whose quotient U'/T is semi-geometric and a compact complex space.*

Proof. U' is obviously T-invariant and contained in $X - X^T$. Lemma (1.3.1) of [**B** − **B** + **S**] shows that $\bigcup X_k^-$ is a closed set and thus U' is an open constructible set and therefore is Zariski open.

For all $q \in Q$ we can consider $\mu(Z_q) \cap U'$ which is contained in $\mu(Z_q) \cap U$. If $\mu(Z_q) \cap U = Ty$ for some $y \in X - X^T$ then y is not an element of X_k for any F_k contained in U and so $\mu(Z_q) \cap U' = Ty$. If $\mu(Z_q) \cap U = Ty_1 \cup \phi^-(y_1) \cup Ty_2$, for some $y_1, y_2 \in X - X^T$, we have that $y_1 \in X_k^-$ for some F_k contained in U but that y_2 is not an element of X_k^- for any F_k contained in$U$ and thus that $\mu(Z_q) \cap U' = Ty_2$. Hence we have that for every $q \in Q$ there is an $y \in X - X^T$ sjuch that $\mu(Z_q) \cap U' = Ty$. Applying Lemma (1.2) of [**B** − **B** + **X**] gives the desired result. □

REMARK (2.10). In [**B** − **B** + **S**] it is shown that a Class I T-invariant Zariski open subset U of X has a compact complex space as quotient if and only if $(X - U)$ has two connected components, one which contains the source, F_1, and the other which contains the sink, F_r. We will use this fact in the next theorem.

THEOREM (2.11). *Let U be a T-invariant Zariski open subset of X whose quotient U/T is semi-geometric and a compact complex space. Then U is a semi-cross sectional set with respect to some semi-cross section (A^-, A^0, A^+).*

Proof. Given U let U' be the corresponding Class I set given by Lemma (2.9), i.e. $U' = U - \cup(X_{k_i}^-)$ where $\{F_{k_1},\ldots,F_{k_n}\}$ is the set of fixed point components contained in U. As noted in Remark (2.10), $(X - U')$ has two connected components, one containing F_1 and the other F_r. Since we assume that U does not contain either F_1 or F_r we must have that $\cup(X_{k_i}^-)$ does not contain them either. Therefore $(X - U') - \cup(X_{k_i}^-)$ must be disconnected, since F_1 and F_r are still in different components. But $(X - U') - \cup(X_{k_i}^-) = (X - U)$. Thus we have that $(X - U)$ is disconnected and that F_1 and F_r are in different components.

Let A_1 be the connected component of $X - U$ which contains F_1 and let A_2 be the connected component of $X - U$ which contains F_r. Assume there was another connected component of $X - U$ besides A_1 and A_2, call it A_3. Let $x \in A_3$ and choose a $q \in Q$ such that $x \in \mu(Z_q)$. By Lemma (2.7) we know that $\mu(Z_q) \cap U$ is either Ty for some $y \in X - X^T$ or $Ty_1 \cup \phi^-(y_1) \cup Ty_2$, for some $y_1, y_2 \in X - X^T$. In either case (1.6) implies that $\mu(Z_q) \cap (X - U)$ has two connected components, one which intersects F_1 and another which intersects F_r. Thus x must be in the same connected component of $X - U$ as F_1 or F_r, i.e. $A_3 = A_1$ or $A_3 = A_2$. Therefore, $X - U$ has exactly two connected components.

Let $\{F_1,\ldots,F_r\}$ be the set of connected components of X^T. Set $A^- = \{j\colon F_j \text{ is contained in } A_1\}$, $A^0 = \{k\colon F_k \text{ is contained in } U\}$ and $A^+ = \{i\colon F_i \text{ is contained in } A_2\}$. We claim that (A^-, A^0, A^+) forms a semi-cross section of $\{1,\ldots,r\}$. Let $j \in A^-$ and suppose j' is directly less than j. We can find $x_j \in F_j$, $x_{j'} \in F_{j'}$, and $x \in X$ such that $\phi^+(x) = x_{j'}$, and $\phi^{-}(x) = x_j$. Thus we can find a $q \in Q$ with $\mu(Z_q)$ containing $\{x_j, x_{j'}, x\}$. Looking at $\mu(Z_q) \cap (X - U)$ we see that $F_{j'}$ is contained in A_1, i.e. $j' \in A^-$. A finite application of the above step shows that $j' \in A^-$ for any $j' < j$. Now let $k \in A^0$ and suppose j is directly less than k. Proposition (2.2) shows that F_j is not contained in U and (1.6) implies that it must be contained in A_1. Thus $j \in A^-$ and therefore so is any $j' < k$. Now suppose there was a $k' \in A^0$ with $k < k'$, the above implies that $k \in A$. Therefore we see that if $k \in A^0$ and if k' is related to k then k' is not an element of A^0. It is also obvious by what we have shown that if $k \in A^0$ and if $k < i$ then $i \in A^+$. Hence (A^-, A^0, A^+) forms a semi-cross section of $\{1, \ldots, r\}$.

Let $x \in U$. Then either $x \in (X_k^+ \cup X_k^-)$, for some $k \in A^0$, or $x \in X_i^- \cap X_j^+$, for some $i \in A^+$ and $j \in A^-$. Therefore since U, A_1 and A_2 are T-invariant and the points of U satisfy the conditions stated above we have that U is given by:

$$U = X - \bigcup_{i \in A^+} X_i^+ - \bigcup_{j \in A^-} X_j^- .$$

Hence U is a semi-sectional set with respect to the semi-cross section (A^-, A^0, A^+).

Combining (2.8) and (2.11) yields the Main Theorem for Class III sets. □

3. An Example. Let T act on $\mathbf{P}^1 \times \mathbf{P}^1$ by $t([z_0 : z_1], [w_0 : w_1]) = ([z_0 : tz_1], [w_0 : tw_1])$. There are four fixed points of this section, $F_1 = ([1 : 0], [1 : 0])$, $F_2 = ([1 : 0], [0 : 1])$, $F_3 = ([0 : 1], [1 : 0])$ and $F_4 = ([0 : 1], [0 : 1])$. The plus decomposition is given by

$$X_1^+ = \mathbf{P}^1 \times \mathbf{P}^1 - \{z_0 = 0 \text{ or } w_0 = 0\},$$

$$X_2^+ = \{([z_0 : z_1], [0 : 1]) : z_0 \neq 0\},$$

$$X_3^+ = \{([0 : 1], [w_0 : w_1]) : w_0 \neq 0\} \text{ and}$$

$$X_4^+ = ([0 : 1], [0 : 1]).$$

The minus decomposition is given by

$$X_1^- = ([1 : 0], [1 : 0]),$$

$$X_2^- = \{([1 : 0], [w_0 : w_1]) : w_1 \neq 0\},$$

$$X_3^- = \{([z_0 : z_1], [0 : 1]) : z_1 \neq 0\} \text{ and}$$

$$X_4^- = \mathbf{P}^1 \times \mathbf{P}^1 + \{z_1 = 0 \text{ or } w_1 = 0\}.$$

Hence $([1 : 0], [1 : 0])$ is the source and $([0 : 1], [0 : 1])$ is the sink.

The following chart describes the possible T-invariant open subsets U of X whose quotient U/T is semi-geometric and a compact complex space:

Class	A^-	A^0	A^+	U	U/T
I	$\{1,2,3\}$	$\varnothing$	$\{4\}$	$\mathbf{C}^2 - 0$	$\mathbf{P}^1$
I	$\{1,2\}$	$\varnothing$	$\{3,4\}$	$\mathbf{C}^* \times \mathbf{P}^1$	$\mathbf{P}^1$
I	$\{1,3\}$	$\varnothing$	$\{2,4\}$	$\mathbf{C}^* \times \mathbf{P}^1$	$\mathbf{P}^1$
I	$\{1\}$	$\varnothing$	$\{2,3,4\}$	$\mathbf{C}^2 - 0$	$\mathbf{P}^1$
II	$\varnothing$	$\{1\}$	$\{2,3,4\}$	$\mathbf{C}^2$	point
II	$\{1,2,3\}$	$\{4\}$	$\varnothing$	$\mathbf{C}^2$	point
III	$\{1,2\}$	$\{3\}$	$\{4\}$	$\mathbf{P}^1 \times \mathbf{P}^1 - ([0:1],[0:1])$ $-\{([1:0],[w_0:w_1])\}$	$\mathbf{P}^1$
III	$\{1,3\}$	$\{2\}$	$\{4\}$	$\mathbf{P}^1 \times \mathbf{P}^1 - ([0:1],[0:1])$ $-\{([z_0:z_1],[1:0])\}$	$\mathbf{P}_1$
III	$\{1\}$	$\{2,3\}$	$\{4\}$	$\mathbf{P}^1 \times \mathbf{P}^1 - ([0:1],[0:1])$ $-\{([1:0],[1:0])\}$	$\mathbf{P}_1$
III	$\{1\}$	$\{3\}$	$\{2,4\}$	$\mathbf{P}^1 \times \mathbf{P}^2 - ([1:0],[1:0])$ $-\{([z_0:z_1],[0:1])\}$	$\mathbf{P}^1$
III	$\{1\}$	$\{2\}$	$\{3,4\}$	$\mathbf{P}^1 \times \mathbf{P}^1 - ([1:0],[1:0])$ $-\{([1:0],[w_0:w_1])\}$	$\mathbf{P}^1$

REFERENCES

[B − B] A. Bialynicki-Birula, *Some theorems on actions of algebraic groups*, Ann. of Math., **98** (1973), 480–497.

[B − B + S] A. Bialynicki-Birula, and A. J. Sommese, *Quotients by* $\mathbf{C}^*$ *and* SL(2, C) *Actions*, to appear in Trans. Amer. Math. Soc.

[B − B + Sw] A. Bialynicki-Birula and J. Swiecieka, *Complete quotients by algebraic torus actions*, to appear in Proceedings of the Conference on $\mathbf{C}^*$-actions of the University of British Columbia, Vancouver, Jan. 1981.

[C + S] J. Carrell and A. J. Sommese, *$\mathbf{C}^*$-actions*, Math. Scand., **43** (1978), 49–59.

[C + S_2] ______, *Some topological aspects of $\mathbf{C}^*$-actions on compact Kaehler manifolds*, Comment. Math. Helv., **54** (1979), 567–582.

[Fu] A. Fujiki, *On automorphism groups of compact Kaehler manifolds*, Invent. Math., **44** (1978), 225–258.

[Fu_2] ______, *Fixed points of the actions on compact Kaehler manifolds*, Publ. RIMS, Kyoto Univ., **15** (1979), 797–826.

[G] D. Gross, *On compact categorical quotients by torus actions*, Doctoral dissertation. Mathematics Department, University of Notre Dame, 1982.

[Kon] J. Konarski, *Decompositions of normal algebraic varieties determined by an action of a one-dimensional torus*, Bull. de l'Acad. Pol. des Sci. ser. des sciences math., astr., et phys. **XXVI** (1978), 295–300.

[Kon_2] ______, *A pathological example of an action of k^**, preprint.

[Kor] M. Koras, *Actions of reductive groups*, Doctoral dissertation. Mathematics Department, University of Warsaw, 1980.

[Kor_2] ______, *Linearization of reductive group actions*, preprint.

[L] D. Lieberman, *Compactness of the Chow Scheme: Applications to Automorphisms and Deformations of Kaehler Manifolds*, Sem. Fran Norguet (1977), Lecture Notes in Mathematics **670**. Heidelberg: Springer-Verlag, 1978.

[Sn] D. Snow, *Reductive group actions on Stein spaces*, Math. Ann., **259** (1982), 79–97.

[So] A. J. Sommese, *Extension theorems for reductive group actions on compact Kaehelr manifolds*, Math. Ann., **218** (1975), 107–116.

[Su] H. Sumihiro, *Equivariant completion*, J. Math., Kyoto Univ., **14** (1974), 1–28.

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