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TOPOLOGICAL METHODS FOR C^*-ALGEBRAS III: AXIOMATIC HOMOLOGY

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A homology theory consists of a sequence $\{h_n\}$ of covariant functors from a suitable category of C^*-algebras to abelian groups which satisfies homotopy and exactness axioms. We show that such theories have Mayer-Vietoris sequences and (if additive) commute with inductive limits. There are analogous definitions and theorems in cohomology with one important difference: an additive cohomology theory associates a Milnor $\lim^1$ sequence to an inductive limit of C^*-algebras. As prerequisite to these results we develop the necessary homotopy theory, including cofibrations and cofibre theories.

0. Introduction.

The construction of a homology theory is exceedingly complicated. It is true that the definitions and necessary lemmas can be compressed within ten pages, and the main properties established within a hundred. But this is achieved by disregarding numerous problems raised by the construction, and ignoring the problem of computing illustrative examples. ...

In spite of this confusion, a picture has gradually evolved of what is and should be a homology theory. Heretofore this has been an imprecise picture which the expert could use in his thinking but not in his exposition. A precise picture is needed. It is at just this stage in the development of other fields of mathematics that an axiomatic treatment appeared and cleared the air.

S. Eilenberg and N. Steenrod
Foundations of Algebraic Topology
(1952).

There are several homology theories and cohomology theories defined on suitable categories of C^*-algebras. Here are some examples in rough historical order:

(a) The K-theory groups $K_*(A)$ of Karoubi [7], [8], defined abstractly in terms of modules or concretely via projections and unitaries in matrix algebras over A.

(b) The Ext groups $\operatorname{Ext}^*(A)$ of Brown-Douglas-Fillmore [3], [4] which arise from the classification of extensions of C^*-algebras of the form

$$0 \to \mathcal{K} \to E \to A \to 0$$

where $\mathcal{K}$ denotes the C^*-algebra of compact operators on a separable Hilbert space.

(c) The groups $\mathrm{Ext}^*(X; A)$ of Pimsner-Popa-Voiculescu [7] which arise from the classification of homogeneous extensions of C^*-algebras of the form

$$0 \to C(X, \mathcal{K}) \to E \to A \to 0.$$

Holding the space X fixed yields a cohomology theory.

(d) The groups $\mathrm{Ext}^*(A, B)$ of Kasparov [**11**], [**12**] which arise from the classification of extensions of C^*-algebras of the form

$$0 \to B \otimes \mathcal{K} \to E \to A \to 0.$$

Holding B fixed yields a cohomology theory; holding A fixed yields a homology theory.

There are other less well-behaved possibilities. Fix a C^*-algebra D. The functors $[C_0(\mathbf{R}^n) \otimes D, A]$ and $[D, C_0(\mathbf{R}^n) \otimes A]$ have properties analogous to a homology theory in some respect. (Here $[A, B]$ denotes homotopy classes of C^*-algebra maps from A to B.) Similarly, the functors $[A, C_0(\mathbf{R}^n) \otimes D]$ and $[C_0(\mathbf{R}^n) \otimes A, D]$ have properties analogous to a cohomology theory in some respect. Upon suspension stabilization these yield "cofibre" homology and cohomology theories. (See Example 8.5.)

In this paper we offer a simple system of axioms for homology theories on C^*-algebras and an analogous system of axioms for cohomology theories on C^*-algebras. Theories (a)–(d) satisfy these axioms. The homotopy theories do not, but they satisfy cofibre axioms which carry much the same structure.

The axioms are strong enough to enable one to effectively compute homology in diverse situations. As evidence we offer theorems on

(a) the homology of a triple

(b) the homology of a pullback (i.e., a Mayer-Vietoris theorem)

(c) the homology of an inductive limit

and analogous results in cohomology. The main theorems of [**22**] are also available.

Here are the axioms for a homology theory.

DEFINITION. A *homology theory* is a sequence $\{h_n\}$ of covariant functors from an admissible category $\mathcal{C}$ of C^*-algebras to abelian groups which satisfies the following axioms:

Homotopy axiom. Let $h\colon A \to C([0,1], B)$ be a homotopy from $f_0 = p_0 h$ to $f_1 = p_1 h$ in $\mathcal{C}$, where $p_t(\xi) = \xi(t)$. Then

$$f_{0_*} = f_{1_*} : h_n(A) \to h_n(B) \quad \text{for all } n.$$

Exactness axiom. Let

$$0 \to J \xrightarrow{i} A \xrightarrow{j} B \to 0$$

be a short exact sequence in $\mathcal{C}$. Then there is a map $\partial\colon h_n(B) \to h_{n-1}(J)$ and a long exact sequence

$$\to h_n(J) \xrightarrow{i_*} h_n(A) \xrightarrow{j_*} h_n(B) \xrightarrow{\partial} h_{n-1}(J) \to \cdots.$$

The map ∂ is natural with respect to morphisms of short exact sequences.

A further axiom is sometimes assumed. If so then the homology theory is said to be *additive*.

Additivity Axiom. Let $A = \bigoplus_{i=0}^{\infty} A_i$ in $\mathcal{C}$. Then the natural maps $h_n(A_i) \to h_n(A)$ induce an isomorphism

$$\bigoplus_i h_n(A_i) \to h_n(A).$$

The axioms for cohomology are quite similar.

DEFINITION. A *cohomology theory* is a sequence $\{h^n\}$ of contravariant functors from an admissible category $\mathcal{C}$ of C^*-algebras to abelian groups which satisfies the following axioms:

Homotopy axiom. Let $h\colon A \to C([0,1], B)$ be a homotopy from $f_0 = p_0 h$ to $f_1 = p_1 h$. Then $f_0^* = f_1^*\colon h^n(B) \to h^n(A)$ for all n.

Exactness axiom. Let

$$0 \to J \xrightarrow{i} A \xrightarrow{j} B \to 0$$

be a short exact sequence in $\mathcal{C}$. Then there is a map $\delta\colon h^n(J) \to h^{n+1}(B)$ and a long exact sequence

$$\cdots \to h^n(B) \xrightarrow{j^*} h^n(A) \xrightarrow{i^*} h^n(J) \xrightarrow{\delta} h^{n+1}(B) \to \cdots.$$

The map δ is natural with respect to morphisms of short exact sequences.

The following axiom is sometimes assumed. If so then the cohomology theory is said to be *additive*.

Additivity axiom. Let $A = \bigoplus_{i=0}^{\infty} A_i$. Then the natural maps

$$h^n(A) \to h^n(A_i)$$

induce an isomorphism

$$h^n(A) \to \prod_i h^n(A_i).$$

We have not yet formally defined "admissible category of C^*-algebras". Roughly, a category of C^*-algebras is admissible if it is closed under the various homotopy constructions required to develop elementary homotopy theory. Here are some admissible categories:

(1) all C^*-algebras
(2) separable C^*-algebras
(3) separable nuclear C^*-algebras

In each case we allow all C^*-maps as morphisms. This is a provisional definition. It should be replaced by the closed model categories of Quillen [**18**]. However, there is as yet no good notion of loop space or fibration (the definitions by Karoubi [**9**] are not helpful here). Thus we proceed in an ad hoc manner and return to this question at (2.12).

As indicated previously, there is another possible axiom framework. It revolves about the notion of cofibration.

DEFINITION. A map of C^*-algebras $p\colon A \to B$ is a *cofibration* if it satisfies the homotopy lifting property: any homotopy $h\colon D \to C([0,1], B)$ of a composite fp, $f\colon D \to A$, can be extended to a homotopy $H\colon D \to C([0,1], A)$ of f. That is, the diagram

$$\begin{tikzcd} & A \\ D \arrow[ur, "f"] \arrow[r, "H"] \arrow[dr, "h"'] & C([0,1], A) \arrow[u, "p_0"'] \arrow[d, "p"] \\ & C([0,1], B) \end{tikzcd}$$

commutes.

A map $X \to Y$ of compact topological spaces is a cofibration in the classical sense [**25**, p. 7] if and only if the map $C(Y) \to C(X)$ is a cofibration in the sense above. A cofibration $p\colon A \to B$ must be surjective.

DEFINITION. A *cofibre homology theory* $\{h_n\}$ is a sequence of covariant functors from an admissible category of C^*-algebras to abelian groups which satisfies the homotopy axiom and the following axioms:

Cofibre axiom. Let $p\colon A \to B$ be a cofibration, and let

$$Cp = \{(\xi, a) \in C([0,1], B) \oplus A \mid \xi(1) = 0, \xi(0) = p(a)\}$$

be the mapping cone, with $\pi(p)\colon Cp \to A$ by $\pi(p)(\xi, a) = a$. Then the sequence

$$h_n(Cp) \xrightarrow{\pi(p)_*} h_n(A) \xrightarrow{p_*} h_n(B)$$

is exact for each n.

Suspension axiom. There is a natural isomorphism

$$\sigma_A\colon h_n(A) \xrightarrow{\cong} h_{n-1}(SA)$$

where $SA = C_0((0,1), A)$ is the suspension of A.

These axioms imply the existence of a long exact sequence in homology for the sequence

$$0 \to J \to A \to B \to 0$$

provided that $A \to B$ is a cofibration.

There is an analogous definition in cohomology.

DEFINITION. A *cofibre cohomology theory* is a sequence $\{h^n\}$ of contravariant functors from an admissible category of C^*-algebras to abelian groups which satisfies the homotopy axiom and the following axioms:

Cofibre axiom. If $p\colon A \to B$ is a cofibration then the sequence

$$h^n(B) \xrightarrow{p^*} h^n(A) \xrightarrow{\pi(p)^*} h^n(Cp)$$

is exact for each n.

Suspension axiom. There is a natural isomorphism

$$\sigma^A\colon h^n(SA) \xrightarrow{\cong} h^{n+1}(A).$$

We shall demonstrate that a homology theory *is* a cofibre homology theory and that a cohomology theory *is* a cofibre cohomology theory.

The paper is organized as follows.

Section 1 is devoted to homotopy theory. In it we introduce the basic equipment—mapping cylinders, mapping cones, cofibrations, pullbacks, and we verify their elementary properties.

Section 2 concerns cofibration sequences. The climax perhaps is Verdier's axiom, which to a composite $A \to C \to B$ associates a weave of cofibre sequences.

Section 3 is devoted to demonstrating that the elementary properties of homology theories are implied by our axioms. For example, we show how to deduce the suspension axiom from the homotopy and exactness axioms. Also included is the classical "homology of a triple" theorem and analogous results for cofibre theories.

Section 4 is devoted to the Mayer-Vietoris theorem. Suppose given a pullback diagram

$$\begin{array}{ccc} P & \xrightarrow{g_1} & A_1 \\ {\scriptstyle g_2}\downarrow & & \downarrow{\scriptstyle f_1} \\ A_2 & \xrightarrow{f_2} & B \end{array}$$

with f_1 and f_2 surjective or f_1 a cofibration and f_2 arbitrary, and suppose given a cofibre theory h_*. Then there is a long exact sequence

$$\cdots \to h_n(P) \xrightarrow{(g_{1*},\, g_{2*})} h_n(A_1) \oplus h_n(A_2) \xrightarrow{(-f_{1*} + f_{2*})} h_n(B) \to h_{n-1}(P) \to \cdots .$$

If h_* is a homology theory then it suffices to assume that f_1 is surjective and f_2 is arbitrary.

Section 5 concerns limits. If $A = \varinjlim A_i$ is the inductive limit of a sequence of C^*-algebras and h_* is an additive homology theory then there is a natural isomorphism

$$\varinjlim{}_i h_n(A_i) \to h_n(A).$$

In §6 we turn to cohomology and establish results analogous to those of §3 and the Mayer-Vietoris theorem of §4. Section 7 is devoted to limits in cohomology. Here there is a real difference in the outcome (as is predicted by the results in topology). If $A = \varinjlim A_i$ is an inductive limit of a sequence of C^*-algebras and h^* is an additive cohomology theory then the natural map $h^n(A) \to \varprojlim h^n(A_i)$ is *not* an isomorphism. Rather, there is a short exact Milnor $\lim^1$ sequence

$$0 \to \varprojlim{}^1 h^{n-1}(A_i) \to h^n(A) \to \varprojlim h^n(A_i) \to 0.$$

Finally, in §8 we discuss the various examples mentioned above and show that the appropriate axioms are satisfied.

There are several topics which are not covered here, and deliberately so: uniqueness, stability and periodicity.

Uniqueness. Eilenberg-Steenrod show that two ordinary homology theories which satisfy the dimension axiom and which coincide on spheres must coincide on finite CW-complexes. However, the analogous statement for generalized homology theories (i.e., no dimension axiom) on spaces is not true. To make it true one needs a natural transformation between the theories. Given that additional hypothesis, the relevant uniqueness theorem does hold for theories on C^*-algebras as is shown in [**22**, Theorem 4.2]. The analogous result holds in cohomology.

Stability. One might require that $h_n(A \otimes M_k(\mathbf{C})) \cong h_n(A)$ for $k = 2$, for all k, or that $h_n(A \otimes \mathcal{K}) \cong h_n(A)$. We never use such assumptions. We believe them to be independent of our other axioms. Of course the limit theorems and the assumption "$h_n(A \otimes M_2(\mathbf{C})) \cong h_n(A)$" imply "$h_n(A \otimes \mathcal{K}) \cong h_n(A)$" and thus the fact that h_* is really defined on the homotopy category of Morita equivalences of C^*-algebras. We regard this sort of stability as still rather mysterious. For example, what can be said about the sequence

$$\begin{aligned}[A, B] &\to [A \otimes M_2(\mathbf{C}), B \otimes M_2(\mathbf{C})] \\ &\to [A \otimes M_4(\mathbf{C}), B \otimes M_4(\mathbf{C})] \to \cdots \to [A \otimes \mathcal{K}, B \otimes \mathcal{K}].\end{aligned}$$

Does it stabilize for k large, given some assumptions on A and B?

Periodicity. We have not assumed that $h_n(A) \cong h_n(S^2A)$, as is the case with K-theory and the other examples (a)–(d) considered. This also we presume to be independent of exactness and homotopy. Periodicity is quite rare in algebraic topology, and we believe it to be quite rare for C^*-algebras, though there is a counter-argument [**19**]. The homotopy situation is not well-understood. Is there a Freudenthal suspension theorem to tell us that for suitable B the sequence

$$[A, B] \to [SA, SB] \to [S^2A, S^2B] \to \cdots$$

eventually becomes a sequence of isomorphisms? Let $\{A, B\} = \lim [S^nA, S^nB]$. How are $\{A, B\}$ and $\{A \otimes \mathcal{K}, B \otimes \mathcal{K}\}$ related? Rosenberg [**19**] has some interesting observations and speculations on these matters.

Much of the material in §1 is scattered in the literature, particularly in papers of Karoubi [**7**], [**8**], [**9**], P. Kohn [**13**], and L. G. Brown, cf. [**2**].

Kohn's (unpublished) thesis [**13**] contains the basic definitions, and it contains the construction of the mapping cone. A preprint by J. Hilgert (received as this paper was being typed) establishes Theorem 4.5 and a variant of Theorem 3.8 for K-theory by somewhat different methods. Our exposition in §§1 and 2 follows that of J. P. May [**14**].

Many ideas in this paper may be traced back to papers, correspondence or discussions with L. G. Brown. He was the first functional analyst to take seriously the notion that there should be a subject called "noncommutative algebraic topology", particularly in his influential "Rome paper" [**2**]. J. Rosenberg's paper [**19**] was also quite influential, particularly in pointing out the problems which arise in the absence of cofibrations. We are deeply grateful to L. G. Brown and to J. Rosenberg for their assistance and stimulation.

This paper is philosophically related to its predecessors [**22**], [**23**] but there is no serious mathematical link between them.

1. Homotopy. We begin the paper by presenting the basic working tools of homotopy theory for C^*-algebras. As indicated before, much of this has been in the folklore for some time.

Fix an admissible category $\mathcal{C}$ of C^*-algebras and C^*-algebra maps. For a C^*-algebra A in $\mathcal{C}$, define the *cylinder* of A as $IA = C([0, 1], A)$ with canonical maps $p_t\colon IA \to A$ given by $p_t(\xi) = \xi(t)$.

DEFINITION 1.1. For A and B in $\mathcal{C}$, define a *homotopy* $h\colon f \simeq g$ between C^*-algebra maps $f, g\colon A \to B$ to be a map $h\colon A \to IB$ in $\mathcal{C}$ such that $p_0 h = f$ and $p_1 h = g$. Let $[A, B]$ denote the set of homotopy classes of maps $A \to B$ in $\mathcal{C}$ and let $h\mathcal{C}$ denote the resulting homotopy category; its objects are the objects of $\mathcal{C}$ and its morphisms from A to B are the elements of $[A, B]$. A map in $\mathcal{C}$ is said to be an *equivalence* if it is an isomorphism in $h\mathcal{C}$ (i.e., homotopy equivalence of C^*-algebras.)

The term *isomorphism* will mean isomorphism in $\mathcal{C}$ (i.e., isomorphism of C^*-algebras.)

DEFINITION 1.2. A map $p\colon A \to B$ is said to be a *cofibration* if it satisfies the homotopy lifting property: any homotopy $h\colon D \to IB$ of a composite fp, $f\colon D \to A$, can be extended to a homotopy $H\colon D \to IA$ of f.

$$\begin{tikzcd}
 & A \\
D \arrow[ur, "f"] \arrow[r, dashed, "H"] \arrow[dr, "h"'] & IA \arrow[u, "p_0"'] \arrow[d, "Ip"] \\
 & IB
\end{tikzcd}$$

Write $h_t(d) = h(d)(t)$. Then a map is a cofibration if whenever given a homotopy $h_t\colon D \to B$ and a lift f of h_0 to A, then the entire homotopy lifts to $H_t\colon D \to B$ with $pH_t = h_t$ and $H_0 = f$.

$$\begin{tikzcd} & A \arrow[d, "p"] \\ D \arrow[ur, "f"] \arrow[ur, dashed, bend left, "H_t"] \arrow[r, "h_t"] & B \end{tikzcd}$$

The simplest examples of a cofibration are the maps $p_r\colon IB \to B$.

LEMMA 1.3. *The map $p_r\colon IB \to B$ is a cofibration for each $r \in [0, 1]$.*

Proof. Suppose given maps $h\colon D \to IB$ and $f\colon D \to IB$ satisfying $p_r f = p_0 h$. We must find some $H\colon D \to I(IB)$ with $p_0 H = f$ and $I(p_r) \circ H = h$. Regard $I(IB)$ as $C([0, 1] \times [0, 1], B)$; then $H(d)(x, y)$ must satisfy

$$H(d)(0, y) = f(d)(y)$$

and

$$H(d)(x, r) = h(d)(x).$$

Thus $H(d)$ is a priori determined on the subspace $(\{0\} \times [0, 1]) \cup ([0, 1] \times \{r\})$. Let ν be some retraction of $[0, 1] \times [0, 1]$ onto that subspace. Define $H(d)$ by

$$H(d)(x, y) = \begin{cases} f(d)(\nu(x, y)) & \text{if } \nu(x, y) \in \{0\} \times [0, 1] \\ h(d)(\nu(x, y)) & \text{if } \nu(x, y) \in [0, 1] \times \{r\}. \end{cases}$$

Then $H(d)$ satisfies the conditions above pointwise, and hence as a C^*-map. □

In general the natural map $A^+ \to A^+/A = \mathbf{C}$ is not a cofibration, where A^+ denotes the unitalization of A. For instance, let $X = \{x,\ x\sin(1/x) \in \mathbf{R}^2 \mid 0 < x \leq 1\}$. Then $C(X)^+ = C(X^+)$ where $X^+ = X \cup \{(0,0)\}$. The map $C(X)^+ \to \mathbf{C}$ given by $g \to g(0,0)$ is not a cofibration. We suspect that the natural map $\mathfrak{L} \to \mathfrak{L}/\mathfrak{K}$ is not a cofibration.

DEFINITION 1.4. Given a diagram

$$\begin{tikzcd} & C \arrow[d, "q"] \\ A \arrow[r, "p"'] & B \end{tikzcd}$$

the *pullback* is defined by

$$P = \{(a, c) \in A \oplus C \mid pa = qc\}$$

as the evident C^*-subalgebra of $A \oplus C$. There is a natural diagram

$$\begin{tikzcd} P \arrow[r, "\bar{p}"] \arrow[d, "\bar{q}"'] & C \arrow[d, "q"] \\ A \arrow[r, "p"'] & B \end{tikzcd}$$

Note that p is injective/surjective if and only if $\bar{p}$ is injective/surjective, and that $\operatorname{Ker} p = \operatorname{Ker} \bar{p}$. The pullback has the following universal property. Given a C^*-algebra D in $\mathcal{C}$ and maps f: $D \to A$, g: $D \to C$ in $\mathcal{C}$ with $pf = qg$, then there is a unique map in $\mathcal{C}$ making the diagram

$$\begin{tikzcd} D \arrow[dr, dashed] \arrow[drr, bend left, "g"] \arrow[ddr, "f"'] & & \\ & P \arrow[r, "\bar{p}"] \arrow[d, "\bar{q}"] & C \arrow[d, "q"] \\ & A \arrow[r, "p"'] & B \end{tikzcd}$$

commute.

PROPOSITION 1.5. *If p: $A \to B$ is a cofibration and q: $C \to B$ is a map then the map $\bar{p}$: $P \to C$ in the pullback diagram*

$$\begin{tikzcd} P \arrow[r, "\bar{p}"] \arrow[d, "\bar{q}"'] & C \arrow[d, "q"] \\ A \arrow[r, "p"'] & B \end{tikzcd}$$

is a cofibration. Thus the pullback of a cofibration (by an arbitrary map) is a cofibration.

Proof. Suppose given maps h: $D \to IC$ and f: $D \to P$ such that $p_0 h = \bar{p} f$. Then qh: $D \to IB$, $\bar{q}f$: $D \to A$ and $p_0 qh = p\bar{q}f$. Since p: $A \to B$ is a cofibration, there is a map k: $D \to IA$ such that $p_0 k = \bar{q}f$ and $pk = qh$. The diagram

$$\begin{tikzcd} D \arrow[rr, bend left, "h"] \arrow[dr, dashed, "H"] \arrow[dd, "k"'] & & IC \arrow[dd, "q"] \\ & IP \arrow[ur, "\bar{p}"] & \\ IA \arrow[ur, "\bar{q}"] \arrow[rr, "p"'] & & IB \end{tikzcd}$$

commutes, since $pk = qh$. Since IP is a pullback, there is a (dotted) map $H\colon D \to IP$ making the diagram commute. Then

$$p_0 H = f \quad \text{since } p_0 k = \bar{q} f \quad \text{and} \quad p_0 h = \bar{p} f$$

and $\bar{p} H = h$ as required. □

DEFINITION 1.7. The *mapping cylinder* Mf of a map $f\colon A \to B$ is defined by the pullback diagram

$$\begin{tikzcd} Mf \arrow[r] \arrow[d, "p"] & IB \arrow[d, "p_0"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

The map $Mf \to A$ is a cofibration since p_0 is a cofibration.

Let $c\colon A \to IB$ be the map defined by setting $c(a)$ to be the constant path at $f(a)$. Then $p_0 c = f$. As Mf is a pullback, there is a map r making the diagram

$$\begin{tikzcd} A \arrow[dr, "r"] \arrow[drr, bend left, "c"] \arrow[ddr, "1_A"'] & & \\ & Mf \arrow[r] \arrow[d, "p"] & IB \arrow[d, "p_0"] \\ & A \arrow[r, "f"] & B \end{tikzcd}$$

commute. Thus $pr = 1_A$. In fact p is an equivalence: $rp \simeq 1_{Mf}$. To see this, write $Mf = \{(\xi, a) \in IB \oplus A \mid p_0 \xi = fa\}$. Then $r(a) = (c(a), a)$. Define $h\colon Mf \to IMf$ by $h(\xi, a)(j) = (\xi_j, a)$, where ξ_j is the path $\xi_j(t) = \xi(jt)$. Then

$$h(\xi, a)(0) = (p_0 \xi, a) = (c(a), a) = rp(\xi, a)$$

and

$$h(\xi, a)(1) = (\xi, a)$$

so that h is a homotopy $h\colon rp \simeq 1_{Mf}$. Thus r and p are inverse equivalences. In fact the map rp is a deformation retraction (see 1.12) of Mf onto $rA \subseteq Mf$, since the homotopy h fixes rA at each time s:

$$\begin{aligned} p_s h r(a) &= p_s h(c(a), a) = h(c(a), a)(s) \\ &= h(c(a)_s, a) \qquad \text{but } c(a)_s = c(a) \\ &= r(a). \end{aligned}$$

Proposition 1.8. *Let $f\colon A \to B$ with mapping cylinder Mf. Define $j\colon Mf \to B$ by $j(\xi, a) = \xi(1)$ (i.e., j is the composite $Mf \to IB \xrightarrow{p_1} B$). Then j is a cofibration.*

Proof. Suppose given a homotopy $h\colon C \to IB$ and a map $k\colon C \to Mf$ with $p_0 h = jk$. Write $k = (k', k'')$ with respect to $Mf \subset IB \oplus A$. Define $\tilde{h}\colon C \to IMf$ as follows. Let $\tilde{h}'\colon C \to I(IB)$ by

$$\tilde{h}'(c)(s,t) = \begin{cases} k'(c)\left(\dfrac{2s}{2-t}\right) & \text{if } 2s \le 2-t \\ h(c)(2s+t-2) & \text{if } 2s \ge 2-t \end{cases}$$

and let $\tilde{h}''\colon C \to IA$ by $\tilde{h}''(c)(t) = k''(c)$. Let

$$\tilde{h} = (\tilde{h}', \tilde{h}'')\colon C \to I(IB \oplus A).$$

Then in fact $\tilde{h}\colon C \to I(Mf)$. The verifications which remain are routine:

$$\tilde{h}(c)(s,0) = (k'(c)(s), k''(c)) = k(c)$$

and

$$j\tilde{h}(c)(s,t) = \tilde{h}'(c)(1,t) = h(c)(t). \qquad \square$$

Corollary 1.9. *Any map $f\colon A \to B$ in $\mathcal{C}$ factors canonically in $\mathcal{C}$ as*

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "r"'] & & B \\ & Mf \arrow[ur, "j"'] & \end{tikzcd}$$

where j is a cofibration and r is the natural equivalence. $\square$

Proposition 1.10. *A map $p\colon A \to B$ is a cofibration if and only if the natural map $v\colon IA \to Mf$ determined by p_0 and f splits.*

Proof. The map $v\colon IA \to Mf$ is determined by the pullback diagram

$$\begin{tikzcd} IA \arrow[dr, "v"] \arrow[drr, bend left, "p_0"] \arrow[ddr, bend right, "f"'] & & \\ & Mf \arrow[r, "\tilde{p}_0"] \arrow[d, "\tilde{f}"] & A \arrow[d, "f"] \\ & IB \arrow[r, "p_0"] & B \end{tikzcd}$$

Suppose that f is a cofibration. Then $\tilde{p}_0$: $Mf \to A$, $\tilde{f}$: $Mf \to IB$, and $p_0\tilde{f} = \tilde{f}p_0$. By the cofibration property, there is some w: $Mf \to IA$ with $fw = \tilde{f}$ and $p_0 w = \tilde{p}_0$. But this says that $vw = 1_{Mf}$.

Conversely, suppose that v is split by some map w: $Mf \to IA$. Suppose given h: $C \to IB$, g: $C \to A$, and $fg = p_0 h$. By the pullback property the maps h and g determine a map $\hat{h}$: $C \to Mf$ with $\tilde{f}\hat{h} = h$, $\tilde{p}_0\hat{h} = g$. Define $k = w\hat{h}$: $C \to IA$. Then

$$fk = fw\hat{h} = \tilde{f}\hat{h} = h \quad \text{and} \quad p_0 k = p_0 w\hat{h} = \tilde{p}_0\hat{h} = g$$

as required. Thus f is a cofibration. □

Note that if f: $A \to B$ is a cofibration then the proposition implies that v is surjective.

PROPOSITION 1.11. *For fixed nuclear $F \in \mathcal{C}$, the functor $A \to A \otimes_{\min} F$ preserves pullbacks and cofibrations (provided that $\mathcal{C}$ is closed under $\otimes_{\min} F$).*

Proof. The first preservation property is formal, since $(\ ) \otimes_{\min} F$ preserves direct sums and kernels. In particular, if f: $A \to B$ and $f \otimes 1_A$: $A \otimes_{\min} F \to B \otimes_{\min} F$ then $M(f \otimes 1_F) \cong Mf \otimes_{\min} F$. Thus the natural map

$$v = v_{A \otimes F}: I(A \otimes_{\min} F) \to M(f \otimes 1_F)$$

is simply the map

$$v_A \otimes 1_F: (IA) \otimes_{\min} F \to Mf \otimes_{\min} F.$$

If f: $A \to B$ is a cofibration with splitting map w_A: $Mf \to IA$ then $v_{A \otimes F}$ is split via $w_A \otimes 1_F$, so $f \otimes 1_A$ is a cofibration. □

DEFINITION 1.12. A surjection f: $A \to B$ is a *deformation retraction* if there is some F: $A \to IA$ such that

(1) $p_0 F = 1$
(2) $p_t(If)F = f$
(3) $p_1 F(\operatorname{Ker} f) = 0$.

If f: $A \to B$ is a deformation retraction then $p_1 F$ factors as

$$\begin{tikzcd} A \arrow[rr, "p_1 F"] \arrow[dr, "f"'] & & A \\ & B \arrow[ur, dashed, "r"'] & \end{tikzcd}$$

with

$$rf = p_1 F \simeq 1_A.$$

It is easy to see that

$$fr = 1_B$$

and thus f is an equivalence with homotopy inverse r. We cite the following facts and refer the reader to G. W. Whitehead [**25**, I. 5] for detailed proofs.

PROPOSITION 1.13.

(a) *Let f: $A \to B$ be a cofibration. Then f is an equivalence if and only if f is a deformation retraction.*

(b) *The pullback of a deformation retraction is a deformation retraction.*

(c) *Suppose given a pullback diagram*

$$\begin{tikzcd} P \arrow[r, "g_1"] \arrow[d, "g_2"'] & A_1 \arrow[d, "f_1"] \\ A_2 \arrow[r, "f_2"] & B \end{tikzcd}$$

such that f_1 is a cofibration and f_2 is an equivalence. Then g_1 is an equivalence.

Proof. Part (a) follows as in Whitehead [**25**, I., 5.9], and part (b) follows as in [**25**, I., 5.4.]. For part (c) suppose first that f_2 and hence g_1 are cofibrations. Note that f_2 is a deformation retraction, by (a), hence g_1 is a deformation retraction, by (b), and hence g_1 is an equivalence, by (a) again. This proves (c) in the special case where f_2 is a cofibration. In general, factor f_2 as

$$A_2 \xrightarrow{r} M \xrightarrow{j} B$$

where j is a cofibration and r is an equivalence. Consider the expanded commutative diagram with $f_2 = jr$:

$$\begin{tikzcd} P \arrow[r, "\tilde{r}"] \arrow[d, "g_2"'] & \tilde{M} \arrow[r, "\tilde{j}"] \arrow[d] & A_1 \arrow[d, "f_1"] \\ A_2 \arrow[r, "r"] & M \arrow[r, "j"] & B \end{tikzcd}$$

Since f_2 is an equivalence (by assumption) and r is an equivalence, the map j is an equivalence and a cofibration. Apply the special case already proved to conclude that $\tilde{j}$ is an equivalence. Finally, $g_1 = \tilde{j}\tilde{r}$ is the composite of equivalences and hence an equivalence. □

We close this section with two more basic definitions.

DEFINITION 1.14. Define the *cone* CA and the *suspension* SA by

$$CA = \{\xi \in IB | \xi(1) = 0\}$$
$$SA = \{\xi \in IB | \xi(0) = \xi(1) = 0\}.$$

There are evident natural exact sequences

$$0 \to CA \to IA \xrightarrow{p_1} A \to 0$$

and

$$0 \to SA \to CA \xrightarrow{p_0} A \to 0.$$

2. Cofibre sequences. In this section the analogue of the Barratt-Puppe cofibration sequences are developed and Verdier's axiom is established.

DEFINITION 2.1. Define the *cofibre* or *mapping cone* Cf of the map $f: A \to B$ via the pullback diagram

$$\begin{tikzcd} Cf \arrow[r, "\pi(f)"] \arrow[d] & A \arrow[d, "f"] \\ CB \arrow[r, "p_0"] & B. \end{tikzcd}$$

For example, CB is the cofibre of the identity map $B \to B$. The map p_0: $CB \to B$ is a cofibration and thus the natural map $\pi(f)$: $Cf \to A$ is a cofibration. Note that if $f \equiv 0$ then $Cf \cong A \oplus SB$. In general there are natural sequences

$$0 \to Cf \to Mf \xrightarrow{p_1} B \to 0$$

and

$$0 \to SB \to CF \xrightarrow{\pi(A)} A \to 0.$$

PROPOSITION 2.2. *If f: $A \to B$ is the inclusion of an ideal, then there is a natural diagram*

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & CA \arrow[d] \arrow[dr, "p_0"] & & \\
0 \arrow[r] & SB \arrow[r] & Cf \arrow[r, "p"] \arrow[d] & A \arrow[r] & 0 \\
 & & S(B/A) \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

Proof. Regard Cf as

$$Cf = \{(\xi, a) \in CB \oplus A | \xi(0) = fa\}.$$

Define q: $Cf \to S(B/A)$ by $q(\xi, a) = \pi\xi$, where π: $B \to B/A$ is the projection. The map q is well-defined since $\pi\xi(1) = \pi f(a) = 0$. The map q is surjective since $S\pi$: $SB \to S(B/A)$ is surjective, and

$$\begin{aligned}
\operatorname{Ker} q &= \{(\xi, a) \in Cf \mid \pi\xi = 0\} \\
&\cong \{\xi \in CB \mid \pi\xi = 0\} \quad \text{since } f \text{ is injective.} \\
&= \{\xi \in CB \mid \xi(t) \in A \text{ for all } t\} = CA. \qquad \square
\end{aligned}$$

PROPOSITION 2.3. *If f: $A \to B$ is surjective with $J = \operatorname{Ker} f$ then there is a natural exact diagram*

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & J \arrow[d] \arrow[dr] & & \\
0 \arrow[r] & SB \arrow[r] \arrow[dr] & Cf \arrow[r] \arrow[d] & A \arrow[r] & 0 \\
 & & CB \arrow[d, dash] & & \\
 & & 0 & &
\end{tikzcd}$$

Proof. Let q: $Cf \to CB$ be defined by $q(\xi, a) = \xi$. The map q is surjective, for a path $\xi \in CB$ is hit by (ξ, a) where a is any element of

$f^{-1}(\xi(0))$. Further,

$$\begin{aligned}\operatorname{Ker} q &= \{(\xi, a) \in Cf \mid \xi = 0\} \\ &= \{(0, f) \in CB \oplus A \mid fa = 0\} \cong J. \end{aligned} \qquad \square$$

PROPOSITION 2.4. *Let $f: A \to B$ be a cofibration with kernel J. Then the inclusion $J \to A$ is the composite of the inclusion $\iota: J \to Cf$ and the natural cofibration $Cf \to A$. Moreover, ι is an equivalence.*

Proof. Let $w: Mf \to IA$ be a splitting of the natural map $v: IA \to Mf$, which exists by Proposition 1.10. Then $p_1 w: Mf \to A$. Let κ denote the restriction of $p_1 w$ to $Cf \subset Mf$. Then $\kappa: Cf \to J$, since $w(\xi, a)(1) \in J$. This map is the required homotopy inverse to ι. The retraction w preceeded by ι induces a homotopy $1 \simeq \kappa\iota$, while w together with the map $h: CA \to I(CA)$ specified by $h(\xi)(s, t) = \xi(\max(s, t))$ induces a homotopy $1 \simeq \iota\kappa$. $\square$

PROPOSITION 2.5. *let $f: A \to B$ with associated short exact sequences*

$$0 \to Cf \to Mf \xrightarrow{p_1} A \to 0$$

$$0 \to SB \xrightarrow{i(f)} Cf \xrightarrow{\pi(f)} A \to 0.$$

If $\psi: C\pi(f) \to SB$ is the equivalence of Proposition 2.4 then the right triangle commutes and the left triangle homotopy commutes in the diagram

$$\begin{tikzcd} SA \arrow[r, "-Sf"] \arrow[dr, "i(\pi f)"'] & SB \arrow[r, "i(f)"] \arrow[d, "\psi"] & Cf \\ & C\pi(f) \arrow[ur, "\pi(\pi f)"'] & \end{tikzcd}$$

where $-g: SA \to SB$ is defined by $(-g)(s) = g(1 - s)$.

Proof. Consider the right triangle, and write $C(\pi f) \subset CA \oplus CB \oplus A$. Then

$$\pi(\pi f)(\psi\xi) = \pi(\pi f)(0, \xi, 0) = (\xi, 0) = i(f)(\xi)$$

so that $\pi(\pi f)\psi = i(f)$ as required. For the left triangle we compute:

$$\psi(-Sf)(\eta) = \psi(-(f\eta)) = (0, f\eta(1 - s), 0)$$

whereas

$$i(\pi f)(\eta) = (\eta(s), 0, 0).$$

Define h: $SA \to I(C(\pi f))$ by $h_t(\eta) = (\eta(s + t - st), f\eta(t - st), 0)$. This is the required homotopy h: $\psi(-Sf) \simeq i(\pi f)$. □

Recall that $[A, SB]$ is a group and $[A, S^2B]$ is an abelian group for any C^*-algebras A, B.

PROPOSITION 2.6. *Let f: $A \to B$ and let C be a fixed C^*-algebra. Then the sequence of maps*

$$\cdots \xrightarrow{S^2 f} S^2B \xrightarrow{-Si(f)} SCf \xrightarrow{-S\pi(f)} SA \xrightarrow{-Sf} SB \xrightarrow{i(f)} Cf \xrightarrow{\pi(f)} A \xrightarrow{f} B$$

gives rise to a long exact sequence of pointed sets, or of groups from the fourth term on, upon application of the functor $[C, -]$.

Proof. By Proposition 2.5 it suffices to prove exactness at

$$[C, Cf] \to [C, A] \to [C, B].$$

Suppose that g: $C \to A$ and that fg: $C \to B$ is null-homotopic. Then fg lifts to h: $C \to CB$ and the diagram

$$\begin{tikzcd} C \arrow[r, "h"] \arrow[d, "g"'] & CB \arrow[d, "p_0"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

commutes. Thus there is a map k: $C \to Cf$ making the diagram

$$\begin{tikzcd} C \arrow[dr, "k"] \arrow[drr, "h"] \arrow[ddr, "f"'] & & \\ & Cf \arrow[r] \arrow[d, "\pi(f)"] & CB \arrow[d, "p_0"] \\ & A \arrow[r, "f"] & B \end{tikzcd}$$

commute. In particular, $\pi(f)k = f$, so that f is in the image of $\pi(f)_*$. □

REMARK 2.7. The sequence resulting in Proposition 2.6 has the following additional properties; we refer the reader to May [**14**] for proofs in the commutative case.

(1) The group $[C, SB]$ acts from the left on the set $[C, Cf]$.
(2) $i(f)_*: [C, SB] \to [C, Cf]$ is a map of left $[C, SB]$-sets.
(3) $i(f)_*(x) = i(f)_*(x')$ if and only if $x = (Sf)_*(y)x'$ for some $y \in [C, SA]$.
(4) $\pi(f)_*(z) = \pi(f)_*(z')$ if and only if $z = xz'$ for some $x \in [C, SB]$.
(5) The image of $[C, S^2B]$ in $[C, SCf]$ is a central subgroup.

LEMMA 2.8. *Suppose given maps $f: A \to IB$, $g: B \to IC$ and for each $t \in [0, 1]$, pullback diagrams*

$$\begin{tikzcd} P_t \arrow[r, "\bar{g}_t"] \arrow[d, "\bar{f}_t"'] & A \arrow[d, "f_t"] \\ C \arrow[r, "g_t"] & B \end{tikzcd}$$

where $f_t = p_t f$, and $g_t = p_t g$. Then there is a natural equivalence $\varphi: P_0 \to P_1$ such that the diagram

$$\begin{tikzcd} P_1 \arrow[r, "\bar{g}_1"] \arrow[d, "\bar{f}_1"'] & A \\ C & P_0 \arrow[l, "\bar{f}_0"] \arrow[u, "\bar{g}_0"'] \arrow[ul, "\varphi"'] \end{tikzcd}$$

homotopy commutes (and in fact φ may be chosen so that the diagram commutes.)

Proof. Let P be the pullback of the diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & A \arrow[d, "f"] \\ C \arrow[r, "g"] & IB \end{tikzcd}$$

The evaluation map $p_t: IB \to B$ induces natural maps $m_t: P \to P_t$ for each t. As p_0 and p_1 are deformation retractions, the resulting maps m_0 and m_1 are equivalences with canonical choices for homotopy inverses. Set $\varphi = m_1 m_0^{-1}: P_0 \to P_1$. $\square$

The following proposition explores the naturality of the cofibre construction.

PROPOSITION 2.9. *If $\beta f \simeq f'\alpha$ in the following diagram, then there exists a map γ which makes the middle square commute and the left square homotopy commute.*

$$\begin{tikzcd}
SB \arrow[r, "i(f)"] \arrow[d, "S\beta"'] & Cf \arrow[r, "\pi(f)"] \arrow[d, "\gamma"] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[d, "\beta"] \\
SB' \arrow[r, "i(f')"] & Cf' \arrow[r, "\pi(f')"] & A' \arrow[r, "f'"] & B'
\end{tikzcd}$$

Moreover, the following statements hold.

(a) *If α and β are equivalences, then γ is an equivalence.*

(b) *The cofibres of homotopic maps are equivalent.*

(c) *If $f \simeq *$ then $SB \to Cf \to A$ is equivalent to $SB \to SB \oplus A \to A$.*

(d) *If f is a cofibration and B is contractible then the inclusion* $\operatorname{Ker} f \hookrightarrow A$ *is an equivalence.*

(e) *If $\beta f = f'\alpha$ and the natural maps $C\beta \to CB'$ and α are cofibrations then the canonical choice of γ, namely the map determined by $A \to A'$ and $CB \to CB'$, is also a cofibration.*

Proof. Let h: $\beta f \simeq f'\alpha$ and define γ: $Cf \to Cf'$ by

$$\gamma(\xi, a) = \begin{cases} (h_{2s}(a), \alpha a) & s \leq \frac{1}{2} \\ (\beta\xi(2s-1), \alpha a) & s \geq \frac{1}{2}. \end{cases}$$

Clearly $\alpha\pi(f) = \pi(f')\gamma$ and $\gamma i(f) \simeq i(f')S\beta$. Observe that γ is obtained by restriction from a map γ: $Mf \to Mf'$ specified by the same formulas, and that $j\gamma = \alpha j$: $Mf \to A'$. By 1.7, 2.4 and a diagram chase, it follows that (a) will hold if it does so when $\beta f = f'\alpha$ and γ is the canonical choice. Therefore (a) is a special case of the homotopy invariance of pullbacks, proved in Lemma 2.8. Statement (b) is a special case of (a), and (c) follows via (2.1) from (b). In (d), $\pi(f)$: $C(f) \to A$ is an equivalence, by (c), and thus $\ker f \to A$ is an equivalence, by Proposition 2.4. In (e), γ is the composite

$$Cf \to C(f'\alpha) \to Cf'$$

and is thus a cofibration. □

The following proposition is known as Verdier's axiom (c.f. Adams [**1**]).

PROPOSITION 2.10. *If $ge \simeq f$ in the following diagram, then there exist maps α and β which make the diagram homotopy commutative.*

$$\begin{tikzcd}
SCg \arrow[rr, bend left, "i(e)S\pi(g)"] \arrow[dr, "S\pi(g)"'] & & Ce \arrow[rr, bend left, "\pi(e)"] \arrow[dr, dashed, "\beta"] & & A \arrow[rr, bend left, "f"] \arrow[dr, "e"] & & B \\
& SC \arrow[ur, "i(e)"] \arrow[dr, "Sg"'] & & Cf \arrow[ur, "\pi(f)"] \arrow[dr, dashed, "\alpha"] & & C \arrow[ur, "g"'] & \\
& & SB \arrow[ur, "i(f)"] \arrow[rr, bend right, "i(g)"'] & & Cg \arrow[ur, "\pi(g)"'] & &
\end{tikzcd}$$

Moreover, there is a canonical equivalence ψ: $C\alpha \to Ce$ such that $\beta\psi \simeq \pi(a)$ and $\psi i(\alpha) \simeq i(e)S\pi(g)$. Thus, up to equivalence, the diagram is a braid of cofibre sequences.

Proof. Let h: $ge \simeq f$. Define α and β by

$$\beta(\xi, a) = \begin{cases} (h_{2s}(a), a) & s \leq \frac{1}{2} \\ (g\xi(2s-1), a) & s \geq \frac{1}{2} \end{cases}$$

$$\alpha(\eta, a) = \begin{cases} (h_{1-2s}(a), ea) & s \leq \frac{1}{2} \\ (\eta(2s-1), ea) & s \geq \frac{1}{2}. \end{cases}$$

(Actually α and β are both special cases of the maps γ of Proposition 2.9.) It is immediate that $\pi(f)\beta = \pi(e)$, $\beta i(e) \simeq i(f)Sg$, $\alpha i(f) \simeq i(g)$, and $\alpha\pi(g) = e\pi(f)$. Thus the diagram homotopy commutes.

The natural map $CC_g \oplus Cf \to CC \oplus A$ induces ψ: $C\alpha \to Ce$ and it is easy to verify that $\beta\psi \simeq \pi(a)$ and that $\psi i(\alpha) \simeq i(e)S\pi(g)$. Observe that α and β are obtained by restrictions to subalgebras from the maps β: $Ce \to Cf$ and α: $Cf \to Cg$ defined by the same formulas. By 1.7, 2.4 and a diagram chase, ψ will be an equivalence as desired if it is so when $ge = f$, $\beta(\xi, a) = (g\xi, a)$, and $\alpha(\eta, a) = (\eta, ea)$. Here it is easy to see that there is a deformation. Define h_t: $C\alpha \to C\alpha$ by $h_t(\theta, \xi, a) = (\theta_t, \xi, a)$, where $\theta_t(s) = \theta(s - ts)$. Then

$$h_1(\theta, \xi, a) = (c(\xi, ea), \xi, a)$$

and

$$h_0(\theta, \xi, a) = \text{identity}.$$

Let $D = \mathrm{Im}(h_1)$. Then h_t fixes D for each t, so that h_t is a deformation retract of $C\alpha$ to D (fixing D). But the natural map $\psi\colon C\alpha \to Ce$ given by $\psi(\theta, \xi, a) = (g\xi, a)$ induces a C^*-isomorphism $\psi_1\colon D \to Ce$. Let $\psi^{-1}\colon Ce \to C\alpha$ be the composition of ψ_1^{-1} with the inclusion $D \to C\alpha$. Then ψ^{-1} is a homotopy inverse to ψ. $\square$

COROLLARY 2.11. *Suppose given a C^*-algebra A and ideals $J \subset H \subset A$. Then there is a cofibre weave*

$$\begin{tikzcd}
SCg \arrow[rr, bend left] \arrow[dr] & & Ce \arrow[rr, bend left] \arrow[dr] & & A \arrow[rr, bend left, "f"] \arrow[dr, "e"] & & A/H \\
& S(A/J) \arrow[ur] \arrow[dr] & & Cf \arrow[ur] \arrow[dr] & & A/J \arrow[ur, "g"'] & \\
& & S(A/H) \arrow[ur] \arrow[rr, bend right] & & Cg \arrow[ur] & &
\end{tikzcd}$$

If the projection maps e, g, and (hence) f are cofibrations then the weave becomes up to equivalence a cofibre weave of the form

$$\begin{tikzcd}
S(H/J) \arrow[rr, bend left] \arrow[dr] & & J \arrow[rr, bend left] \arrow[dr] & & A \arrow[rr, bend left, "f"] \arrow[dr, "e"] & & A/H \\
& S(A/J) \arrow[ur] \arrow[dr] & & H \arrow[ur] \arrow[dr] & & A/J \arrow[ur, "g"'] & \\
& & S(A/H) \arrow[ur] \arrow[rr, bend right] & & H/J \arrow[ur] & &
\end{tikzcd}$$

REMARK 2.12. What is an admissible category? There are essentially two requirements. First, if $A \in \mathcal{C}$ then $IA \in \mathcal{C}$. Second, $\mathcal{C}$ should be closed under taking some pullbacks. A minimalist view would suggest enumerating the necessary pullbacks. For instance, CB and SB are the kernels of cofibrations and hence a very special sort of pullback. A maximalist would require that $\mathcal{C}$ be closed under all pullbacks. This is sometimes easy to check. For instance, separable C^*-algebras are closed under arbitrary pullbacks. A middle route would be to ask for closure for pullbacks when at least one of the maps is a cofibration. In that case we have

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d] & P \arrow[r, "g_1"] \arrow[d] & A_1 \arrow[r] \arrow[d, "f_1"] & 0 \\
0 \arrow[r] & K \arrow[r] & A_2 \arrow[r, "f_2"] & B \arrow[r] & 0
\end{tikzcd}$$

If f_2 is a cofibration then so is g_1. So this closure requirement is equivalent to closure under kernels of cofibrations and closure under extensions by cofibre maps. We prefer to leave the matter unsettled at this time.

3. Homology: first properties. In this section we derive the elementary properties of a homology theory from the axioms. Throughout this section h_* is understood to be a homology theory on an admissible category of C^*-algebras. Thus the homotopy and exactness axioms are satisfied. No assumption is made concerning additivity until §4.

PROPOSITION 3.1. *Let*

$$A_1 \underset{p_1}{\overset{i_1}{\rightleftarrows}} A_1 \oplus A_2 \underset{p_2}{\overset{i_2}{\leftrightarrows}} A_2$$

be the canonical maps. Then there is a natural isomorphism

$$(i_1, i_2)_*: h_n(A_1) \oplus h_n(A_2) \xrightarrow{\cong} h_n(A_1 \oplus A_2).$$

Proof. The split short exact sequence

$$0 \to A_1 \xrightarrow{i_1} A_1 \oplus A_2 \xrightarrow{p_2} A_2 \to 0$$

induces a split short exact sequence

$$0 \to h_n(A_1) \to h_n(A_1 \oplus A_2) \to h_n(A_2) \to 0$$

and a similar sequence if A_1 and A_2 are reversed. This implies the result. □

Let m: $SB \oplus SB \to SB$ by

$$m(\xi_1, \xi_2)(t) = \begin{cases} \xi_1(2t) & t \leq \frac{1}{2} \\ \xi_2(2t-1) & t \geq \frac{1}{2}. \end{cases}$$

This is the map which induces the group structure on $[A, SB]$. The following proposition shows that the group structure on $h_n(SB)$ is determined by (3.1) and the map m_*.

PROPOSITION 3.2. *The diagram*

$$\begin{tikzcd} h_n(SB) \oplus h_n(SB) \arrow[r, "(i_1, i_2)_*"] \arrow[dr, "+"'] & h_n(SB \oplus SB) \arrow[d, "m_*"] \\ & h_n(SB) \end{tikzcd}$$

commutes.

Proof. Let $x_1, x_2 \in h_n(SB)$. Then

$$m_*\left(i_{1_*}, i_{2_*}\right)(x_1, x_2) = m_*\left(i_{1_*}x_1, +i_{2_*}x_2\right) = (mi_1)_*x_1 + (mi_2)_*x_2$$

since m_* is a homomorphism. The maps mi_1 and mi_2 are homotopic to the identity, so $(mi_1)_* = (mi_2)_* = 1$. Thus

$$m_*(i_1, i_2)_*(x_1, x_2) = x_1 + x_2$$

as required. □

PROPOSITION 3.3. *The natural map*

$$[A, SB] \to \hom(h_n(A), h_n(SB))$$

is a homomorphism of groups.

Proof. Let f_1, f_2: $A \to SB$. Then $f_1 + f_2$ is the homotopy class of the composite

$$A \overset{(f_1, f_2)}{\to} SB \oplus SB \overset{m}{\to} SB.$$

Thus

$$\begin{aligned}(f_1 + f_2)_*(x) &= [m(f_1, f_2)]_*(x) = m_*(f_1, f_2)_*(x)\\ &= m_*\left(i_{1_*}f_1(x) + i_{2_*}f_2(x)\right) = f_{1_*}(x) + f_{2_*}(x) \quad \text{by (3.2).}\end{aligned}$$

□

COROLLARY 3.4. (a) *If f: $A \to B$ is the constant map $f(a) \equiv 0$ then* $f_* = 0$.

(b) $h_*(CA) = 0$.

Proof. (a) Write f as $A \overset{f'}{\to} SB \to B$, the composite of constant maps. We have $f'_* = 0$ by Proposition 3.3 and thus $f_* = 0$.

(b) This is immediate from (a) and the homotopy axiom. □

THEOREM 3.5. (a) *The natural suspension map σ_A: $h_n(A) \to h_{n-1}(SA)$ from the sequence*

$$0 \to SA \to CA \to A \to 0$$

is an isomorphism.

(b) *If f: $A \to B$ is a map with cofibre sequence*

$$0 \to SB \to Cf \to A \to 0$$

then the diagram

$$\begin{tikzcd} h_n(A) \arrow[rr, "\partial"] \arrow[dr, "f_*"'] & & h_{n-1}(SB) \\ & h_n(B) \arrow[ur, "\cong", "\sigma_B"'] & \end{tikzcd}$$

commutes. Thus f_* corresponds to ∂ up to isomorphism.

(c) *If $f\colon A \to B$ is a map then $f_*\colon h_*(A) \to h_*(B)$ is an isomorphism if and only if $h_*(Cf) = 0$.*

Proof. Part (a) is immediate from the exactness axiom and the fact that $h_*(CA) = 0$. Part (b) follows from exactness, and (c) is immediate from (b) and exactness. □

PROPOSITION 3.6. (a) *Let $f\colon A \to B$ be a surjection with $J = \operatorname{Ker} f$. Then the natural map $J \to Cf$ induces an h_*-isomorphism.*

(b) *Let $f\colon A \to B$ be the inclusion of an ideal. Then the natural map $Cf \to S(B/A)$ induces an h_*-isomorphism.*

Proof. There are short exact sequences

$$0 \to J \to Cf \to CB \to 0$$

and

$$0 \to CA \to Cf \to S(B/A) \to 0$$

respectively. As $h_*(CB) = h_*(CA) = 0$, exactness implies the result. □

Note that if f is a cofibration then the natural map $J \to Cf$ is an equivalence, by Proposition 2.4. In general the maps $J \to Cf$ and $Cf \to S(B/A)$ are not equivalences; counterexamples exist even for commutative C^*-algebras. (See Remark 8.6.)

As indicated in the introduction, there is another possible choice of axioms. We have assumed homotopy and exactness. Instead, one could assume that h_* satisfies homotopy and the following two axioms:

Suspension. The natural map $\sigma_A\colon h_n(A) \to h_{n-1}(SA)$ is an isomorphism.

Cofibre. If $f\colon A \to B$ is a cofibration then for each n the sequence

$$h_n(Cf) \to h_n(A) \to h_n(B)$$

is exact.

These axioms are weaker than exactness. We have shown that suspension is implied by exactness and homotopy (3.5). The following proposition shows that the cofibre axiom is also implied by the exactness and homotopy axioms. In fact a stronger result obtains; f is not required to be a cofibration.

PROPOSITION 3.7. *Let h_* be a homology theory. Then for each n the sequence*

$$h_n(Cf) \to h_n(A) \xrightarrow{f_*} h_n(B)$$

is exact for any surjection $f\colon A \to B$.

Proof. Let $J = \operatorname{Ker} f$. Then the sequence

$$h_n(J) \to h_n(A) \xrightarrow{f_*} h_n(B)$$

is exact. The natural map $h_n(J) \to h_n(Cf)$ is an isomorphism by (3.6a). □

We consider next a cofibre theory h_*.

PROPOSITION 3.8. *Let h_* be a cofibre theory and let*

$$\cdots \to A_k \to A_{k-1} \to A_{k-2} \to \cdots$$

be a cofibre sequence. Then for each n there is a long exact sequence

$$\cdots \to h_n(A_k) \to h_n(A_{k-1}) \to h_n(A_{k-2}) \to \cdots .$$

In particular, for any cofibration $f\colon A \to B$ with $J = \operatorname{Ker} f$ there is a long exact sequence

$$\to h_n(J) \to h_n(A) \to h_n(B) \to h_{n-1}(J) \to \cdots .$$

Proof. The first part of the proposition is immediate from the cofibre axiom. Let $f\colon A \to B$ be a cofibration. Then there is a natural cofibre sequence

$$\to SA \to SB \to Cf \to A \to B.$$

Apply the functor h_n and the suspension axiom to obtain the exact sequence

$$h_{n+1}(A) \to h_{n+1}(B) \to h_n(Cf) \to h_n(A) \to h_n(B).$$

Since f is a cofibration, the map $J \to Cf$ is an equivalence. Replace $h_*(Cf)$ by $h_*(J)$ and the proof is complete. □

COROLLARY 3.9. *Let h_* be a cofibre theory. Then*:

(a) $h_*(CA) = 0$.

(b) *If $f: A \to B$ then $f_*: h_*(A) \to h_*(B)$ is an isomorphism if and only if $h_*(Cf) = 0$.*

(c) *If $f: A \to B$ is a cofibration with $J = \operatorname{Ker} f$ then f_* is an isomorphism if and only if $h_*(J) = 0$.* □

PROPOSITION 3.10. *If h_* is a cofibre theory then*

(a) *the natural map*

$$h_*(A_1) \oplus h_*(A_2) \to (A_1 \oplus A_2)$$

is an isomorphism.

(b) *The diagram*

$$\begin{tikzcd} h_*(SA) \oplus h_*(SA) \arrow[r] \arrow[dr, "+"'] & h_*(SA \oplus SA) \arrow[d, "m_*"] \\ & h_*(SA) \end{tikzcd}$$

commutes.

(c) *The map*

$$[A, SB] \to \hom(h_n(A), h_n(SB))$$

is a homomorphism. □

Finally we consider for homology theories the analogue of the "homology of a triple" sequence. That the sequence (3.12) below is exact is almost immediate. The real point of the theorem is the identification of the boundary homomorphism.

THEOREM 3.11. *Homology of a triple. Let h_* be a homology theory. Suppose given (closed) ideals $J \subset H \subset A$ with maps*

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "e"'] & & A/H \\ & A/J \arrow[ur, "g"'] & \end{tikzcd}$$

and short exact sequence

$$0 \to H/J \xrightarrow{i} A/J \xrightarrow{j} A/H \to 0.$$

Then the boundary homomorphism b in the resulting exact sequence

$$(3.12) \qquad \cdots \to h_n(H/J) \to h_n(A/J) \to h_n(A/H) \xrightarrow{b} h_{n-1}(H/J) \to \cdots$$

arises as the composite

$$h_n(A/H) \xrightarrow{\partial} h_{n-1}(H) \xrightarrow{k_*} h_{n-1}(H/J)$$

where ∂ *is the natural boundary map and* $k\colon H \to H/J$ *is the projection. If* h_* *is but a cofibre theory and* e, g, *and (hence)* f *are cofibrations then there is still an exact sequence* (3.12) *and the boundary homomorphism* b *still satisfies* $b = k_*\partial$.

Proof. Refer to Corollary 2.11 where we see that the diagram

$$\begin{tikzcd} S(A/H) \arrow[rr] \arrow[dr] & & H/J \\ & H \arrow[ur, "k"'] & \end{tikzcd}$$

commutes. □

4. The Mayer-Vietoris Theorem. This section is devoted to the proof of the following theorem.

THEOREM 4.1. (*Mayer-Vietoris.*) *Let* h_* *be a cofibre homology theory on* $\mathcal{C}$. *Suppose given a pullback diagram*

$$\begin{tikzcd} P \arrow[r, "g_1"] \arrow[d, "g_2"'] & A_1 \arrow[d, "f_1"] \\ A_2 \arrow[r, "f_2"] & B \end{tikzcd}$$

in $\mathcal{C}$. *Suppose that* f_1 *and* f_2 *are surjective or that* f_1 *is a cofibration and* f_2 *is arbitrary. Then there is a long exact sequence*

$$\to h_n(P) \xrightarrow{(g_{1*},\, g_{2*})} h_n(A_1) \oplus h_n(A_2) \xrightarrow{(-f_{1*}\, +f_{2*})} h_n(B) \to h_{n-1}(P) \to \cdots$$

which is natural with respect to maps of pullback diagrams in $\mathcal{C}$.

Proof. Assume initially that f_1 and f_2 are surjective. We perform some homotopy constructions in order to reduce to a situation covered by Theorem 3.11. Note that for any map $f\colon A \to SB$, the map $\hat{f}\colon A \to SB$ defined by $\hat{f}(a)(t) = f(a)(1-t)$ satisfies $(\hat{f})_* = -(f_*)$ by (3.3). Thus it is reasonable to write "$-f$" for $\hat{f}$, and then $(-f)_* = -(f_*)$.

Let $g\colon P \to A_1 \oplus A_2$ be given by $g(x) = (g_1(x), g_2(x))$. Then

$$Cg = \{(\xi_1, \xi_2, x) \in CA_1 \oplus CA_2 \oplus P \mid \xi_1(0) = g_1(x),\ \xi_2(0) = g_2(x)\}.$$

There is a natural diagram

(4.3)
$$\begin{tikzcd}
Cg \arrow[r] \arrow[d] & Cg_1 \arrow[r] \arrow[d, "\pi(g_1)"] & CA_1 \arrow[d, "p_0"] \\
Cg_2 \arrow[r, "\pi(g_2)"] \arrow[d] & P \arrow[r, "g_1"] \arrow[d, "g_2"'] & A_1 \arrow[d, "f_1"] \\
CA_2 \arrow[r, "p_0"] & A_2 \arrow[r, "f_2"] & B
\end{tikzcd}$$

where each small square is a pullback. Hence the outer square of (4.3), namely

$$\begin{tikzcd}
Cg \arrow[r, "h_1"] \arrow[d, "h_2"'] & CA_1 \arrow[d, "f_1 p_0"] \\
CA_2 \arrow[r, "f_2 p_0"'] & B
\end{tikzcd}$$

is a pullback, where $h_i(\xi_1, \xi_2, x) = \xi_i$. The maps f_i: $A_i \to B$ induce maps $\tilde{f}_i$: $CA_i \to CB$, and

$$p_0\tilde{f}_1(\xi_1) = p_0(f_1\xi_1) = f_1\xi_1(0) = f_1 g_1(x) = p_0\tilde{f}_2(\xi_2) \quad \text{by symmetry.}$$

Thus the diagram

(4.4)
$$\begin{tikzcd}
Cg \arrow[rrr, "h_1"] \arrow[dr, dashed] \arrow[ddd, "h_2"'] & & & CA_1 \arrow[dl, "\tilde{f}_1"] \arrow[ddl, bend left, "f_1 p_0"] \\
& Q \arrow[r] \arrow[d] & CB \arrow[d, "p_0"] & \\
& CB \arrow[r, "p_0"] & B & \\
CA_2 \arrow[ur, "\tilde{f}_2"] \arrow[urr, bend right, "f_2 p_0"'] & & &
\end{tikzcd}$$

commutes, where Q is the pullback, and there is a natural map $Cg \to Q$, as indicated, making the diagram commute.

The C^*-algebra Q is isomorphic to SB but we must be quite careful in the identification. A priori,

$$Q = \{(\eta_1, \eta_2) \in CB \oplus CB | \eta_1(0) = \eta_2(0)\}.$$

Define φ: $Q \to SB$ by

$$\varphi(\eta_1, \eta_2)(t) = \begin{cases} \eta_1(1 - 2t) & t \le \frac{1}{2} \\ \eta_2(2t - 1) & t \ge \frac{1}{2}. \end{cases}$$

The map φ is well-defined since $\eta_1(0) = \eta_2(0)$, and φ is evidently an isomorphism. Replacing Q by SB in the diagram (4.4) yields a diagram

$$\begin{tikzcd} Cg \arrow[r, "\psi"] & SB \arrow[r, "r_1"] \arrow[d, "r_2"] & CB \arrow[d] \\ & CB \arrow[r] & B \end{tikzcd}$$

where $r_1(\xi)(t) = \xi(1 - \frac{1}{2}t)$, $r_2(\xi) = \xi((t+1)/2)$, and

$$\psi(\xi_1, \xi_2, x)(t) = \begin{cases} f_1\xi_1(1-2t) & t \le \frac{1}{2} \\ f_2\xi_2(2t-1) & t \ge \frac{1}{2}. \end{cases}$$

It is easy to see that ψ is surjective. (This is where the assumption that the f_i are surjective is used.)

Write $J_i = \operatorname{Ker} f_i \colon A_i \to B$. Note that

$$Cg = \{(\xi_1, \xi_2) \in CA_1 \oplus CA_2 | f_1 p_0 \xi_1 = f_2 p_0 \xi_2\}$$

and $\psi(\xi_1, \xi_2)$ is as above. Then

$$\operatorname{Ker} \psi = \{(\xi_1, \xi_2) \in CA_1 \oplus CA_2 | f_1\xi_1 = f_2\xi_2 = 0\} \cong C(J_1) \oplus C(J_2).$$

We have established the following proposition.

PROPOSITION 4.5. *Let*

$$\begin{tikzcd} P \arrow[r, "g_1"] \arrow[d, "g_2"'] & A_1 \arrow[d, "f_1"] \\ A_2 \arrow[r, "f_2"] & B \end{tikzcd}$$

be a pullback diagram, with all maps surjective and $J_i = \operatorname{Ker} f_i$. *Let* $g = (g_1, g_2)\colon P \to A_1 \oplus A_2$. *Then there is a natural short exact sequence*

$$0 \to C(J_1) \oplus C(J_2) \to Cg \xrightarrow{\psi} SB \to 0$$

where ψ *is given by*

$$\psi(\xi_1, \xi_2)(t) = \begin{cases} f_1\xi_1(1-2t) & t \le \frac{1}{2} \\ f_2\xi_2(2t-1) & t \ge \frac{1}{2}. \end{cases} \qquad \square$$

The diagram

$$\begin{tikzcd} 0 \arrow[r] & S(A_1 \oplus A_2) \arrow[r] \arrow[d, equal] & Cg \arrow[r, "\pi(g)"] \arrow[d, "\psi"] & P \arrow[r] & 0 \\ & SA_1 \oplus SA_2 \arrow[r, bend right, "k"] & SB & & \end{tikzcd}$$

commutes if k is chosen properly: take

$$k(\xi_1, \xi_2) = \begin{cases} f_1\xi_1(1-2t) & t \leq \frac{1}{2} \\ f_2\xi_2(2t-1) & t \geq \frac{1}{2}. \end{cases}$$

Note that $t \rightsquigarrow f_1\xi_1(1-2t)$ is homotopic to $t \rightsquigarrow f_1\xi_1(1-t) \equiv -f_1\xi_1(t)$ and that $t \rightsquigarrow f_2\xi_2(2t-1)$ is homotopic to $t \rightsquigarrow f_2\xi_2(t)$. Thus the diagram

$$\begin{tikzcd} SA_1 \arrow[r] \arrow[dr, "-f_1\xi_1"'] & SA_1 \oplus SA_2 \arrow[d, "k"] & SA_2 \arrow[l] \arrow[dl, "f_2\xi_2"] \\ & SB & \end{tikzcd}$$

homotopy commutes. The long exact sequence associated to the cofibration

$$Cg \to P \overset{g}{\to} A_1 \oplus A_2$$

reads

$$\begin{tikzcd} h_{n+1}(P) \arrow[r, "\partial"] & h_n(SA_1 \oplus SA_2) \arrow[r] \arrow[d, "\cong"'] & h_n(Cg) \arrow[r] \arrow[d, "\cong"', "\psi_*"] & h_n(P) \arrow[r] & \cdots \\ & h_n(SA_1) \oplus h_n(SA_2) \arrow[r, "k_*"] & h_n(SB) & & \end{tikzcd}$$

where $k_*(x, y) = -f_{1_*}x + f_{2_*}x$. Applying the suspension isomorphism leaves us with the long exact sequence

$$\longrightarrow h_{n+1}(P) \to h_{n+1}(A_1) \oplus h_{n+1}(A_2) \overset{k_*}{\to} h_{n+1}(B) \to h_n(P) \longrightarrow$$

The map $h_{n+1}(P) \to h_{n+1}(A_1) \oplus h_{n+1}(A_2)$ is simply (g_{1_*}, g_{2_*}), since $\sigma f_* = \partial$. This proves the theorem in the case that the f_j are surjective.

Finally, suppose given a pullback diagram in $\mathcal{C}$ of the form

$$\begin{tikzcd} P \arrow[r] \arrow[d] & A_1 \arrow[d, "f_1"] \\ A_2 \arrow[r, "f_2"] & B \end{tikzcd}$$

such that f_1 is a cofibration but f_2 is arbitrary. Factor f_2 as

$$A_2 \xrightarrow{s} Mf_2 \xrightarrow{t} B$$

by Corollary 1.8, where t is a cofibration and s is an equivalence. Consider the resulting expanded pullback diagram

$$\begin{tikzcd} P \arrow[r, "s'"] \arrow[d, "f''"] & P_1 \arrow[r] \arrow[d, "f'"] & A_1 \arrow[d, "f_1"] \\ A_2 \arrow[r, "s"] & Mf_2 \arrow[r, "t"] & B. \end{tikzcd}$$

Since f_1 is a cofibration, so are f' and f'' by Proposition 1.5. Proposition 1.6 implies that s' is an equivalence. Thus it suffices to prove the Mayer-Vietoris theorem for the pullback square

$$\begin{tikzcd} P_1 \arrow[r] \arrow[d] & A_1 \arrow[d, "f_1"] \\ Mf_2 \arrow[r, "t"] & B. \end{tikzcd}$$

Since f_1 and t are surjective, the theorem has previously been established for this square. This completes the argument. □

Note that some assumption is necessary on the pullback squares in order for Mayer-Vietoris to hold. For example, if

$$\begin{tikzcd} P \arrow[r] \arrow[d] & A_1 \arrow[d, "f_1"] \\ A_2 \arrow[r, "f_2"] & B \end{tikzcd}$$

is a pullback square for which the Mayer-Vietoris theorem holds and if $h_*(B) \not\cong h_*(B) \oplus h_*(B)$ (which is generally the case) then the Mayer-Vietoris theorem will *not* hold for the pullback square

$$\begin{tikzcd} P \arrow[r] \arrow[d] & A_1 \arrow[d, "f_1 \oplus f_1"] \\ A_2 \arrow[r, "f_2 \oplus f_2"] & B \oplus B \end{tikzcd}$$

(Note that it *is* a pullback!)

On the other hand, if h_* is a homology theory then a better result obtains.

THEOREM 4.5. *If h_* is a homology theory, f_1 is surjective and f_2 is arbitrary then Mayer-Vietoris holds.*

The proof is an elaborate diagram chase as in Eilenberg-Steenrod [5, Chapter I.]

5. Limits in homology. This section centers about the following theorem.

THEOREM 5.1. *Let h_* be an additive homology theory. Let $A = \varinjlim A_i$ be the inductive limit of a sequence*

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \to \cdots$$

of C^*-algebra maps in $\mathcal{C}$. Then the natural maps $h_n(A_i) \to h_n(A)$ induce an isomorphism

$$\varinjlim h_n(A_i) \to h_n(A).$$

Note that the f_i are not assumed to be inclusions. The case

$$A_1 \hookrightarrow A_2 \hookrightarrow A_3 \hookrightarrow \cdots$$

is somewhat simpler; we remark on this case later.

The proof proceeds via a mapping telescope construction as follows. Fix a sequence t_i of real numbers of the form

$$0 = t_0 < t_1 < t_2 < \cdots < 1$$

which converges to 1. If ξ is some function defined on some interval including t_i then we let $p_i\xi = \xi(t_i)$ generically. Let

$$\tilde{I}A_{i+1} = C([t_i, t_{i+1}], A_{i+1}).$$

Let $f_{ij}\colon A_i \to A_j$ and $f_{i\infty}\colon A_i \to A$ be the natural maps.

Let

$$\tilde{T}(A_i) = \{\xi = (\xi_i) | \xi_i \in \tilde{I}A_{i+1}, f_i p_i \xi_{i-1} = p_i \xi_i, \text{ and } \|\xi_i\| \text{ is bounded independent of } i\}.$$

(The more precise notation $\tilde{T}(\{A_i\})$ adds more clutter than clarity and is thus avoided.) Then $\tilde{T}(A_i)$ is a C^*-algebra with $\|\xi\| = \sup_i(\|\xi_i\|)$. We may regard ξ as a function on $[0, 1)$: if $t_i < t \leq t_{i+1}$ then set $n(t) = i + 1$ and $\xi(t) = \xi_{n(t)} \in A_{i+1}$. This is not continuous. However the function $\xi^\infty\colon [0, 1) \to A$ given by $\xi^\infty(t) = f_{n(t),\infty}\xi_{n(t)}(t)$ is obviously continuous.

DEFINITION 5.2. (L. G. Brown.) The *mapping telescope* $T(A_i)$ is defined by

$$T(A_i) = \left\{(\xi, a) \in \tilde{T}(A_i) \oplus A \middle| \text{ condition } (*) \text{ holds}\right\}.$$

Define $\xi(1) = a$. Condition $(*)$ then reads as follows. Given $\varepsilon > 0$, then there is some $\delta > 0$ such that

$$(*) \qquad \left\|\xi(t) - f_{n(s),n(t)}\xi(s)\right\| < \varepsilon$$

provided that $1 - \delta \leq s \leq t \leq 1$.

Here is an alternate description of $T(A_i)$ which is probably more intuitive. Define

$$T_m(A_i) = \Big\{(\xi, a) \in \tilde{T}(A_i) \oplus A \Big| \text{ if } t \geq t_m \text{ then } \xi(t) = f_{m,n(t)}\xi(t_m)$$
$$\text{and } \xi(1) = f_{m,\infty}\xi(t_m)\Big\}.$$

That is, $T_m(A_i)$ consists of sequences which are "constant" in the only sense possible beyond t_m. Then it is clear that $T_m(A_i) \subset T_{m+1}(A_i)$ and that

$$(5.3) \qquad \overline{\bigcup_m T_m(A_i)} = T(A_i).$$

LEMMA 5.4. *$T(A_i)$ is a contractible C^*-algebra.*

Proof. Define h: $T(A_i) \to I(\tilde{T}(A_i) \oplus A)$ by

$$h_j(\xi, a)(t) = (\xi(jt), \xi^\infty(j)).$$

Then $h_0 \equiv 0$, $h_1 = \mathrm{id}$, and continuity is clear. It remains to show that $h_j(T(A_i)) \subset T(A_i)$. It is easy to see that $h_j(T_m(A_i)) \subset T_m(A_i)$ for all j and m; this implies the result. □

Define e: $T(A_i) \to A$ by $e(\xi, a) = a$. This is a surjection; let $J = \operatorname{Ker} e$. Define M_i by

$$M_i = \left\{(\xi, a) \in \tilde{I}A_{i+1} \oplus A_i \middle| p_i\xi = f_i(a)\right\}.$$

This is of course a simple variant of the mapping cylinder construction. In particular, the map $M_i \to A_i$ given by $(\xi, a) \to a$ is an equivalence.

Define r_i: $T(A_i) \to M_i$ by restriction:

$$r_i(\xi, a) = (\xi_i, p_i\xi_{i-1}).$$

As $J \subset T(A_\iota)$ the maps p_ι combine to induce a natural map

$$\prod p_i : J \to \prod A_\iota$$

given by $(\prod p_i)(\xi)_i = p_i \xi_{\iota - 1}$.

LEMMA 5.5. *The image of* $\prod p_\iota$ *lies in* $\oplus A_\iota$, *so that evaluation yields a natural map* $J \to \oplus_\iota A_i$.

Proof. Let $(\xi, 0) \in J$. We must show that $\|\xi_{i-1}(t_\iota)\| \equiv \|p_\iota \xi_{i-1}\|$ approaches zero as $i \to \infty$. In fact we show that $\xi(t) \to 0$ as $t \to 1$, which suffices. Fix $\varepsilon > 0$ and choose $\delta = 1 - t_0 = 1 - s > 0$ such that $(*)$ holds for ξ. Then

$$\left\|\xi(t) - f_{n(t_0)n(t)}\xi(t_0)\right\| < \varepsilon$$

for all $t \geq t_0$. In particular, this holds if $t = 1$. However, $\xi(1) = 0$ since $(\xi, 0) \in J$. Thus

$$\left\|f_{n(t_0),\infty}\xi(t_0)\right\| < \varepsilon$$

for $t \geq t_0$. By the definition of the norm in A, there is some integer $k > n(t_0)$ such that

$$\left\|f_{n(t_0),k}\xi(t_0)\right\| < \varepsilon.$$

Choose $u \in [t_{k-1}, t_k]$. Then $(*)$ implies that

$$\left\|\xi(u) - f_{n(t_0),k}\xi(t_0)\right\| < \varepsilon$$

and hence $\|\xi(u)\| < 2\varepsilon$. Returning to $(*)$ again we see that

$$\left\|\xi(t) - f_{k,n(t)}\xi(u)\right\| < \varepsilon$$

for all $t \geq u$. Thus

$$\left\|\xi(t)\right\| < 3\varepsilon \quad \text{for all } t \geq u$$

which implies that $\xi(t) \to 0$ as $t \to 1$, completing the proof. □

By virtue of the previous lemma there is a natural commuting diagram

$$\begin{tikzcd} J \arrow[r, "\oplus r_{2\iota}"] \arrow[d, "\oplus r_{2\iota+1}"'] & \oplus M_{2\iota} \arrow[d, "\oplus p_{2\iota}"] \\ \oplus M_{2i+1} \arrow[r, "\oplus p_{2\iota+1}"'] & \oplus A_\iota \end{tikzcd} \tag{5.6}$$

LEMMA 5.7. *Diagram 5.6 is a pullback diagram with surjective maps.*

Proof. It is easy to see that the maps are surjective. Let P be the pullback. There is an obvious map $J \to P$; we construct its inverse. A typical element of P is of the form (ν^1, ν^2), where

$$\nu^1 = \left(\nu^1_{2i+1}\right), \quad \nu^2 = \left(\nu^2_{2i}\right)$$

with $\nu_i^\varepsilon \in \tilde{Mf}_i$ and

$$p_i \nu_i^\varepsilon = p_i \nu_{i+1}^{\varepsilon \pm 1}.$$

Further, $\|\nu_i^\varepsilon\| \leq \max(\|\nu^1\|, \|\nu^2\|)$. Thus the elements yield $\nu = (\nu_i) \in \tilde{T}(A_i)$. Since $\nu_i(t_i) \to 0$ as $i \to \infty$, the element $(\nu, 0) \in J$. This defines a map $P \to J$ which is visibly an isomorphism. $\square$

The short exact sequence

$$0 \to J \to T(A_i) \xrightarrow{e} A \to 0$$

yields a long exact sequence in homology. However $T(A_i)$ is contractible, by Lemma 5.3, and so $h_*(T(A_i)) = 0$. Thus the boundary homomorphism

$$\partial \colon h_n(A) \to h_{n-1}(J)$$

is an isomorphism for each n. The Mayer-Vietoris sequence associated to (5.6) yields a diagram with exact row

(5.7)

$$\begin{tikzcd}
{} \arrow[r] & h_n(J) \arrow[r] \arrow[d, "\partial", "\cong"'] & h_n(\oplus M_{2i}) \oplus h_n(\oplus M_{2i+1}) \arrow[r, "\phi_n"] & h_n(\oplus A_i) \arrow[r] & h_{n-1}(J) \arrow[r] \arrow[d, "\partial", "\cong"'] & {} \\
 & h_{n+1}(A) & & & h_n(A) &
\end{tikzcd}$$

where $\phi = -(\oplus p_{2i})_* + (\oplus p_{2i+1})_*$. There are natural isomorphisms

$$\bigoplus h_n(A_i) \to h_n\left(\bigoplus A_i\right)$$

and

$$\bigoplus h_n(A_i) \to h_n\left(\bigoplus M_{2i}\right) \oplus h_n\left(\bigoplus M_{2i+1}\right)$$

where $h_n(A_i) \to h_n(M_i)$ is the isomorphism $(p_i)_*^{-1}$. Thus there is a natural long exact sequence

$$(5.8) \qquad \cdots \to h_{n+1}(A) \to \bigoplus h_n(A_i) \xrightarrow{\Phi_n} \bigoplus h_n(A_i) \to h_n(A) \to \cdots$$

with

$$(5.9) \qquad \Phi(x_i) = \left((f_{i-1})_* x_{i-1} - x_i\right).$$

Unsplicing yields short exact sequences of the form

$$(5.10) \qquad 0 \to \operatorname{Cok} \Phi_n \to h_n(A) \to \operatorname{Ker} \Phi_{n+1} \to 0.$$

Since $\operatorname{Cok} \Phi_n \cong \varinjlim h_n(A_i)$ and $\operatorname{Ker} \Phi_{n+1} = 0$, the theorem is established. □

REMARK 5.11. If each $f_i\colon A_i \to A_{i+1}$ is an inclusion then we may regard each A_i as a subalgebra of A. The telescope may be defined in a simpler way:

$$T(A_i) = \left\{\xi\colon [0, 1] \to A \mid \xi(t) \in A_i \text{ if } t \leq t_i\right\}.$$

The entire argument goes through with much less fuss. Our original proof of Theorem 5.1 in this case went via this route and we had a convoluted argument to derive the general case. L. G. Brown suggested the definition of $T(A_i)$ given in (5.2) with this use in mind.

6. Cohomology theories. In this section and the next we enter the results in cohomology analogous to those previously established in homology. For the most part this is an exercise in translation and we omit proofs. Where there is a significant difference we explore it. This occurs primarily as a result of the different additivity axioms. The relevant homology theorems are listed for comparison. Thus "(= H4.3)" refers to the homology Proposition 4.3.

Fix an admissible category $\mathcal{C}$ of C^*-algebras and C^*-algebra maps. Let h^* be a cohomology theory on $\mathcal{C}$.

PROPOSITION 6.1. (= H3.1.) *The evident map induces a natural isomorphism*

$$h^n(A_1 \oplus A_2) \xrightarrow{(i_1, i_2)^*} h^n(A_1) \oplus h^n(A_2).$$

PROPOSITION 6.2. (> H3.2.)
(a) *The diagram*

$$\begin{tikzcd} h^n(SB) \arrow[r, "m^*"] \arrow[dr, "d"'] & h^n(SB \oplus SB) \arrow[d, "(i_1, i_2)^*"] \\ & h^n(SB) \oplus h^n(SB) \end{tikzcd}$$

commutes, where $d(x) = (x, x)$.

(b) *The diagram*

$$\begin{tikzcd} h^n(A \oplus A) \arrow[rr, "\Delta^*"] & & h^n(A) \\ h^n(A) \oplus h^n(A) \arrow[u, "(p_1, p_2)^*"] \arrow[urr, "+"'] & & \end{tikzcd}$$

commutes, where $\Delta(a) = (a, a)$.

Proof. Part (a) is immediate from the fact that $i_j m$ is homotopic to the identity. Part (b) follows from the fact that $p_j \Delta$ is homotopic to the identity. □

PROPOSITION 6.3. (= H3.3.) *The natural map*

$$[A, SB] \to \hom(h^n(SB), h^n(A))$$

is a homomorphism of groups.

Proof. Let f_1, f_2: $A \to SB$. Then $f_1 + f_2$ is the homotopy class of the composite

$$A \xrightarrow{\Delta} A \oplus A \xrightarrow{(f_1, f_2)} SB \oplus SB \xrightarrow{m} SB.$$

Thus

$$\begin{aligned} (f_1 + f_2)^*(x) &= \Delta^*(f_1, f_2)^* m^*(x) \\ &= \Delta^*(p_1, p_2)^*(f_1, f_2)^*(i_1, i_2)^* m^*(x) \\ &= \Delta^*(p_1, p_2)^*(f_1, f_2)^*(x, x) \quad \text{by (6.2a)} \\ &= \Delta^*(p_1, p_2)^*(f_1^* x, f_2^* x) = f_1^* x + f_2^* x. \quad \text{by (6.2b).} \end{aligned}$$ □

COROLLARY 6.4. (= H3.4.)
(a) *If* f: $A \to B$ *is the constant map* $f(a) \equiv 0$ *then* $f^* = 0$.
(b) $h^*(CA) = 0$. □

THEOREM 6.5. (= H3.5.) (a) *The natural suspension map*

$$\sigma^A: h^n(SA) \to h^{n+1}(A)$$

from the sequence

$$0 \to SA \to CA \to A \to 0$$

is an isomorphism.

(b) *If f: $A \to B$ is a map with cofibre sequence*

$$0 \to SB \to Cf \to A \to 0$$

then the diagram

$$\begin{tikzcd} h^n(SB) \arrow[rr, "\delta"] \arrow[dr, "\sigma^B"'] & & h^{n+1}(A) \\ & h^{n+1}(B) \arrow[ur, "f^*"'] & \end{tikzcd}$$

commutes.

(c) *If f: $A \to B$ then f^*: $h^*(B) \to h^*(A)$ is an isomorphism if and only if $h^*(Cf) = 0$.* □

PROPOSITION 6.6. (= H3.6.)

(a) *Let f: $A \to B$ be a surjection with $J =$ Ker f. Then the natural map $J \to Cf$ induces an h^*-isomorphism.*

(b) *Let f: $A \to B$ be the inclusion of an ideal. Then the natural map $Cf \to S(B/A)$ induces an h^*-isomorphism.*

Just as in homology, there is another possible choice of axioms. One could replace the exactness axiom by suspension and cofibre axioms. (As in homology, exactness plus homotopy imply suspension and cofibre.)

PROPOSITION 6.7. (= H3.8.) *Let h^* be a cofibre theory. If*

$$\cdots \to A_k \to A_{k-1} \to A_{k-2} \to \cdots$$

is a cofibre sequence then there is a long exact sequence

$$h^n(A^{k-2}) \to h^n(A^{k-1}) \to h^n(A^k) \to \cdots .$$

In particular, for any cofibration f: $A \to B$ with $J =$ Ker f there is a long exact sequence

$$\cdots \to h^n(B) \to h^n(A) \to h^n(J) \to h^{n+1}(B) \to \cdots .$$

COROLLARY 6.8 (= H3.9.) *If h^* is a cofibre theory then*

(a) *if f: $A \to B$ then f^* is an isomorphism if and only if $h^*(Cf) = 0$;*

(b) $h^*(CA) = 0$;

(c) *if f: $A \to B$ is a cofibration with $J =$ Ker f then f^* is an isomorphism if and only if $h^*(J) = 0$.* □

PROPOSITION 6.9 (= H3.10.) *If h^* is a cofibre theory, then*
(a) *the natural maps*

$$h^n(A_1 \oplus A_2) \to h^n(A_1) \oplus h^n(A_2)$$

are isomorphisms for each n;
(b) *the diagrams*

$$\begin{tikzcd} h^n(SB) \arrow[r, "m^*"] \arrow[dr, "d"'] & h^n(SB \oplus SB) \arrow[d, "(i_1, i_2)^*"] \\ & h^n(SB) \oplus h^n(SB) \end{tikzcd}$$

and

$$\begin{tikzcd} h^n(A \oplus A) \arrow[r, "\Delta^*"] \arrow[d, "(p_1, p_2)^*"] & h^n(A) \\ h^n(A) \oplus h^n(A) \arrow[ur, "+"'] & \end{tikzcd}$$

commute;
(c) *the natural map*

$$[A, SB] \to \mathrm{hom}(h^n(SB), h^n(A))$$

is a homomorphism. □

THEOREM 6.10. (*Cohomology of a triple* (= H3.11).) *Suppose given ideals $J \subset H$ contained in A. Then there is a natural long exact sequence*

$$(*) \qquad h^n(A/H) \xrightarrow{i^*} h^n(A/J) \xrightarrow{j^*} h^n(H/J) \xrightarrow{k} h^{n+1}(A/H)$$

where i and j are the natural maps

$$0 \to H/J \xrightarrow{i} A/J \xrightarrow{j} A/H \to 0$$

and k is the connecting homomorphism for $(*)$ which coincides with the composite

$$\begin{tikzcd} h^n(H/J) \arrow[rr, "k"] \arrow[dr] & & h^{n+1}(A/H) \\ & h^n(H) \arrow[ur, "\delta"'] & \end{tikzcd}$$

□

Mayer-Vietoris Theorem 6.11. (= H4.1.) *Suppose given a pullback diagram*

$$\begin{tikzcd} P \arrow[r, "g_1"] \arrow[d, "g_2"'] & A_1 \arrow[d, "f_1"] \\ A_2 \arrow[r, "f_2"] & B \end{tikzcd}$$

with f_1 and f_2 surjective or f_1 a cofibration and f_2 arbitrary. Suppose given a cofibre cohomology theory h^*. Then there is a natural long exact sequence

$$\to h^n(B) \xrightarrow{(-f_1^*,\, f_2^*)} h^n(A_1) \oplus h^n(A_2) \xrightarrow{g_1^* + g_2^*} h^n(P) \to h^{n+1}(B) \to \cdots .$$

Proof. We use the notation of the homology Mayer-Vietoris theorem. After identifying Cg via Proposition 4.5, the long exact sequence associated to $Cg \to P \to A_1 \oplus A_2$ takes the form

$$\begin{tikzcd} \arrow[r] & h^n(P) \arrow[r] & h^n(Cg) \arrow[r] & h^n(SA_1 \oplus SA_2) \arrow[r, "\delta"] & h^{n+1}(P) \arrow[r] & {} \\ & & h^n(SB) \arrow[u, "\psi^*"', "\cong"] \arrow[r, "k^*"] & h^n(SA_1) \oplus h^n(SA_2) \arrow[u, "\cong"'] & & \end{tikzcd}$$

where $k^*(x) = (-f_1^*(x), f_2^*(x))$. Apply the suspension isomorphism and the fact that $f^*\sigma = \delta$ to identify the map $g_1^* + g_2^*$. $\square$

As in Theorem 4.5, we note that if h^* is a cohomology theory then it suffices to assume that f_1 is surjective and f_2 is arbitrary.

7. Limits in cohomology. In this section we prove that an additive cohomology theory has a Milnor $\lim^1$-sequence for limits.

THEOREM 7.1. (= H5.1.) *Let h^* be an additive cohomology theory. Let $A_1 \to A_2 \to \cdots \to A_n \to \cdots$ be a sequence of C^*-algebras and let $A = \varinjlim A_i$. Then the natural maps $h^n(A) \to h^n(A_i)$ induce a map $h^n(A) \to \varprojlim h^n(A_i)$ which fits into a natural long exact sequence*

$$0 \to \varprojlim{}^1 h^{n-1}(A_i) \to h^n(A) \to \varprojlim h^n(A_i) \to 0.$$

(Recall that if $G_1 \xleftarrow{\alpha_1} G_2 \xleftarrow{\alpha_2} G_3 \leftarrow \cdots$ is an inverse sequence of abelian groups and $\psi\colon \prod G_i \to \prod G_i$ by $\psi(g_i) = (g_i - \alpha_i g_{i+1})$ then $\operatorname{Ker} \psi \cong \varprojlim G_i$ and $\operatorname{Cok} \psi \cong \varprojlim{}^1 G_i$. For further information on $\lim^1$ the reader is referred to [**15**], [**16**], [**6**]).

Proof. We adopt the notation of the homology Theorem (5.1). Recall that we had defined the mapping telescope $T(A_\iota)$ and had observed that it was contractible. There were restriction maps r_i: $T(A_i) \to M_\iota$, evaluation maps p_ι, an exact sequence

$$0 \to J \to T(A_\iota) \xrightarrow{e} A \to 0 \tag{7.2}$$

and a pullback diagram

$$\begin{tikzcd} J \arrow[r, "\oplus r_{2\iota}"] \arrow[d, "\oplus r_{2\iota+1}"'] & \bigoplus_0^\infty M_{2i} \arrow[d] \\ \bigoplus_0^\infty M_{2\iota+1} \arrow[r, "\oplus P_{2\iota+1}"] & \bigoplus_0^\infty A_\iota \end{tikzcd}$$

The exact sequence (7.2) implies that

$$\delta\colon h^{n-1}(J) \to h^n(A)$$

is an isomorphism. The Mayer-Vietoris Theorem produces a long exact sequence

$$\to h^{n-1}(J) \to h^n(\oplus A_i) \xrightarrow{\phi^n} h^n(\oplus M_{2\iota}) \oplus h^n(\oplus M_{2\iota+1}) \to h^n(J) \to$$

and thus an exact sequence

$$0 \to \operatorname{Cok} \Phi^{n-1} \to h^n(A) \to \operatorname{Ker} \Phi^n \to 0.$$

So we must examine the maps Φ^n. By additivity and the fact that M_ι is equivalent to A_ι, the map Φ^n is up to isomorphism a map

$$\Phi^n\colon \textstyle\prod_\iota h^n(A_\iota) \to \prod_\iota h^n(A_\iota).$$

The Mayer-Vietoris map

$$h^n(\oplus A_i) \to h^n(\oplus M_{2\iota}) \oplus h^n(\oplus M_{2\iota+1})$$

is given by $(-\oplus p_{2\iota}^*, \oplus p_{2\iota+1}^*)$. Identifying M_ι with A_ι via the equivalence p_ι: $M_\iota \to A_\iota$ we see that

$$\Phi(x_\iota) = (-x_\iota + f_\iota^* x_{\iota-1}).$$

Thus $\operatorname{Ker} \Phi^n \cong \varprojlim h^n(A_i)$ and $\operatorname{Cok} \Phi^{n-1} \cong \varprojlim{}^1 h^{n-1}(A_\iota)$. The map

$$h^n(A) \to \operatorname{Ker} \Phi^n \xrightarrow{\cong} \varprojlim h^n(A_\iota)$$

is easily seen to be the obvious map, and the theorem is established. □

8. Some examples. In this section we briefly discuss some examples of homology and cohomology theories.

EXAMPLE 8.1. *K-Theory.* The group $K_0(A)$ may be defined for any ring. Karoubi (cf. [7], [**8**]) was apparently the first to set up the correct definition for $K_1(A)$ to obtain K_* as a homology theory on Banach algebras (though Wood [**26**] proved, at Atiyah's instigation, the critical extension of the Bott periodicity theorem to the setting of Banach algebras.) The homotopy axiom appears in Karoubi's thesis [7, 1.2.21] as does the exactness axiom [7, 2.3.1 and 2.3.3]. Additivity does not seem to appear explicitly anywhere in the literature, though it is a well-established folklore theorem used extensively. The theory K_* has all C^*-algebras as an admissible category.

EXAMPLE 8.2. *BDF-theory.* Brown-Douglas-Fillmore establish the homotopy, exactness, and additivity axioms in their seminal paper [**4**; 2.14, 2.19 + 6.8, 7.3], for the category of separable nuclear C^*-algebras. Thus $\mathrm{Ext}^*(A)$ is an additive cohomology theory on that category.

EXAMPLE 8.3. *PPV-theory.* Fix some finite-dimensional compact metric space X and regard the Pimsner-Popa-Voiculescu functors $\mathrm{Ext}_*(X; A)$ as functors in the A-variable on separable nuclear C^*-algebras. PPV establish the homotopy [**17**, 5.12] and exactness [**17**, 8.4] axioms. They do not show additivity, but it follows from [**20**] provided that X is a finite complex or, more generally, provided that $K^*(X)$ is finitely generated, by [**20**], [**21**]. In their treatment they insist that A be generalized quasi-diagonal. This is no longer a necessary restriction, as Rosenberg and Schochet [**21**] show that the PPV groups correspond to certain Kasparov groups (see below) which are known to be homotopy-invariant.

EXAMPLE 8.4. *The Kasparov groups.* Let A be a separable nuclear C^*-algebra and let B have a countable approximate unit. G. G. Kasparov [**11**] has defined groups $KK_*(A, B) \cong \mathrm{Ext}_*(A, B)$ which seem to be crucial in many areas of C^*-algebras. (The previous examples are all essentially special cases of the Kasparov groups.) Fixing A, one obtains covariant functors $h_*^A(B) = KK_*(A, B)$. Fixing B, one obtains contravariant functors $h_B^*(A) = KK_*(A, B)$. The homotopy axiom and the exactness axiom are satisfied by h_*^A and by h_B^* ([**11**], §4, Theorem 3 and §7, Theorems 2 and 3). The theory h_B^* is additive on the (large) category $\mathfrak{N}$ (which includes the direct limit of type I algebras) by [**20**]. The theory h_*^A is additive provided that $A \in \mathfrak{N}$ and $K_*(A)$ is finitely generated, by [**20**].

EXAMPLE 8.5. *Stable homotopy.* Fix some C^*-algebra D. Define functors $D_q(A)$ and $D^q(A)$ for $q \in \mathbf{Z}$ by

$$D_q(A) = \lim_{k \to \infty} [S^{k-q}D, S^kA]$$
$$D^q(A) = \lim_{k \to \infty} [S^{k+q}A, S^kD].$$

These are obvious candidates for homology and cohomology theories respectively. The homotopy axiom is satisfied trivially. The theories are defined so as to make the suspension axiom hold. Exactness does not hold in general, but it is easy to see that the cofibre axiom holds. Thus D_* and D^* are cofibre homology and cohomology theories respectively.

The natural map

$$[D, \oplus A_i] \to \prod [D, A_i]$$

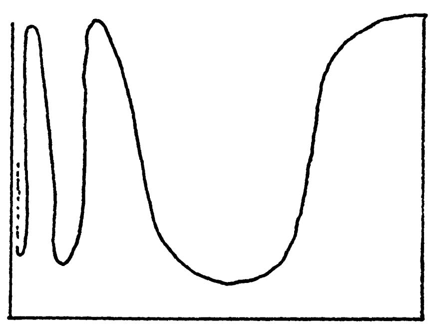
does not have $\oplus[D, A_i]$ as its image in general. Rosenberg [**19**, 3.6] points out that an example is obtained by taking $D = \oplus A_i$ with each A_i non-contractible. Then 1_D is not null-homotopic, and its image in $\prod[D, A_i]$ is non-zero in every coordinate. Thus D_* does not in general satisfy the additivity axiom.

If $D \cong D \otimes \mathcal{K}$ (i.e. D is stable) then Rosenberg shows [**19**, 3.4] that

$$[\oplus A_i, D] \cong \prod [A_i, D].$$

Thus D^* is an additive cofibre theory when D is stable.

REMARK 8.6. The theories D_* and D^* are cofibre homology (cohomology) theories but in general they are *not* homology (cohomology) theories. Here is an example at the level of spaces. Let X be the following closure of the $\sin(1/x)$ curve:



and let A be the left vertical side. The inclusion map $\iota\colon A \to X$ is *not* a cofibration, even though $\iota_*\colon \pi_*(A) \to \pi_*(X)$ is an isomorphism. (If $\iota\colon A \to X$ were a cofibration then it would be an equivalence. But $\check{H}^1(A) = 0$

and $\check{H}^1(X) = \mathbf{Z}$, so A and X are not equivalent.) Let $\pi\colon X \to X/A$. The space X/A is homeomorphic to a circle $S^1 = \mathbf{R}^+$. Thus there is a short exact sequence

$$0 \to C_0(\mathbf{R}) \xrightarrow{\pi^\#} C(X) \xrightarrow{\iota^\#} C(A) \to 0.$$

Since A is contractible, the map $\pi^\#$ must induce an isomorphism on any cohomology theory. However $\pi_1(X) = 0$ and thus exactness fails for the cofibre theory $\mathbf{C}^*$ which is the extension of $\pi_*^S(-)$ to C^*-algebras.

REMARK 8.7. Let π_*^S denote the stable homotopy ring. That is,

$$\pi_n^S = \lim_{k\to\infty} \pi_{n+k}(S^k).$$

If h_* is a cofibre homology theory then $h_*(A)$ is a graded module over π_*^S and if $f\colon A \to B$ then $f_*\colon h_*(A) \to h_*(B)$ is a map of π_*^S-modules. The action

$$\pi_n^S \otimes h_m(A) \to h_{n+m}(A)$$

is given as follows. Let $\alpha \in \pi_n^S$ be represented by a proper map $\alpha\colon \mathbf{R}^{n+k} \to \mathbf{R}^k$. This induces $\tilde{\alpha}\colon S^kA \to S^{n+k}A$. The composite

$$\begin{tikzcd} h_m(A) \arrow[r, dashed] \arrow[d, "\cong"] & h_{m+n}(A) \arrow[d, "\cong"] \\ h_{m-k}(S^kA) \arrow[r, "(\tilde{\alpha})_*"] & h_{m-k}(S^{n+k}A) \end{tikzcd}$$

gives a map in $\hom(h_m(A), h_{n+m}(A))$ as required. There are various elementary checks which use (3.3); these we delete. Similarly, if h^* is a cofibre cohomology theory then there is a pairing

$$\pi_n^S \otimes h^m(A) \to h^{m-n}(A)$$

with the same properties. Here is an easy consequence. Let $\phi\colon S^k \to S^k$ be a map of degree d, and let $f\colon A \to B$ be any C^*-algebra map. Then

$$f \otimes \phi^*\colon A \otimes C_0(\mathbf{R}^k) \to A \otimes C_0(\mathbf{R}^k)$$

and the diagram

$$\begin{tikzcd} h_n(A) \arrow[r, "d\cdot f_*"] \arrow[d, "\cong"'] & h_n(B) \arrow[d, "\cong"] \\ h_{n-k}(S^kA) \arrow[r, "(f\otimes\phi^*)_*"] & h_{n-k}(S^kA) \end{tikzcd}$$

commutes, where the vertical maps are suspension isomorphisms and $(d \cdot f_*)(x) = d(f_*(x))$. In particular, the map

$$(1_A \otimes \phi^*)_*: h_*(A \otimes C_0(\mathbf{R}^k)) \to h_*(A \otimes C_0(\mathbf{R}^k))$$

is simply multiplication by the integer d.

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