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RECURSIVELY ENUMERABLE SETS AND VAN DER WAERDEN'S THEOREM ON ARITHMETIC PROGRESSIONS

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Subsets of the set ω of nonnegative integers which possess some algebraic structure are interesting since they are most likely to give number-theoretic information. Arithmetic progressions are one of the simplest structures to observe. Effectiveness of any kind of information is of course an important factor. It seems that a study of possible interrelationships between combinatoric and number-theoretic properties of recursively enumerable (r.e.) subsets of ω might be interesting. In this paper we study van der Waerden's theorem on arithmetic progressions in this light.

1. Introduction. Schur, while working on the distribution of quadratic residues in $\mathbf{Z}_p$, had conjectured that: if ω is split into two disjoint sets, then an arithmetic progression of any desired length may be found in at least one of the sets (see [**1**].)

Baudet, a student at Göttingen, had mentioned this conjecture to van der Waerden. In 1927, van der Waerden published an elementary but insightful proof of the conjecture to which he referred as Baudet's conjecture (see [**5**].)

Van der Waerden [**6**] gives an interesting account of how in an afternoon of 1926, he, Artin and Schreier discussed the conjecture and found a proof for it. In [**5**] (and [**6**]), van der Waerden gives a proof of a more general form of the conjecture: if ω is equal to the disjoint union of the sets $A_1,\ldots,A_n$, then given an arbitrary k, at least one A_i contains an arithmetic progression of length k. Moreover, for a proof he demonstrated the existence of a function f such that given n (the number of cells in the partition) and k (the desired length of arithmetic progression), then a mere partitioning of $\{0,1,\ldots,f(n,k)\}$ into n cells in any manner, yields an arithmetical progression of length k in at least one of the cells. Furthermore, the given function f is indeed a recursive one. For our purposes we state

VAN DER WAERDEN'S THEOREM. *Let $A \subset \omega$. Pick k. Then either A or $\omega - A$ has an arithmetic progression of length k.*

From a recursion theoretic point of view it is reasonable to ask whether van der Waerden's theorem holds when only r.e. sets are considered. This question may not at first appear to make sense because $\omega - A$ need not be r.e. even when A is r.e. To remedy this difficulty, we let the role of $\omega - A$ be taken over by arbitrary r.e. sets disjoint from A. To avoid trivialities we also require that a set have infinitely many pairwise disjoint arithmetic progressions of length k rather than just a single such progression. In §3 we use the finite form of van der Waerden's theorem mentioned above to show that for any r.e. set A, either A contains arbitrarily long arithmetic progressions or for each k there is an r.e. set B_k disjoint from A which contains infinitely many pairwise disjoint arithemtic progressions of length k. One might hope that this result could be strengthened to show that for any r.e. set A, either A or some r.e. set disjoint from A contains arbitrarily long arithmetic progressions. Our main result, proved in §4 using the priority method, refutes this conjecture by giving an example of an r.e. set which contains no arithmetic progression of length three and yet intersects every r.e. set which contains arbitrarily long arithmetic progressions. Section 5 gives a corollary based on Szemerédi's generalization of van der Waerden's theorem. Section 2 contains the key definitions and §6 generalizes our main result. Finally some open questions are listed in §7.

2. Definitions and some notation. By the *length* of an arithmetic progression we mean the number of the terms in it. Let $A \subset \omega$.

DEFINITION 1. A is *thick* if A contains arithmetic progressions of any length $k \in \omega$.

DEFINITION 2. Let $k \in \omega$. A is *k-thin* if A does not contain an arithmetic progression of length k. (This notion will also be used for subsets of rationals.)

DEFINITION 3. Let $k \in \omega$. A is *k-thick* if there is no finite set $F \subset \omega$ such that $A - F$ is k-thin.

DEFINITION 4. A is *thin* if A is k-thin for some $k \in \omega$.

Some trivial observations about these notions will be useful:

(a) A is thick iff A is k-thick for all $k \in \omega$.

(b) A is thick iff A is not k-thin for any $k \in \omega$.

If $a, b \in \omega$, then $(a, b]$ denotes the set of elements of ω between a and b together with b. The set of rational numbers is denoted by $\mathbf{Q}$. For any finite sequence $q_1, \ldots, q_n$ of rational numbers define

$$\Delta q_i = q_{i+1} - q_i \quad \text{for } 1 \le i < n$$

and

$$\Delta^2 q_i = \Delta q_{i+1} - \Delta q_i = q_{i+2} - 2q_{i+1} + q_i \quad \text{for } 1 \le i < n - 1.$$

These are the usual first and second difference operators. Note that Δ and Δ^2 are linear in the obvious sense. $\mathscr{P}(\mathbf{Q})$ denotes the collection of all subsets of $\mathbf{Q}$ while $\mathscr{P}_{\text{fin}}(\mathbf{Q})$ denotes the collection of all finite subsets of $\mathbf{Q}$. For $C \subseteq \mathbf{Q}$, define the operator

$$\Phi_C : \mathscr{P}(\mathbf{Q}) \to \mathscr{P}(\mathbf{Q})$$

by

$$\Phi_C(B) = \left\{ \sum_{i=1}^{n} c_i b_i \middle| c_i \in C \text{ and } b_1, \ldots, b_n \text{ are distinct elements of } B \right\}.$$

Φ_C is *finitely based* if C is finite. Note that finitely based operators map finite sets to finite sets.

Finally, W_e denotes the eth r.e. subset of ω in an acceptable enumeration of all such sets. For all other background on recursion theory, we refer the reader to Rogers [**3**].

3. The positive direction.

THEOREM 5. *If A is r.e. and not thick, then for each k there is an r.e., k-thick set B_k which is disjoint from A.*

Proof. Suppose A is n-thin. It suffices to construct B_k for $k > n$. Fix $k > n$. By van der Waerden's theorem, there is a bound b_k such that whenever $(0, b_k]$ is partitioned into two sets, one of the sets is not k-thin. For each i, let

$$F_i = (b_k i, b_k(i + 1)].$$

So $F_0, F_1, \ldots$ are pairwise disjoint translates of $F_0 = (0, b_k]$. Thus whenever any F_i is partitioned into two sets, one of the sets is not k-thin.

Let

$$m_k = \text{the largest number } m \text{ such that } |A \cap F_i| = m \text{ for infinitely many } i.$$

Note that m_k exists because $|A \cap F_i| \le |F_i| = b_k$ for all i.

Choose i_k such that

$$i \geq i_k \Rightarrow |A \cap F_i| \leq m_k.$$

(Of course m_k and i_k are not obtained uniformly in k, but this is no obstacle since we are not claiming that B_k is r.e. uniformly in k.)

Let A^s be a recursive enumeration of A and define

$$B_k = \{ b \mid \exists s (\exists i \geq i_k)[|A^s \cap F_i| = m_k \;\&\; b \in F_i - A^s] \}.$$

Clearly, B_k is r.e. Also, $B_k \cap A = \varnothing$ since otherwise $|A \cap F_i| > m_k$ for some $i \geq i_k$ contrary to the choice of i_k.

To show that B_k is k-thick, it suffices to show that $B_k \cap F_i$ is not k-thin whenever $|A \cap F_i| = m_k$ and $i \geq i_k$, since there are infinitely many such i by the choice of m_k. To see this, let $|A \cap F_i| = m_k$ and $i \geq i_k$. Then $A \cap F_i$ and $B_k \cap F_i$ partition F_i into two sets, one of which is not k-thin by the choice of b_k. However, A is n-thin, $k > n$, and $A \cap F_i \subseteq A$; hence $A \cap F_i$ is k-thin. Therefore $B_k \cap F_i$ is not k-thin. □

We observe an interesting corollary. Recall A is *simple* if A is r.e. and coinfinite, but it meets every infinite r.e. set.

COROLLARY 6. *Every simple set is thick.*

Corollary 6 may also be proved easily without using van der Waerden's theorem. The observation needed is that for any recursive sequence $\{F_i\}$ of pairwise disjoint finite sets of uniformly bounded cardinality and any simple set A, F_i is a subset of A for infinitely many i. (See [**2**], pp. 116–117.)

4. The negative direction. We now prove our main result using a priority construction. The construction is a finite injury one in the sense that each positive requirement contributes at most one element to the r.e. set A being constructed. However, it is a bit unusual because an individual negative requirement may (permanently) restrain infinitely many numbers from A. The positive requirements are satisfied nonetheless because any finite set of negative requirements together restrain only a *thin* set of numbers from A.

THEOREM 7. *There is a* 3-*thin r.e. set A which intersects every thick r.e. set.*

Proof. It is necessary and sufficient to satisfy the following requirements in the construction of the r.e. set A:

$$N_{-1}: \quad A \text{ is 3-thin;}$$
$$P_e: \quad W_e \text{ is thick} \to A \cap W_e \neq \varnothing.$$

To satisfy N_{-1}, it suffices to keep out of A, with highest priority, any numbers which form an arithmetic progression of length 3 with numbers already in A (and add at most one new element to A at a time).

Let

$$A^* = \{n \mid A \cup \{n\} \text{ is not 3-thin}\}.$$

Hence, to satisfy N_{-1} we must arrange $A \cap A^* = \varnothing$. Now since A is r.e., A^* is also an r.e. set; say $A^* = W_j$. Clearly, to satisfy P_j it is necessary that $A^* = W_j$ be k-thin for some k. The following proposition shows that we may not choose $k = 3$.

PROPOSITION 8. *If A is r.e. and 3-thin and A intersects every thick r.e. set, then A^* is 3-thick.*

Proof. Clearly A has to be infinite. Suppose $a, b \in A$, $a < b \leq 2a$ and $b - a$ is even. Then the numbers $2b - a$, $(a + b)/2$ and $2a - b$ are all in A^* and form an arithmetic progression of length 3. Thus if A^* is not 3-thick, there are only finitely many pairs (a, b) as above. But then

$$W = \{b \mid (\exists a \in A)[a < b \leq 2a \,\&\, b - a \text{ is even}]\}$$

is a thick r.e. set having only finite intersection with A. This is a contradiction. Hence A^* is 3-thick.

In view of the above proposition it is reasonable to ask whether, under the same hypothesis, it can be shown that A^* is thick (which would refute the theorem we are trying to prove). The answer is no. The explanation lies in the fact that every element of A^* may be written as $c_1a_1 + c_2a_2$ where a_1 and a_2 are distinct elements of A and c_1 and c_2 are elements of the fixed finite set $C = \{-1, \frac{1}{2}, 2\}$. The arithmetic progressions of length 3 in A^* arise only because C contains an arithmetic progression of length 3. On the other hand, if we consider the fact that every element of A^* may be written as $\sum_{i=1}^{n} c_i a_i$ where $a_1, \ldots, a_n$ are distinct elements of A and $c_1, \ldots, c_n$ are elements of the fixed finite set $C \cup \{0\}$, it would be possible to keep A^* 4-thin because $C \cup \{0\}$ does not contain an arithmetic progression of length 4. (Actually, because of the way the priorities are about to be arranged, A^* will only be k_0-thin, where k_0 is obtained by recursive approximations.)

Keeping A^* k_0-thin is a new negative requirement N_0. The negative requirements N_{-1} and N_0 together will keep a certain set of numbers out of A; clearly, we must introduce a new negative requirement N_1 that this set be k_1-thin for a suitable number k_1. Continuing inductively, we obtain negative requirements $N_0, N_1, \ldots$. We rank the requirements with priorities as follows: $N_{-1}, P_0, N_0, P_1, N_1, \ldots$.

The strategy for each positive requirement P_e is the obvious one:

> if $A^s \cap W_e^s = \varnothing$ and W_e^s contains an element x not restrained from A at s by any N_i, $i < e$, e is the least such, put the least such x into A at s.

If the negative requirement N_e succeeds in keeping the set of numbers held out of A by the requirements N_i with $i < e$ k_e-thin, and if W_e is thick, then P_e will eventually receive attention and be satisfied permanently.

In order to form a strategy for the negative requirements, we need some preliminary notions. In terms of the operator Φ_C of §2, if $C_0 = \{-1, 0, \frac{1}{2}, 2\}$, then $A^* \subseteq \Phi_{C_0}(A)$. Next, in order to keep track of what is kept out of A, we shall inductively define finite sets $C_0 \subseteq C_1 \subseteq \cdots$ so that every number kept out of A by any N_i, $-1 \leq i < e$, is in $\Phi_{C_e}(A)$. Then N_e will require $\Phi_{C_e}(A)$ to be k_e-thin for a certain number k_e. For any operator Ψ: $\mathscr{P}(\mathbf{Q}) \to \mathscr{P}(\mathbf{Q})$ and any $k \in \omega$, let

$$\Psi^{(k)}(B) = \{ q \in \mathbf{Q} \mid \Psi(B) \text{ is } k\text{-thin} \\ \& \Psi(B \cup \{q\}) \text{ is not } k\text{-thin}\}.$$

The following lemma will be used to obtain C_{e+1} from C_e effectively.

LEMMA 9. *Let B, $C \subseteq \mathbf{Q}$. There is a recursive function F: $\mathscr{P}_{\text{fin}}(\mathbf{Q}) \to \mathscr{P}_{\text{fin}}(\mathbf{Q})$ such that if C is finite, k-thin and $0 \in C$, then $\Phi_C^{(k)}(B) \subseteq \Phi_{F(C)}(B)$.*

Proof. Let C be a given k-thin finite set of rational numbers containing 0. Suppose $q \in \Phi_C^{(k)}(B)$; hence $\Phi_C(B)$ is k-thin, but $\Phi_C(B \cup \{q\})$ is not k-thin. We now carry out some elementary computations to express q as a linear combination of elements of B with the coefficients from a finite set $F(C)$ not dependent on q or B. Let $r_1, \ldots, r_k$ be a (nonconstant) arithmetic progression of length k in $\Phi_C(B \cup \{q\})$. Hence each r_i may be written as

$$r_i = b_i + c_i q \quad \text{where } b_i \in \Phi_C(B) \text{ and } c_i \in C.$$

(Note that here the fact that $0 \in C$ allows us to cover the case that $r_i \in \Phi_C(B)$; also note that b_i's are not necessarily distinct.) Since $r_1, \ldots, r_k$ is an arithmetic progression and Δ^2 is linear, we have

(1) $$0 = \Delta^2 r_i = \Delta^2 b_i + q\Delta^2 c_i \quad \text{for } 1 \leq i < k - 1.$$

First assume for the sake of a contradiction that $\Delta^2 c_i = 0$ for all i, $1 \leq i < k - 1$. Then $c_1 = c_2 = \cdots = c_k$ since otherwise $c_1, \ldots, c_k$ would be an arithmetic progression of length k in C. Next, since $r_1, \ldots, r_k$ is not a constant sequence, neither is $b_1, \ldots, b_k$. But $\Delta^2 b_i = 0$ for $1 \leq i < k - 1$ from (1) and by our hypothesis that $\Delta^2 c_i = 0$. Therefore $b_1, \ldots, b_k$ is an arithmetic progression of length k in $\Phi_C(B)$, which is the desired contradiction.

Fix i, $1 \leq i < q - 1$, so that $\Delta^2 c_i \neq 0$. From (1) we have

(2) $$q = -\Delta^2 b_i / \Delta^2 c_i.$$

Since $b_j \in \Phi_C(B)$, $i \leq j \leq i + 2$, there are distinct numbers $a_1, \ldots, a_n \in B$ such that $b_j = \sum_{t=1}^{n} c_{j,t} a_t$ for $i \leq j \leq i + 2$ and certain numbers $c_{j,t} \in C$. (Here, we use the fact that $0 \in C$ to make $a_1, \ldots, a_n$ independent of j.) By linearity of Δ^2,

(3) $$\Delta^2 b_i = \sum_{t=1}^{n} a_t \Delta^2 c_{it}$$

where

(4) $$\Delta^2 c_{i,t} = c_{i+2,t} - 2c_{i+1,t} + c_{i,t}.$$

Since we are in search of $F(C)$ in order to have $q \in \Phi_C^{(k)}(B) \subseteq \Phi_{F(C)}(B)$, (2), (3), and (4) suggest that we may take $F(C)$ to be the set of all numbers of the form $-u/v$ where $v \neq 0$ and u, v are each of the form $c - 2d + e$ with $c, d, e \in C$. □

Let

$$C_0 = \{-1, 0, \tfrac{1}{2}, 2\} \quad \text{and} \quad C_{e+1} = C_e \cup F(C_e)$$

where F is the function from Lemma 9. We enumerate A in stages by defining A^s for any s. At the end of stage s we define k_e^s so that C_e and $\Phi_{C_e}(A^s)$ are k_e^s-thin. During the construction we need to satisfy the positive requirements P_e.

DEFINITION. P_e requires attention at stage $s + 1$ if there exists x such that

(1) $$A^s \cap W_e^s = \varnothing,$$

$$(2) \qquad x \in W_e^s - \Phi_{C_e}(A^s).$$

Construction.

Stage 0. To initialize, let

$$\begin{aligned} A^0 &= \varnothing, \quad \text{and, for each } e, \\ k_e^0 &= \text{the least } k \text{ such that } C_e \text{ is } k\text{-thin,} \end{aligned}$$

Stage $s+1$. If no P_e requires attention, let

$$A^{s+1} = A^s \qquad \text{and} \qquad k_e^{s+1} = k_e^s \quad \text{for all } e$$

and go on to the next stage. Otherwise, let j be the least e such that P_e requires attention and choose the least corresponding x. Let

$$A^{s+1} = A^s \cup \{x\}$$

and

$$k_e^{s+1} = \begin{cases} k_e^s & \text{if } e < j \\ \text{the least } k \text{ such that} & \\ \qquad C_e \text{ and } \Phi_{C_e}(A^{s+1}) \text{ are } k\text{-thin} & \text{if } j \leq e. \end{cases}$$

(Note that the k in the second case exists because C_e and $\Phi_{C_e}(A^{s+1})$ are finite sets.)

Since A is clearly r.e., if the requirements $N_{-1}, P_0, N_0, \ldots$ are satisfied, the theorem is established. In the following we state and prove some lemmas which show that the requirements are indeed satisfied.

LEMMA 10. *A is 3-thin.*

Proof. Note that

$$(A^s)^* \subseteq \Phi_{C_0}(A^s) \subseteq \Phi_{C_e}(A^s)$$

for all e and s, so no element of $(A^s)^*$ enters A at s and $|A^{s+1} - A^s| \leq 1$. Hence as remarked before the constructions, it follows that A is 3-thin.

LEMMA 11. *For each e, $\lim_s k_e^s$ exists.*

Proof. If $k_e^s \neq k_e^{s+1}$, then some P_j with $j \leq e$ must have needed attention at stage $s+1$. But each P_j receives attention at most once.

LEMMA 12. *For each e and s, C_e and $\Phi_{C_e}(A^s)$ are k_e^s-thin.*

Proof. This is proved by induction on s. It is clear if $s = 0$. Suppose C_e and $\Phi_{C_e}(A^s)$ are k_e^s-thin. Consider stage $s + 1$; if no P_e requires attention then the result for $s + 1$ is immediate. Otherwise, assume P_j requires attention at stage $s + 1$ via x. By construction, x is the unique element of $A^{s+1} - A^s$. If $e \geq j$, then C_e and $\Phi_{C_e}(A^{s+1})$ are k_e^{s+1}-thin by the choice of k_e^{s+1}. Suppose $e < j$. For the sake of a contradiction, assume that $\Phi_{C_e}(A^{s+1})$ is not k_e^{s+1}-thin, and hence not k_e^s-thin. Therefore $x \in \Phi_{C_e}^{(k_e^s)}(A^s)$. Since $\Phi_{C_e}(A^s)$ and C_e are k_e^s-thin, it follows by choice of C_{e+1} that $x \in \Phi_{C_{e+1}}(A^s)$. But $e + 1 \leq j$, so $C_{e+1} \subseteq C_j$ and hence $x \in \Phi_{C_j}(A^s)$, contrary to the construction.

LEMMA 13. *If W_e is thick, then $A \cap W_e \neq \varnothing$.*

Proof. Let $k_e = \lim_s k_e^s$. From Lemma 12 and the monotonicity and continuity of Φ_{C_e}, it follows that $\Phi_{C_e}(A)$ is k_e-thin. Assume e is such that W_e is thick and $A \cap W_e = \varnothing$. Then $W_e \not\subseteq \Phi_{C_e}(A)$, so choose $x \in W_e - \Phi_{C_e}(A)$. By monotonicity, $x \in W_e^s - \Phi_{C_e}(A^s)$ for all sufficiently large s, so P_e will require attention at all sufficiently large stages and will eventually receive it, since each P_i, $i < e$, receives attention at most once.

This completes the proof of the theorem. $\square$

5. A corollary.

To see a corollary consider:

DEFINITION 14. Let $B \subseteq \omega$. B is said to have *positive upper density* if

$$\lim_{n \to \infty} \sup \frac{1}{n} |B \cap \{1, \ldots, n\}| > 0.$$

It can be seen that a thick set does not have to have positive upper density. However, Erdös-Turán conjectured a phenomenon, later proved intricately by Szemerédi [3], one of whose consequences is that *every set of positive upper density is thick.*

Now, clearly the van der Waerden Theorem is a special case of Szemerédi's theorem since if a set B is k-thin for any k then it is not thick and therefore not of positive upper density. Hence it can be seen that $\omega - B$ has to be of positive upper density and hence it must have a thick subset.

Theorem 7 and Szemerédi's theorem may be used to give a counter-example to effectiveness of the Szemerédi's theorem.

COROLLARY 15. *There exists an r.e. set A which is thin but intersects every r.e. set of positive upper density.*

6. Generalization. It may be the case that Theorem 7 can be generalized in different ways. In this section we give one version which has a proof structurally the same as our proof of Theorem 7.

DEFINITION 16. Let $A \subset \omega$ and $k, d \in \omega$. A is *k-thin for degree d* if there is no nonconstant polynomial $p(x)$, with coefficients in $\mathbf{Q}$ and of degree $\leq d$ with $p(0),\ldots,p(k-1) \in A$.

Before we give the theorem, we state some useful facts. Clearly A is k-thin for degree 1 iff A is k-thin. Also, if A is k-thin for degree d and A has more than 1 element, then $k \geq d+2$. This is because Lagrange's Interpolation Formula applied to given pairs $(x_0, y_0),\ldots,(x_d, y_d)$ with distinct $x_0,\ldots,x_d$ produces a polynomial p of degree $\leq d$ such that $p(x_i) = y_i$ for $0 \leq i \leq d$.

DEFINITION 17. A is *sparse for degree d* if for all $d' \leq d$, $0 < d'$, A is $(d'+2)$-thin for degree d'. Note that A is sparse for degree 1 iff A is 3-thin, and if A is sparse for degree d and $d' \leq d$, then A is sparse for degree d'.

DEFINITION 18. A is *thick for degree d* if A is not k-thin of degree d for any k.

Finally we are able to generalize Theorem 7 whose intuitive content is 'there are sparse r.e. sets which meet every thick r.e. set'.

THEOREM 19. *For each d, there is an r.e. set A which is sparse for degree d and intersects every r.e. set which is thick for degree d.*

We leave the proof to the reader with a hint: in proof for Theorem 7, replace Δ^2 by Δ^{d+1} and modify C_0 appropriately.

Observe that Theorem 19 reduces to Theorem 7 when $d = 1$ and that the strength of Theorem 19 increases as d increases.

7. Open questions. Upon further reflection, it becomes clear that Theorems 5 and 7 are about "controlling" thinness. This raises the following questions:

(1) Is there a recursive function f and a 3-thin r.e. set A such that $A \cap W_e \neq \varnothing$ whenever W_e is $f(e)$-thick? By Theorem 5, f may not be

chosen to be constant. By Theorem 7, there is such an f recursive in $0'$, i.e. $f(e) = k_e$.

In Theorem 19 we stated a generalized form of Theorem 7. In that light

(2) Is there an r.e. set A which is sparse for *all* degrees d and which meets every r.e. set which is thick for *any* degree d?

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