GLOBAL POSITIVE SOLUTIONS OF SEMILINEAR ELLIPTIC PROBLEMS

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The existence of bounded positive solutions of semilinear elliptic boundary value problems of the type

(1.1) $\qquad Lu = \lambda f(x, u), \quad x \in \Omega,$

(1.2) $\qquad u(x) = 0, \qquad x \in \partial\Omega,$

will be proved in unbounded domains $\Omega \subset R^n$, $n \geq 2$, with boundary $\partial\Omega \in C^{2+\alpha}$, $0 < \alpha < 1$, where λ is a positive constant and

$$(1.3) \quad Lu = -\sum_{i,j=1}^{n} D_i \big[a_{ij}(x) D_j u \big] + m(x)u, \qquad x \in \Omega,$$

$D_i = \partial/\partial x_i$, $i = 1, \dots, n$. The existence of a bounded positive solution of (1.1) in the entire space R^n is proved also by the same procedure. The regularity and additional hypotheses H1–H5 to be imposed on L and f are stated in §2. In particular, the assumption $f(x,0) = 0$ for all $x \in \Omega$ implies that the boundary value problem (1.1), (1.2) always has the trivial solution.

1. **Introduction.** Generally a positive solution of (1.1), (1.2) in Ω exists only if λ is sufficiently large, as might be expected from known results for bounded domains, see e.g. Rabinowitz [23]. In fact, under the extra hypotheses H8 and H9, we prove the Uniqueness Theorem 5.1: There exists a positive interval $(0, \lambda_*]$ such that (1.1), (1.2) has no nontrivial solution $u(x, \lambda)$ for any λ in this interval. However, under different conditions H6 and H7, Theorems 4.4 and 4.6 yield bounded positive solutions of, respectively, the boundary value problem (1.1), (1.2) and the differential equation (1.1) in all of R^n for arbitrary positive λ.

The physically important case [9, 25]

$$(1.4) \qquad -\Delta u + m(x)u = p(x)u^\gamma - q(x)u^\beta, \qquad x \in \Omega,$$

is included, where $1 < \gamma < \beta$ and p, q, m are nonnegative functions in Ω. Solutions $u(x)$ of (1.4) provide stationary states $e^{i\omega t}u(x)$ of the corresponding wave equation, often called the Klein-Gordon equation. In the case of constants p, q, m, the existence of positive solutions of (1.4) in the

entire space R^n which decay exponentially at ∞ (if $m > 0$) has been proved by Berestycki and Lions [7], Berestycki, Lions, and Peletier [8], Berger [9], and Strauss [25]. These results of course are not concerned with boundary conditions on $\partial\Omega$. Our method establishes, in particular, the existence of positive solutions of the Dirichlet problem for (1.4) in unbounded domains Ω without the requirement of any symmetry conditions on either Ω or the coefficients. The earlier methods cited, depending critically on the radial symmetry of the coefficients, could not be extended to solve the present problems.

However, we are unable to prove in general that the solution has limit zero at ∞. This is proved in §5 only in cases for which the nonlinear terms in (1.4) have limit zero at ∞.

Existence theorems for boundary value problems on *bounded* domains are contained in papers by Ako [1], Amann [2], Amann and Crandall [3], Bandle [4], Keller [16], Nagumo [18], Rabinowitz [23], and Schmitt [24]. Extensive bibliographies appear in [23] and [24]. For unbounded domains, the existence theorems of Benci and Fortunato [5, 6], Berger and Schechter [10], Bose [11], Edmunds and Evans [13], Edmunds and Webb [14], Noussair [19] and Ogata [21] either apply only to special cases of our problem, or do not guarantee nontrivial solutions, or both. The complex theory of weighted Sobolev spaces, as developed and exploited in [5, 6, 10, 13, 14], requires restrictions on the "size" of the domain Ω at ∞ and does not aim at positivity or asymptotic behavior of the solutions.

Section 2 describes the notation and principal hypotheses. In §3 some results of Rabinowitz [23] for bounded domains are adapted to our structure. The main Theorem 4.3 establishes the existence of a bounded positive solution of (1.1), (1.2) in Ω for all sufficiently large λ if H1–H5 hold. The modifications in Theorem 4.5 and 4.6 yield bounded positive solutions of (1.1) in R^n. Section 5 concerns the existence of positive solutions $u(x, \lambda)$ of (1.1), (1.2) which have limit 0 as $|x| \to \infty$. This is proved in two cases: (i) $m(x)$ is uniformly positive and $f(x, t) \to 0$ as $|x| \to \infty$; and (ii) in dimensions $n \geq 3$, $f(x, t)$ satisfies a Dini condition of the Meyers and Serrin type [17]. Section 6 contains modifications for $m(x) \leq 0$.

The case of nonnegative functions $f(x, u)$ for all x, u is not being considered here. We solve (1.1), (1.2) in this case elsewhere by completely different methods.

2. **Preliminaries.** Let $|x|$ denote the Euclidean norm of a point $x = (x_1, \ldots, x_n)$ in real Euclidean n-space R^n, $n \geq 2$. The notation below

will be used throughout:

$$S_t = \{x \in R^n : |x| = t\}, \qquad t > 0,$$
$$G_t = \{x \in R^n : |x| > t\}, \qquad t > 0,$$
$$G_{s,t} = \{x \in R^n : s < |x| < t\}, \quad 0 < s < t,$$
$$\Omega_t = \{x \in \Omega : |x| < t\}, \qquad t > 0.$$

An *exterior domain* $\Omega \subset R^n$ has the property that $G_\delta \subset \Omega$ for some $\delta > 0$, fixed in the sequel.

For a bounded domain $M \subset \Omega$, $C^{m+\alpha}(\overline{M})$ denotes the usual Hölder space, with norm $\| \cdot \|_{m+\alpha,\overline{M}}$, $0 < \alpha < 1$, $m = 0, 1, 2, \ldots$. The abbreviation $C_{\text{loc}}^{m+\alpha}(\Omega)$ is used for the set of all $u \in C^{m+\alpha}(\overline{M})$ for every bounded subdomain M of Ω. The space $\mathring{W}_2^1(M)$ is defined to be the completion of $C_0^\infty(M)$ in the norm

$$\|u\| = \left[\int_M \sum_{|i| \leq 1} |D^i u(x)|^2 \, dx \right]^{1/2}$$

where multi-index notation has been used.

The conditions below will be imposed on the functions a_{ij}, m, and f in (1.1), (1.3):

H1. Each $a_{ij} \in C_{\text{loc}}^{1+\alpha}(\Omega)$, $m \in C_{\text{loc}}^\alpha(\Omega)$, $m(x) \geq 0$ in Ω, and $(a_{ij}(x))$ is symmetric and uniformly positive definite in every bounded subdomain of Ω, where $\alpha \in (0, 1)$ is fixed throughout.

H2. f is locally Lipschitz continuous on $(\Omega \cup \partial\Omega) \times R^+$, where $R^+ = [0, \infty)$, and $f(x, 0) = 0$ for all $x \in \Omega$.

H3. There exists a positive number T such that $f(x, t) < 0$ for all $t > T$ and for all $x \in \Omega$.

H4. There exist $x_0 \in \Omega$ and $T_0 \in [0, T)$ such that $F(x_0, T_0) > 0$, where

$$(2.1) \qquad F(x, t) = \int_0^t f(x, \tau) \, d\tau, \qquad 0 \leq t < \infty.$$

H5. For every bounded domain $M \subset \Omega$ and for every $t_0 > 0$, there corresponds a positive constant $K = K(M, t_0)$ such that $\lambda f(x, t) + Kt$ is a nondecreasing function of t on $0 \leq t \leq t_0$ for each fixed $x \in \overline{M}$.

H5 is satisfied, for example, if $f(x, t)$ is continuously differentiable with respect to t at every $(x, t) \in \Omega \times R^+$. In particular H2 implies that $f \in C^\alpha_{\text{loc}}(\Omega \times R^+)$.

These hypotheses are all satisfied in the case that (1.1) is specialized to equation (1.4) under the following conditions: (i) $1 < \gamma < \beta$; (ii) p, q, m are locally Lipschitz continuous, nonnegative, and bounded; and (iii) p/q is bounded and bounded away from zero in Ω.

3. **Construction of subsolutions.** Let T be as in H3 and let f_T: $\Omega \times R \to R$ be defined by

$$(3.1) \qquad f_T(x, t) = \begin{cases} 0 & \text{if } t < 0, \\ f(x, t) & \text{if } 0 \le t \le T, \\ f(x, T) & \text{if } t > T. \end{cases}$$

Also define F_T by

$$(3.2) \qquad F_T(x, t) = \int_0^t f_T(x, \tau) \, d\tau.$$

Then, by (2.1) and (3.2), $F_T(x, t) = F(x, t)$ for $0 \le t \le T$, $x \in \Omega$. Associated with (1.1), consider the following boundary value problem on bounded domains Ω_R:

$$(3.3) \qquad \begin{cases} Lu = \lambda f(x, u), & x \in \Omega_R, \\ u(x) = 0, & x \in \partial\Omega_R. \end{cases}$$

Let $I_R(\phi, \lambda)$ be the functional on $\mathring{W}_2^1(\Omega_R)$ defined by

$$(3.4) \qquad I_R(\phi, \lambda) = I_{1,R}(\phi) - \lambda I_{2,R}(\phi),$$

where

$$(3.5) \qquad I_{1,R}(\phi) = \frac{1}{2} \int_{\Omega_R} \left[\sum_{i,j=1}^n a_{ij}(x) D_i \phi D_j \phi + m(x) \phi^2(x) \right] dx$$

$$(3.6) \qquad I_{2,R}(\phi) = \int_{\Omega_R} F_T(x, \phi(x)) \, dx.$$

THEOREM 3.1 (*Rabinowitz*). *If* H1–H4 *hold, there exist positive constants* R *and* $\lambda^* = \lambda^*(R)$ *such that the boundary value problem* (3.3) *has a positive solution* $u_R = u_R(x, \lambda)$ *in* Ω_R *for all* $\lambda \ge \lambda^*$. *Furthermore, the functional* $I_R(\phi, \lambda)$ *on* $\mathring{W}_2^1(\Omega_R)$ *attains its minimum at* $\phi = u_R(\cdot, \lambda)$ *for* $\lambda \ge \lambda^*$.

H4 implies that R can be chosen large enough (and fixed in the sequel) so that there exists a point $x_0 \in \Omega_R$ with $F(x_0, T_0) > 0$. Then Theorem 3.1 follows from a theorem of Rabinowitz [23, p. 177] applied to the bounded domain Ω_R.

Theorem 3.1 is strengthened to the result below under the following additional hypotheses:

H6. Each $a_{ij}(x)$ and $f(x, t)$ are bounded in Ω for each fixed $t > 0$, and $m(x) = 0$ identically.

H7. There exists a positive constant C such that

$$\limsup_{R \to \infty} R^{1-n} \int_{\Omega_R} F(x, C) \, dx = +\infty,$$

where $F(x, t)$ is defined by (2.1).

THEOREM 3.2. *If* H1–H3, H6, *and* H7 *hold, there exists* $R > 0$ *such that* (3.3) *has a nontrivial nonnegative solution* $u_R(x, \lambda)$ *in* Ω_R *for all* $\lambda > 0$.

Proof. H7 implies that H4 holds for some x_0 in a domain Ω_R for some $R > 0$. The proof given by Rabinowitz [23, pp. 176–177] shows on the basis of H1–H4 in Ω_R that $I_R(\phi, \lambda)$ attains its minimum on $\mathring{W}_2^1(\Omega_R)$ at a point $\phi = u_R(\cdot, \lambda)$ which is a nonnegative classical solution of (3.3) in Ω_R. To show that u_R is nontrivial, consider the piecewise C^1 function on $\overline{\Omega}_R$ defined by

$$w_R(x) = \begin{cases} 0 & \text{if } x \in \overline{\Omega}_a, \\ C(|x| - a) & \text{if } a \le |x| \le a + 1, \\ C & \text{if } a + 1 \le |x| \le R - 1, \\ C(R - |x|) & \text{if } R - 1 \le |x| \le R, \end{cases}$$

where $a \ge 0$ is chosen so that $G_a \subset \Omega$, and $R > a + 2$ without loss of generality. Since $w_R = 0$ on $\partial\Omega_R$, $w_R \in \mathring{W}_2^1(\Omega_R)$. Each $a_{ij}(x)$ and $F_T(x, t)$ are bounded in Ω for each fixed $t > 0$ by H6, (3.1), and (3.2). Then one sees easily from (3.5) and (3.6) that there exist positive constants K_1 and K_2 such that

$$I_{1,R}(w_R) \le K_1 R^{n-1}$$

$$I_{2,R}(w_R) \ge -K_2 R^{n-1} + \int_{G_{a+1,R-1}} F_T(x, C) \, dx,$$

for $R > a + 2$. Since $C < T$ by H3 and H7, $F_T(x, C) = F(x, C)$. Then H7 and (3.4) show that $I_R(w_R, \lambda) < 0$ for some sufficiently large R. However, $I_R(0, \lambda) = 0$ from H2, (3.2), (3.4)–(3.6) and hence u_R is nontrivial from the minimum property of u_R.

REMARK. The boundedness hypothesis H6 can be relaxed to allow unbounded functions $a_{ij}(x)$ and $f(x, \cdot)$ provided H7 is modified. This can be accomplished by a different choice of $w_R(x)$ in the proof of Theorem 3.2.

4. Existence of positive solutions.

A *solution* of (1.1), (1.2) in an unbounded domain $\Omega \subset R^n$ is understood to be a function $u = u(\cdot, \lambda) \in C_{loc}^{2+\alpha}(\Omega)$ such that u satisfies (1.1), (1.2) identically.

Let $u_R(x, \lambda)$ be a nontrivial nonnegative solution of the boundary value problem (3.3) in a bounded domain Ω_R. If H1–H4 are satisfied, Theorem 3.1 shows that $u_R(x, \lambda)$ exists for some $R > 0$ (fixed) and for all $\lambda \geq \lambda^* > 0$. If H1–H3, H6, and H7 hold, Theorem 3.2 shows that $u_R(x, \lambda)$ exists for all $\lambda > 0$. Let $w_0(x, \lambda)$ denote the extension of $u_R(x, \lambda)$ to all of R^n defined by

$$(4.1) \qquad w_0(x, \lambda) = \begin{cases} u_R(x, \lambda) & \text{if } x \in \overline{\Omega}_R, \\ 0 & \text{otherwise.} \end{cases}$$

The Lemma below concerns functions $v \in C_{loc}^{2+\alpha}(\Omega)$ (to be found later) which satisfy

$$(4.2) \qquad Lv \geq \lambda f(x, v) \quad \text{in } \Omega, \qquad v \geq 0 \quad \text{on } \partial\Omega.$$

LEMMA 4.1. *Under hypotheses* H1, H2, *and* H5, *suppose there exist non-trivial functions* $u_R \in C^{2+\alpha}(\overline{\Omega}_R)$ *and* $v \in C_{loc}^{2+\alpha}(\Omega)$ *satisfying* (3.3) *and* (4.2), *respectively, such that* $0 \leq u_R(x, \lambda) \leq v(x, \lambda)$ *for all* $x \in \overline{\Omega}_R$, $R > 0$, $\lambda > 0$. *Then there exists a sequence of functions* $w_j \in C^{2+\alpha}(\overline{\Omega}_{R+j})$ *satisfying*

$$Lw_j \leq \lambda f(x, w_j) \quad \text{in } \Omega_{R+j}, \qquad w_j = 0 \quad \text{on } \partial\Omega_{R+j}$$

such that $w_0(x, \lambda) \leq w_j(x, \lambda) \leq v(x, \lambda)$ *throughout* $\overline{\Omega}_{R+j}$, *where* $w_0(x, \lambda)$ *is defined by* (4.1).

Proof. Define

$$T_j = \sup_{x \in \Omega_{R+j}} v(x), \qquad j = 1, 2, \ldots.$$

By H5, we can choose a positive constant K_j, depending on v, such that $\lambda f(x, t) + K_j t$ is a nondecreasing function of t on $0 \le t \le T_j$ for each fixed $x \in \overline{\Omega}_{R+j}$. Define $w_j = w_j(x, \lambda)$ to be the unique solution of the linear boundary value problem

$$(4.3) \quad \begin{cases} (L + K_j)w_j = \lambda f(x, w_0(x)) + K_j w_0(x) & \text{in } \Omega_{R+j}, \\ w_j(x) = 0 & \text{on } \partial\Omega \cup S_{R+j}. \end{cases}$$

We now verify that w_j has all the properties stated in Lemma 4.1. Since λ is fixed, we delete the λ dependence from the notation $w_j(x, \lambda)$, etc.

First let $F_j(x) = \lambda f(x, w_0(x)) + K_j w_0(x)$, $x \in \overline{\Omega}_{R+j}$. Then $F_j \in C^{\alpha}(\overline{\Omega}_{R+j})$ by H2, and the standard Schauder estimate for the solution of (4.3) [**12**, p. 335] shows that $w_j \in C^{2+\alpha}(\overline{\Omega}_{R+j}), j = 1, 2, \ldots$.

It follows from (4.2) and (4.3) that

$$(L + K_j)(v - w_j) \ge \left[\lambda f(x, v) + K_j v\right] - \left[\lambda f(x, w_0) + K_j w_0\right]$$

in Ω_{R+j}. Since $v \ge u_R$ in $\overline{\Omega}_R$ by hypothesis, so also $v \ge w_0$ in Ω_{R+j}, $(L + K_j)(v - w_j) \ge 0$ in Ω_{R+j}. Furthermore, $v - w_j = v \ge 0$ on $\partial\Omega \cup S_{R+j}$ and therefore $v - w_j \ge 0$ throughout $\overline{\Omega}_{R+j}$ by the maximum principle [**22**].

Similarly, $(L + K_j)w_j \ge 0$ in Ω_{R+j} and hence $w_j \ge 0$ throughout Ω_{R+j}. From (3.3) and (4.3) we obtain

$$\begin{cases} (L + K_j)(w_j - u_R) = 0 & \text{in } \Omega_R, \\ w_j - u_R = 0 & \text{on } \partial\Omega, \\ w_j - u_R = w_j \ge 0 & \text{on } S_R, \end{cases}$$

from which $w_j \ge u_R$ throughout $\overline{\Omega}_R$ by the maximum principle, and hence $w_j \ge w_0$ throughout $\Omega \cup \partial\Omega$ by (4.1).

Finally, from (4.3),

$$Lw_j - \lambda f(x, w_j) = \left[\lambda f(x, w_0) + K_j w_0\right] - \left[\lambda f(x, w_j) + K_j w_j\right] \le 0$$

in $\Omega_{R+j}, j = 1, 2, \ldots$.

LEMMA 4.2. *Under the hypotheses of Lemma* 4.1, *there exists a sequence of functions* u_j *in* $\Omega \cup \partial\Omega$ *with the following properties:*
  (A) $u_j \in C^{2+\alpha}(\overline{\Omega}_{R+j})$;
  (B) $u_j(x, \lambda) = 0$ *if* $x \in \partial\Omega \cup S_{R+j} \cup G_{R+j}$;
  (C) $Lu_j = \lambda f(x, u_j(x, \lambda))$ *if* $x \in \Omega_{R+j}$;
  (D) $w_0(x, \lambda) \le w_j(x, \lambda) \le u_j(x, \lambda) \le v(x, \lambda)$ *in* $\overline{\Omega}_{R+j}$;
  (E) $u_{j+1}(x, \lambda) \ge u_j(x, \lambda)$ *in* $\Omega \cup \partial\Omega, j = 1, 2, \ldots$.

Proof. By Lemma 4.1, w_j is a subsolution and v is a supersolution of the boundary value problem

$$(4.4) \qquad \begin{cases} Lu = \lambda f(x, u), & x \in \Omega_{R+j}, \\ u(x) = 0, & x \in \partial\Omega_{R+j}, \end{cases}$$

with $w_j(x, \lambda) \le v(x, \lambda)$ throughout $\overline{\Omega}_{R+j}$. Then a theorem of Amann [2, p. 283] shows that (4.4) has a *minimal* solution $u = U_j \in C^{2+\alpha}(\overline{\Omega}_{R+j})$ satisfying $w_0 \le w_j \le U_j \le v$ throughout $\overline{\Omega}_{R+j}$, i.e. $W_j \ge U_j$ on $\overline{\Omega}_{R+j}$ for any solution W_j of (4.4) satisfying $w_j \le W_j \le v$ on $\overline{\Omega}_{R+j}$. The extension $u_j(x, \lambda)$ of $U_j(x, \lambda)$ to $\Omega \cup \partial\Omega$ defined to be 0 for $|x| > R + j$ then has properties (A)–(D) of the Lemma.

To prove (E), notice from (A)–(D) that u_{j+1} is a supersolution of (4.4) in Ω_{R+j}. By Amann's Theorem [2] again, there exists a solution W_j of (4.4) satisfying $w_0 \le w_j \le W_j \le u_{j+1}$ in Ω_{R+j}. The minimality of U_j then implies that $W_j \ge U_j$, and hence $u_{j+1} \ge u_j$ in $\overline{\Omega}_{R+j}$. Since the same inequality holds if $|x| > R + j$ by (B), (D), property (E) has been proved.

In the first main theorem we specialize Lemma 4.2 to the case of the constant function $v(x, \lambda) = T$, which evidently satisfies (4.2) if H3 holds.

THEOREM 4.3. *If* H1–H5 *are satisfied, there exists* $\lambda^* > 0$ *such that the boundary value problem* (1.1), (1.2) *has a bounded positive solution* $u(x, \lambda)$ *in* Ω *satisfying* $w_0(x, \lambda) \le u(x, \lambda) \le T$ *for all* $x \in \Omega \cup \partial\Omega$ *and for all* $\lambda \ge \lambda^*$.

Proof. Since H1–H4 hold, Theorem 3.1 guarantees a positive solution $u_R(x, \lambda)$ of (3.3) in Ω_R for all $\lambda \ge \lambda^* > 0$, as required for Lemma 4.2. We need to check first that the constant solution $v(x, \lambda) = T$ of (4.2) satisfies $u_R(x, \lambda) \le T$ for all $x \in \overline{\Omega}_R$. Suppose to the contrary that $u_R(x_0, \lambda) > T$ is a (necessarily positive) maximum of $u_R(x, \lambda)$ in Ω_R. Then, in appropriate coordinates, $0 \le Lu_R(x_0, \lambda) = \lambda f(x_0, u_R(x_0, \lambda))$, contradicting H3. Accordingly it follows from (4.1) that $w_0(x, \lambda) \le T$ for all $x \in \Omega$.

Therefore the sequence $\{u_j(x, \lambda)\}$ in Lemma 4.2 exists. The next (crucial) step is to prove, for every positive integer i, that there exists a positive constant $K(i)$, independent of $j \ge i$, such that

$$\|u_j\|_{2+\alpha, \overline{\Omega}_{R+i}} \le K(i) \quad \text{for all } j \ge i.$$

This can be accomplished by L^p-estimates, Sobolev embedding, and Schauder estimates as in [20, Lemma 3.2]. Then the compactness of the injection $C^{2+\alpha}(\overline{\Omega}_{R+i}) \to C^2(\overline{\Omega}_{R+i})$ enables us to define a subsequence

$\{u_j^i\}$ of $\{u_j^{i-1}\}$ inductively which converges in the $C^2(\overline{\Omega}_{R+i})$ norm to a function $u^i \in C^2(\overline{\Omega}_{R+i})$, $i = 2, 3, \ldots$, where $u_j^1 = u_j$. The proof given in [20, p. 126] shows that the diagonal sequence $\{u_j^j\}$ converges to a solution u of (1.1), (1.2). By property (D) of Lemma 4.2, the solution satisfies $w_0(x, \lambda) \le u(x, \lambda) \le T$ in Ω, and so is bounded.

The pointwise limit of $\{u_j(x, \lambda)\}$ exists by the monotone property (E), and the limit can only be the function $u(x, \lambda)$ constructed above. To prove that $u(x, \lambda) > 0$ for all $x \in \Omega$, let M be any bounded domain with $\overline{M} \subset \Omega$ and choose an integer i such that $\overline{M} \subset \Omega_{R+i}$. By Lemma 4.1, w_i is nontrivial and satisfies

$$(L + K_i)w_i \ge 0 \quad \text{in } \Omega_{R+i}, \qquad w_i = 0 \quad \text{on } \partial\Omega_{R+i}$$

for $K_i > 0$. Then the strong maximum principle [22] shows that $w_i > 0$ throughout Ω_{R+i}. By properties (D) and (E) of Lemma 4.2, $u_j \ge u_i \ge w_i$ throughout Ω_{R+i} for all $j \ge i$, and therefore $u(x, \lambda) = \lim_{j \to \infty} u_j(x, \lambda) > 0$ on $\overline{M}$. Since M is arbitrary, $u(x, \lambda)$ is a positive solution of (1.1), (1.2) in Ω for all $\lambda \ge \lambda^*$.

The following analogue of Theorem 4.3 is proved in the same way by using Theorem 3.2 instead of Theorem 3.1.

THEOREM 4.4. *Suppose that* H1–H3 *and* H5–H7 *are satisfied. Then the boundary value problem* (1.1), (1.2) *has a bounded positive solution* $u(x, \lambda)$ *in* Ω *satisfying* $w_0(x, \lambda) \le u(x, \lambda) \le T$ *for all* $x \in \Omega \cup \partial\Omega$ *and for all* $\lambda > 0$.

The existence of a bounded positive solution of (1.1) in the entire space R^n is proved by exactly the same method. An analogue of Lemma 4.2, with Ω replaced by R^n and $\partial\Omega$ deleted, follows from Theorem 3.1 or Theorem 3.2 without essential change. The hypotheses H'1–H'7 are formally the same as H1–H7 with R^n replacing Ω and $\partial\Omega$ deleted.

THEOREM 4.5. *If* H'1–H'5 *hold, there exists* $\lambda^* > 0$ *such that* (1.1) *has a bounded positive solution* $u(\cdot, \lambda) \in C_{\text{loc}}^{2+\alpha}(R^n)$ *satisfying* $w_0(x, \lambda) \le u(x, \lambda) \le T$ *for all* $x \in R^n$ *and for all* $\lambda \ge \lambda^*$.

THEOREM 4.6. *If* H'1–H'3 *and* H'5–H'7 *are satisfied, then* (1.1) *has a bounded positive solution* $u(x, \lambda)$ *satisfying* $w_0(x, \lambda) \le u(x, \lambda) \le T$ *for all* $x \in R^n$ *and for all* $\lambda > 0$.

5. Behavior of solutions as $|x| \to \infty$. In this section we consider the case that (1.1) is the Schrödinger equation. Then the boundary value

problem (1.1), (1.2) reduces to

$$(5.1) \qquad -\Delta u + m(x)u = \lambda f(x, u), \quad x \in \Omega,$$
$$(5.2) \qquad u(x) = 0, \qquad\qquad\qquad x \in \partial\Omega.$$

We prove in particular, under hypotheses selected from the list below, that (5.1), (5.2) has an exponentially decaying positive solution $u(x, \lambda)$ in Ω as $|x| \to \infty$, provided λ is sufficiently large.

H8. There exists $m_0 > 0$ such that $m(x) \geq m_0 > 0$ for all $x \in \Omega$.

H9. $f(x, t)/t$ is bounded in $\Omega \times (0, T]$, and $\lim_{t \to 0+}(f(x, t)/t) = 0$ for all $x \in \Omega$.

H10. $\lim_{|x| \to \infty} \sup_{0 < t \leq T}(f(x, t)/t) = 0$.

H11. There exists $r_0 > 0$ such that

$$\sup_{|x|=r} \phi(x) \leq \frac{\delta(r)}{r^2} \quad \text{for all } r \geq r_0,$$

where

$$\phi(x) = \sup_{0 \leq t \leq T} f(x, t)$$

and $\delta(r)$ denotes a Dini function, i.e. the improper integral $\int^\infty r^{-1}\delta(r)\, dr$ exists and is finite.

UNIQUENESS THEOREM 5.1. *If* H1–H4, H8, *and* H9 *hold, then there exists* $\lambda_* > 0$ *such that the only nonnegative bounded solution* $u(x, \lambda)$ *of* (5.1), (5.2) *is identically zero in* Ω *for* $0 < \lambda \leq \lambda_*$.

Proof. Define $h(x, t) = f(x, t)/t$ if $t > 0$, $h(x, 0) = 0$ for all $x \in \Omega$, and

$$\mu_* = \sup_{\substack{x \in \Omega \\ 0 \leq t \leq T}} h(x, t), \qquad \lambda_* = \frac{m_0}{2\mu_*}.$$

Thus μ_* is positive and finite by H4 and H9, so $\lambda_* > 0$. For any nonnegative bounded solution $u(x, \lambda)$ of (5.1), (5.2), define

$$(5.3) \qquad M(x, \lambda) = m(x) - \lambda h(x, u(x, \lambda)), \qquad x \in \Omega.$$

If $0 < \lambda \leq \lambda_*$ and $0 \leq u(x, \lambda) \leq T$ it follows that

$$\lambda h(x, u(x, \lambda)) \leq \lambda_* \mu_* = m_0/2, \qquad x \in \Omega,$$

and hence by H8 that

(5.4) $\qquad M(x, \lambda) \geq m(x) - m_0/2 \geq m_0/2 > 0, \qquad x \in \Omega.$

However by H3, $h(x, u(x, \lambda)) < 0$ for all $x \in \Omega$ such that $u(x, \lambda) > T$, and for all $\lambda > 0$, and so $M(x, \lambda) > m(x) \geq m_0$ at all such points. Therefore (5.4) is satisfied for all $x \in \Omega$ and for all λ in $0 < \lambda \leq \lambda_*$.

By (5.1) and (5.3), $u(x, \lambda)$ is a solution of the *linear* differential equation

(5.5) $\qquad\qquad -\Delta u + M(x, \lambda)u = 0, \qquad x \in \Omega.$

Since $u(x, \lambda)$ is bounded, a theorem of Kato [15, p. 415] shows in view of (5.4) that there exists a positive constant $C(\lambda)$ such that

$$0 \leq u(x, \lambda) \leq C(\lambda) \exp\left[-\sqrt{m_0/2}\,|x|\right], \qquad x \in \Omega,$$

for $0 < \lambda \leq \lambda_*$. Because of the boundary condition (5.2) on $\partial\Omega$, $u(x, \lambda)$ must have a local positive maximum at some point $x_0 \in \Omega$ unless $u(x, \lambda)$ is identically zero in Ω. The conclusion of Theorem 5.1 then follows from the maximum principle for (5.5).

THEOREM 5.2. *Suppose* H1–H5, H8, *and* H10 *are satisfied. Then there exists* $\lambda^* > 0$ *such that, for all* $\lambda \geq \lambda^*$, *the boundary value problem* (5.1), (5.2) *has a positive solution* $u(x, \lambda)$ *in* Ω *satisfying* $u(x, \lambda) \leq C(\lambda) \exp[-\sqrt{m_0/2}\,|x|]$ *for some constant* $C(\lambda) > 0$.

Proof. Let $u(x, \lambda)$ be the solution of (5.1), (5.2) constructed in Theorem 4.3: $0 < u(x, \lambda) \leq T$ for all $x \in \Omega$, $\lambda \geq \lambda^*$, λ^* as in Theorem 4.3. Define

$$H(x) = \sup_{0 < t \leq T} h(x, t), \qquad h(x, t) = f(x, t)/t.$$

By H10 there exists $\rho(\lambda) > 0$ such that $\lambda H(x) \leq m_0/2$ for all $x \in \Omega$ with $|x| \geq \rho(\lambda)$. Since $0 < u(x, \lambda) \leq T$ for all $x \in \Omega$, it follows that

$$\lambda h(x, u(x, \lambda)) \leq m_0/2 \quad \text{for } |x| \geq \rho(\lambda), \lambda \geq \lambda^*.$$

Then H8 and (5.3) imply that

(5.6) $\qquad\qquad M(x, \lambda) \geq m_0/2 \quad \text{for } |x| \geq \rho(\lambda), \lambda \geq \lambda^*.$

Since $u(x, \lambda)$ is a solution of the linear equation (5.5) with (5.6) holding in $G_{\rho(\lambda)}$, Kato's Theorem [15, p. 415] implies that there exists a constant

$C_0(\lambda) > 0$ for $\lambda \geq \lambda^*$ such that

$$0 < u(x, \lambda) \leq C_0(\lambda) \exp\left[-\sqrt{m_0/2}\,|x|\right], \qquad |x| \geq \rho(\lambda).$$

The estimate in Theorem 5.2 then follows, where $C(\lambda) = \max[C_0(\lambda), T]$.

A version of the Dirichlet problem for unbounded domains is

(5.7)
$$\begin{cases} -\Delta u + m(x)u = \lambda f(x, u), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \\ u(x) \to 0 & \text{as } |x| \to \infty. \end{cases}$$

Under the hypotheses of Theorem 5.2, a positive solution $u(x, \lambda)$ of (5.7) in Ω exists for all $\lambda \geq \lambda^*$.

We now obtain an analogue of Theorem 5.2 without the uniform positivity hypothesis H8 on the coefficient $m(x)$ in (5.1). In particular, $m(x)$ is allowed to be identically zero.

THEOREM 5.3. *If H1–H5 and H11 hold, and $n \geq 3$, then there exists $\lambda^* > 0$ such that (5.1), (5.2) has a positive solution $u(x, \lambda)$ in Ω such that $u(x, \lambda) \to 0$ as $|x| \to \infty$ for all $\lambda \geq \lambda^*$.*

Proof. With $\phi(x)$ as in H11, consider the linear boundary value problem

(5.8)
$$\begin{cases} -\Delta v + m(x)v = \lambda\phi(x), & x \in \Omega, \\ v(x) = 0, & x \in \partial\Omega. \end{cases}$$

In view of H11, a theorem of Meyers and Serrin [17, Theorem 10, p. 527] shows, if $n \geq 3$, that (5.8) has a nontrivial solution $v = v(x, \lambda)$ in Ω such that $v(x, \lambda) \to 0$ as $|x| \to \infty$.

Let $u_R = u_R(x, \lambda)$ be the solution of (5.1) in the bounded domain Ω_R guaranteed by Theorem 3.1, for $\lambda \geq \lambda^*$:

(5.9)
$$\begin{cases} -\Delta u_R + m(x)u_R = \lambda f(x, u_R(x, \lambda)), & x \in \Omega_R, \\ u_R(x, \lambda) = 0, & x \in \partial\Omega_R. \end{cases}$$

For $x \in \Omega_R$, (5.8) and (5.9) give

(5.10) $(-\Delta + m(x))(v - u_R) = \lambda[\phi(x) - f(x, u_R(x, \lambda))].$

Since $u_R(x, \lambda) \leq T$ for all $x \in \overline{\Omega}_R$ by the proof of Theorem 4.3, it follows from (5.8)–(5.10) and H11 that

$$\begin{cases} (-\Delta + m(x))(v - u_R) \geq 0 & \text{in } \Omega_R, \\ v - u_R = 0 & \text{on } \partial\Omega, \\ v - u_R \geq 0 & \text{on } S_R. \end{cases}$$

Then the maximum principle implies that

$$0 < u_R(x, \lambda) \leq v(x, \lambda), \qquad x \in \Omega_R, \lambda \geq \lambda^* > 0.$$

The sequence $\{u_j(x, \lambda): j = 1, 2, \ldots\}$ constructed in Lemma 4.2 satisfies, in particular,

(D) $$0 \leq w_0(x, \lambda) \leq u_j(x, \lambda) \leq v(x, \lambda)$$

for all $x \in \Omega \cup \partial\Omega$, $\lambda \geq \lambda^*$, $j = 1, 2, \ldots$, where $w_0(x, \lambda)$ is defined by (4.1). The proof of monotone convergence of the sequence $\{u_j(x, \lambda)\}$ to a solution $u(x, \lambda)$ of (5.1), (5.2) is exactly the same as in Theorem 4.3. It also follows from (D) that $u(x, \lambda) \to 0$ as $|x| \to \infty$ since $v(x, \lambda) \to 0$ and the convergence of $\{u_j(x, \lambda)\}$ to $u(x, \lambda)$ holds pointwise in Ω for all $\lambda \geq \lambda^*$.

For example, the results of this section apply to the equation

$$-\Delta u + m(x)u = \lambda p(x)(u^\gamma - \varepsilon u^\beta),$$

$$x \in \Omega, 1 < \gamma < \beta, \varepsilon > 0, \lambda > 0.$$

Hypotheses H3–H5 hold automatically, and the other hypotheses of this section reduce to the following:

H1–H2. m and p are nonnegative, $m, p \in C^\alpha_{\text{loc}}(\Omega)$, and p is not identically zero.

H8. m is uniformly positive.

H9. $p(x)$ is bounded.

H10. $\lim_{|x| \to \infty} p(x) = 0$.

H11. $\max_{|x|=r} p(x) \leq \delta(r)/r^2, r \geq r_0 \geq 0$, where $\delta(r)$ is a Dini function.

Results parallel to Theorems 5.2 and 5.3 hold in the entire space R^n if Hypotheses Hj are replaced by formally identical Hypotheses $H'j$ with R^n replacing Ω and $\partial\Omega$ deleted.

THEOREM 5.4. *Suppose that* H'1–H'5, H'8, *and* H'10 *are satisfied. Then there exists* $\lambda^* > 0$ *such that, for all* $\lambda \geq \lambda^*$, *equation* (5.1) *has a positive solution* $u(x, \lambda)$ *in* R^n *satisfying*

$$u(x, \lambda) \leq C(\lambda) \exp\left[-\sqrt{m_0/2}\,|x|\right], \qquad x \in R^n,$$

for some constant $C(\lambda) > 0$.

THEOREM 5.5. *Suppose that $n \geq 3$, and H'1–H'5 and H'11 are satisfied. Then there exists $\lambda^* > 0$ such that, for all $\lambda \geq \lambda^*$, equation (5.1) has a positive solution $u(x, \lambda)$ in R^n such that $u(x, \lambda) \to 0$ as $|x| \to \infty$.*

Theorem 5.4 is proved from the solution constructed in Theorem 4.5 by application of Kato's theorem. The method used for Theorem 5.3 also proves Theorem 5.5.

6. Existence theorems in the case of negative $m(x)$. It will now be proved, under modified hypotheses, that the existence theorems of §4 remain valid when $m(x)$ has negative values in Ω. Although $m(x)$ will be allowed to change sign for arbitrarily large $|x|$, the set $\{ x \in R^n : m(x) < 0 \}$ will be required to have sufficiently large measure so that the linear operator L in (1.3) is nodally oscillatory in Ω, i.e. for arbitrary $R > 0$ there exists a nonempty bounded domain $M \subset G_R \cap \Omega$ such that the Dirichlet problem

$$(6.1) \qquad Lu = 0 \quad \text{in } M, \qquad u = 0 \quad \text{on } \partial M$$

has a nontrivial nonnegative solution u_M. Many explicit nodal oscillation criteria are known [**26**]; for example, L is nodally oscillatory in R^2 if the entries of the matrix $(a_{ij}(x))$ are bounded and

$$\int_\Omega m(x) \, dx = -\infty, \qquad n = 2.$$

Hypotheses H2 and H5 will be retained while H1 and H3 need to be slightly modified:

H*1. The a_{ij} are as in H1, $m \in C^\alpha_{\text{loc}}(\Omega)$, and L is nodally oscillatory in Ω.

H*3. There exists $T > 0$ such that $f(x, T) = 0$, $f(x, t) \geq 0$ if $0 \leq t < T$, and $f(x, t) < 0$ if $t > T$ for all $x \in \Omega$.

By H*1 and the linearity of L, (6.1) has a nontrivial nonnegative solution u_M in some nonempty bounded domain $M \subset \Omega$ satisfying $u_M(x) \leq T$ for all $x \in M$. Then H*3 implies that

$$(6.2) \qquad f(x, u_M(x)) \geq 0 \quad \text{for all } x \in M.$$

Let R be a positive number such that $M \subset \Omega_R$. By H5 there exists a positive constant K_j, for each $j = 1, 2, \ldots$, such that both
  (i) $m(x) + K_j > 0$ for all $x \in \Omega_{R+j}$; and
  (ii) $\lambda f(x, t) + K_j t$ is a nondecreasing function of t in $[0, T]$ for each fixed $x \in \Omega_{R+j}$.

The lemma below is then easily established by the procedure used for Lemmas 4.1 and 4.2.

LEMMA 6.1. *Suppose that* H*1, H2, H*3, *and* H5 *are satisfied. Let* $u_M \in C^{2+\alpha}(\overline{M})$ *be a nontrivial nonnegative solution of* (6.1) *chosen so that* $u_M(x) \le T$ *for all* $x \in M$, *and suppose that* $M \subset \Omega_R$. *Then there exists a sequence of functions* u_j *in* $\Omega \cup \partial\Omega$ *with properties* (A), (B), (C), (E) *of Lemma* 4.2 *and*

(D*) $w_0(x) \le u_j(x, \lambda) \le T$ *for all* $x \in \Omega \cup \partial\Omega, j = 1, 2, \ldots, \lambda > 0,$

where $w_0(x) = u_M(x)$ *if* $x \in \overline{M}$ *and* $w_0(x) = 0$ *otherwise.*

THEOREM 6.2. *If* H*1, H2, H*3, *and* H5 *hold, then the boundary value problem* (1.1), (1.2) *has a bounded positive solution* $u(x, \lambda)$ *in* Ω *such that* $w_0(x) \le u(x, \lambda) \le T$ *for all* $x \in \Omega \cup \partial\Omega$ *and for all* $\lambda > 0$, *where* $w_0(x)$ *is defined in Lemma* 6.1.

THEOREM 6.3. *Suppose that the hypotheses of Theorem* 6.2 *hold in the entire space* R^n (*with* $\partial\Omega$ *deleted*). *Then equation* (1.1) *has a bounded positive solution satisfying* $w_0(x) \le u(x, \lambda) \le T$ *for all* $x \in R^n$ *and for all* $\lambda > 0$.

The proofs on the basis of Lemma 6.1 are essentially the same as the proof of Theorem 4.3.

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Pacific Journal of Mathematics
Vol. 115, No. 1   September, 1984