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In this paper we show that an arbitrary contractive projection on a J^*-algebra has the properties of a conditional expectation (Theorem 1). This fact is then used to solve the bicontractive projective problem (Theorem 2).

Let M be a J^*-algebra and let θ be an isometry (equivalently a J^*-automorphism [7]) of M of order 2. Then P, defined by $Px = \frac{1}{2}(x + \theta x)$, is a bicontractive projection on M, i.e., $P^2 = P$, $\|P\| \leq 1$, $\|\mathrm{id}_M - P\| \leq 1$. By the bicontractive projection problem we mean the converse of this statement.

An affirmative answer to the bicontractive projection problem imposes strong symmetry properties on the Banach space M, so it cannot be true for a general Banach space.

In Bernau-Lacey [**2**], the problem is solved for the class of Lindenstraus spaces. In [**1**] Arazy-Friedman solved it with M = the C^*-algebra of compact operators on a separable complex Hilbert space. In [**10**], Størmer, influenced by partial results of Robertson-Youngson [**9**], solved it with M an arbitrary C^*-algebra and P assumed positive and unital. Our Theorem 2, specialized to a C^*-algebra, generalizes each of these results of Arazy-Friedman and Størmer. The authors have recently solved the problem for associative Jordan triple systems [**3**].

Both Robertson-Youngson and Størmer expressed the belief that the result is true in the case of a positive unital projection with contractive complement on a JB-algebra. In order to prove Theorem 2, we found it necessary to first prove the conjecture of Robertson-Youngson in the case of a JC-algebra.

As we have pointed out [**6**], a J^*-algebra is the appropriate algebraic model in which to study problems not involving order. The techniques developed by us in [**4**, **5**] can now be used to give a short solution of the bicontractive projection problem.

A simple analysis of this problem leads to a formulation of the conditional expectation properties proved in Theorem 1. As a corollary of Theorem 1 we obtain an analogue of the well known theorem of Tomiyama [**11**].

A *J^*-algebra* is a norm closed complex linear subspace of $\mathscr{L}(H, K)$, the bounded linear operators from a Hilbert space H to a Hilbert space K, which is closed under the operation $a \to aa^*a$. By setting $\{abc\} = \frac{1}{2}(ab^*c + cb^*a)$, one can make a J^*-algebra into a Jordan triple system.

We now recall some notation and results from [**4**, **5**] which will be used in this paper.

Let M be a J^*-algebra. For each f in M' let $v = v(f)$ be the unique partial isometry in M'' occurring in the enveloping polar decomposition of f [**4**: Th. 1]. Then $l(f) = vv^*$ and $r(f) = v^*v$ are projections in the von Neumann algebra A'', where A is any C^*-algebra containing M as a J^*-subalgebra. More generally, for any partial isometry v in M'', the *Peirce projections* are defined by $E(v)x = lxr$, $F(v)x = (1 - l)x(1 - r)$, $G(v)x = lx(1 - r) + (1 - l)xr$, where $l = vv^*$ and $r = v^*v$. We shall write $E(f)$ for $E(v(f))$ and similarly for $G(f)$ and $F(f)$.

The following commutativity formulas from [**4**] are fundamental: let Q be a contractive projection on the dual M' of a J^*-algebra M and let $f \in Q(M')$. Then

$$QE(f) = E(f)QE(f) \qquad ([\mathbf{4}\text{: Prop. 3.3}]); \tag{0.1}$$

$$G(f)Q = QG(f)Q \qquad ([\mathbf{4}\text{: Prop. 4.3}]); \tag{0.2}$$

$$E(f)Q = QE(f)Q \qquad ([\mathbf{4}\text{; Prop. 4.3}]). \tag{0.3}$$

Let Q be a contractive projection on the dual M' of M. By an *atom* of Q is meant any extreme point of the unit ball $Q(M')_1$ of $Q(M')$. The elements $v(f)$, f an atom of Q, are called *minimal tripotents* of Q'. Define

$$L_0 = \sup\{l(f) : f \text{ atom of } Q\}, \qquad R_0 = \sup\{r(f) : f \text{ atom of } Q\}.$$

Then L_0, R_0 are projections in A'' (where A is any C^*-algebra containing M as a J^*-subalgebra) and they define contractive projections $\mathscr{E}_0$ and $\mathscr{T}_0$ on A'' by $\mathscr{E}_0 z = L_0 z R_0$, $\mathscr{T}_0 z = (1 - L_0)z(1 - R_0)$, for $z \in A''$. We reserve the notation L_1, R_1, $\mathscr{E}_1$, $\mathscr{T}_1$ for the objects just defined in the case $Q = \mathrm{id}_{M'}$.

A fundamental result from [**5**] is the following decomposition of functionals with respect to a contractive projection Q [**5**: Theorem 1]

(0.4) Let Q be a contractive projection on the dual M' of a J^*-algebra M. Then $Q(M') = \mathscr{A} \oplus_{l_1} \mathscr{N}$, where $\mathscr{A}$ is the norm closed linear span of the atoms of Q and the unit ball of $\mathscr{N}$ has no extreme points. Moreover $\mathscr{A} = \mathscr{E}_0 Q(M')$ and $\mathscr{N} = \mathscr{T}_0 Q(M')$.

We shall use the following two consequences of this result (cf. [**5**: Cor. 4.4, Lemma 4.5, Prop. 4]).

(0.5) Let M_{fin} be the set of all finite linear combinations of pairwise orthogonal minimal tripotents of Q'. Then M_{fin} is σ-weakly dense in $\mathscr{E}_0 Q'(M'')$.

(0.6) For each x in $Q'(M'')$ we have $x = \mathscr{E}_0 x + \mathscr{T}_0 x$. Then by (0.5), $\mathscr{E}_0 x, \mathscr{T}_0 x \in M''$.

The following fact is a consequence of [**4**: Remark 2.5b] and [**5**: Lemma 4.5].

(0.7) For $x \in M''$, $\mathscr{E}_0 x = 0$ implies $\mathscr{E}_0 Q'x = 0$.

Finally we shall use the following, which is a consequence of [**4**: Remark 3.2]:

(0.8) Let P be a contractive projection on a J^*-algebra M, and let $f \in M'$. Then $E(f)M''$ is a JW^*-algebra and $E(f)P''$ restricted to $E(f)M''$ is a positive unital faithful projection.

1. Conditional expectation without order. In this section we prove the conditional expectation properties of an arbitrary contractive projection and prove the conjecture of Robertson-Youngson for JC-algebras.

THEOREM 1. *Let P be a contractive projection on a J^*-algebra M. Let a, $x \in M$ satisfy $Pa = a$, $Px = 0$. Then*

(i) $P\{aax\} = 0$;

(ii) $P\{axa\} = 0$.

Proof. (i) Let $b = \sum_{i=1}^{n} \alpha_i v_i \in M_{\text{fin}}$ with v_i orthogonal minimal tripotents of P''. We show first that $P''\{bbx\} = 0$. We have

$$\{bbx\} = \sum_{i,j} \alpha_i \bar{\alpha}_j \{v_i v_j x\} = \sum_i |\alpha_i|^2 \{v_i v_i x\},$$

and

$$P''\{v_i v_i x\} = P''\big(E(v_i) + \tfrac{1}{2}G(v_i)\big)x = P''\big(E(v_i) + \tfrac{1}{2}G(v_i)\big)P''x = 0$$

by (0.2) and (0.3). Thus $P''\{bbx\} = 0$ and by linearization we have $P''\{bcx\} = 0$ for $b, c \in M_{\text{fin}}$. By (0.6), $a = \mathscr{E}_0 a + \mathscr{T}_0 a$ so that

$$\{aax\} = \{\mathscr{E}_0 a, \mathscr{E}_0 a, x\} + \{\mathscr{T}_0 a, \mathscr{T}_0 a, x\}.$$

Set $\alpha_1 = \{\mathscr{E}_0 a, \mathscr{E}_0 a, x\}$, $\alpha_2 = \{\mathscr{T}_0 a, \mathscr{T}_0 a, x\}$. Since by Krein-Milman, $\|P\{aax\}\| = \|\mathscr{E}_0 P\{aax\}\|$, it suffices to prove $\mathscr{E}_0 P''\alpha_1 = \mathscr{E}_0 P''\alpha_2 = 0$. Since $\alpha_2 \in M''$ and $\mathscr{E}_0 \alpha_2 = 0$ we have $\mathscr{E}_0 P''\alpha_2 = 0$ by (0.7). On the other hand, with b_n a net in M_{fin} converging σ-weakly to $\mathscr{E}_0 a$, we have $\alpha_1 = \lim_n \lim_m \{b_n b_m x\}$ so that $P''\alpha_1 = 0$.

(ii) With $a = \mathscr{E}_0 a + \mathscr{T}_0 a$ we have $\{axa\} = \beta_1 + \beta_2 + 2\beta_3$ where $\beta_1 = \{\mathscr{E}_0 a, x, \mathscr{E}_0 a\}$, $\beta_2 = \{\mathscr{T}_0 a, x, \mathscr{T}_0 a\}$, $\beta_3 = \{\mathscr{E}_0 a, x, \mathscr{T}_0 a\}$. Since $\|P\{axa\}\| = \|\mathscr{E}_0 P\{axa\}\|$ it suffices to prove $\mathscr{E}_0 P''\beta_1 = \mathscr{E}_0 P''\beta_2 = \mathscr{E}_0 P''\beta_3 = 0$. Since $\beta_2, \beta_3 \in M''$ and $\mathscr{E}_0\beta_2 = \mathscr{E}_0\beta_3 = 0$, we have $\mathscr{E}_0 P''\beta_2 = \mathscr{E}_0 P''\beta_3 = 0$. We now prove that $P''\beta_1 = 0$. By the linearization and approximation argument in the proof of (i), it will suffice to prove $P''\{bxb\} = 0$ for $b \in M_{\text{fin}}$. Setting $b = \sum_{i=1}^n \alpha_i v_i$ with v_i orthogonal minimal tripotents of P'' shows that it suffices to prove that $P''\{vxu\} = 0$ whenever u, v are minimal tripotents of P'' which are either equal or orthogonal.

Let $w = u + v$ (or $w = v$ if $u = v$), let A be the JW^*-algebra $E(w)M''$ with identity element e, and let R be the unital contractive projection $E(w)P''$ on A. Let $z = \{vxu\}$. Since, by (0.3), $P''z = P''E(w)z = P''E(w)P''z = P''Rz$, it suffices to prove that $Rz = 0$. Let $y = E(w)x$ and note that $z = \{vxu\} = \{vyu\}$ and $y \in A$. Note also that e, v, $u \in R(A)$ and that by (0.1) $Ry = E(w)P''E(w)x = E(w)P''x = 0$. It is easy to verify that

$$\{ve\{uey\}\} = \{v\{eye\}u\} + \{v\{eue\}y\}$$

so that $z = \{vye\} = \{ve\{uey\}\} + \{vuy\}$. By (i) applied to R and A, $R(z) = 0$. □

By considering elements x of the form $z - Pz$, and linearizing we obtain:

COROLLARY 1. *Let P be a contractive projection on a J^*-algebra M. For $x, y, z \in M$,*

$$P\{Px, Py, Pz\} = P\{Px, Py, z\} = P\{Px, y, Pz\}.$$

We know from [5: Theorem 2] that $P(M)$ is a Jordan triple system isometric to a J^*-algebra. If $P(M)$ happens to be a J^*-subalgebra of M we obtain the following analogue of a well known of Tomiyama.

COROLLARY 2. *Let N be a J^*-subalgebra of a J^*-algebra M and let P be a norm one projection of M onto N. Then for $a, b \in N$ and $x \in M$,*

(i) $P\{abx\} = \{a, b, Px\}$,

(ii) $P\{axb\} = \{a, Px, b\}$.

We note that (ii) was proved for JB-algebras and unital P in [**8**: Appendix].

Our final corollary solves the problem of Robertson-Youngson in the important cases of a JC-algebra.

COROLLARY 3. *Let R be a unital bicontractive projection on a JC-algebra A. Then R has the form $Rx = \frac{1}{2}(x + \theta x)$ where θ is a Jordan automorphism of A of order* 2.

Proof. As remarked by Robertson-Youngson, such a θ exists if and only if we have the implication: $Ra = 0 \Rightarrow R(a^2) = a^2$. Since the complexification of A is a J^*-algebra we have, with $Q = \mathrm{id} - R$, $Q(a^2) = Q\{a, 1, a\} = 0$ since $Qa = a$ and $Q1 = 0$.

2. Solution of the bicontractive projection problem. In this section we prove the following, which solves the bicontractive projection problem for J^*-algebras.

THEOREM 2. *Let P be a bicontractive projection on a J^*-algebra M. Then there is a J^*-automorphism θ of M of order* 2 *such that*

$$Px = \tfrac{1}{2}(x + \theta x), \qquad x \in M. \tag{2.0}$$

Proof. Let P be a bicontractive projection on a J^*-algebra M and define θ by (2.0). We need only show that

$$\theta(xx^*x) = \theta x(\theta x)^*\theta x, \quad \text{for } x \in M. \tag{2.1}$$

Write $x = x_1 + x_2$, with $x_1 \in P(M)$ and $x_2 \in (\mathrm{id} - P)(M)$. Then $\theta x = x_1 - x_2$ and

$$xx^*x = x_1x_1^*x_1 + x_2x_2^*x_2 + 2\{x_1x_1x_2\} + 2\{x_2x_2x_1\} + x_1x_2^*x_1 + x_2x_1^*x_2, \tag{2.2}$$

$$\theta x(\theta x)^*\theta x = x_1x_1^*x_1 - x_2x_2^*x_2 - 2\{x_1x_1x_2\} + 2\{x_2x_2x_1\} - x_1x_2^*x_1 + x_2x_1^*x_2. \tag{2.3}$$

By Theorem 1 applied to P and $\mathrm{id} - P$ we have

$$P\{x_1x_1x_2\} = 0, \qquad P\{x_1x_2x_1\} = 0; \tag{2.4}$$

$$P\{x_2x_2x_1\} = \{x_2x_2x_1\}, \qquad P\{x_2x_1x_2\} = \{x_2x_1x_2\}. \tag{2.5}$$

Below we shall prove

(2.6) $P(M)$ and $(\mathrm{id} - P)(M)$ are J^*-subalgebras of M.

Applying $\theta = 2P - \mathrm{id}$ to (2.2) and using (2.4)–(2.6) we get (2.1).

Thus Theorem 2 will be proved if we can show that the range of a bicontractive projection on a J^*-algebra is a J^*-subalgebra. This will be done in Proposition 1 below, for which we prepare some lemmas.

We need two technical facts in order to prove Proposition 1. First, P'' fixes the atomic part of P'' (Lemma 4) and second, the decompositions $x = \mathscr{E}_0 x + \mathscr{T}_0 x$ of $x \in P''(M)$ and $x = \mathscr{E}_1 x + \mathscr{T}_1 x$ (defined in the introduction) coincide (Lemma 5). Lemmas 1 and 2 are preliminary to Lemma 3, which is needed to prove Lemma 5.

LEMMA 1. *Let A be a JW-algebra and let R be a normal unital bicontractive projection on A. Then $R(A)$ is a JW-subalgebra of A and if $R(A)$ is purely non-atomic then so is A.*

Proof. The fact that $R(A)$ is a JW-subalgebra follows from [**9**]. By Corollary 3, $R = \frac{1}{2}(\mathrm{id} + \theta)$ with θ a Jordan automorphism of A.

Suppose that φ is a multiple of a normal pure state of A. Then $\psi \equiv R'\varphi = \frac{1}{2}(\varphi + \theta'\varphi)$ is a purely atomic normal positive functional on A and can therefore be written as a linear combination of two orthogonal normal pure states of A. It follows that $E(\psi)A$ is a JW-algebra of rank ≤ 2. Now $E(\psi)R(A)$ is a JW-subalgebra of $E(\psi)A$, hence also of rank ≤ 2. Since ψ is in the range of $(E(\psi)R)'$ it can be written as a linear combination of two atoms of $R'E(\psi)$, which are atoms of R' by [**5**: Remark 1.3]. Since $R(A)$ is purely non-atomic we must have $\psi = 0$. But R is faithful, so $\varphi = 0$. □

In the lemmas that follow, P denotes a bicontractive projection on a J^*-algebra M.

LEMMA 2. *The atoms of P' lie in the convex hull of the extremal points of the unit ball M_1' of M'.*

Proof. Let f be an atom of P'. Let A be the JW-algebra which is the self-adjoint part of $E(f)M''$, and let $R = E(f)P''$ on A. By (0.8) and [**4**: Prop. 3.7], R is a unital bicontractive projection on A with $R(A) = \mathbf{R} \cdot 1_A$. According to [**9**: Prop. 2.6] there are three possible cases: $A = \mathbf{R} \cdot 1_A$, $A = \mathbf{R} \oplus \mathbf{R}$, $A =$ a spin factor. Therefore $E(f)M''$ is a Jordan algebra of rank ≤ 2, and so f is a convex combination of at most two extremal elements of $E(f)M'$, which, by [**5**: Remark 1.3] are extremal points of M_1'. □

We shall now use Lemmas 1 and 2 to show that the decomposition (0.4) of a functional in the image of P' coincides with the decomposition corresponding to the identity projection.

LEMMA 3. *For each φ in $P'(M')$ we have $\mathscr{E}_0\varphi = \mathscr{E}_1\varphi$ and $\mathscr{T}_0\varphi = \mathscr{T}_1\varphi$. Moreover*

$$\mathscr{T}_1 P' \mathscr{E}_0 = 0 \quad \text{and} \quad \mathscr{E}_1 P' \mathscr{T}_1 = 0. \tag{2.7}$$

Proof. Let $\varphi_1 = \mathscr{E}_0\varphi$, $\varphi_2 = \mathscr{T}_0\varphi$, and let $R = E(\varphi_2)P''$ restricted to $A = E(\varphi_2)M''$. By (0.8) $R(A)$ is a JW-subalgebra of A and by the definition of $\mathscr{T}_0\varphi$, $R(A) = E(\varphi_2)P''(M'')$ is purely non-atomic. By Lemma 1, A is purely non-atomic so that $\varphi_2 = \mathscr{T}_1\varphi_2$. On the other hand, by Lemma 2,

$$\mathscr{E}_0\varphi = \mathscr{E}_1\varphi_1 = \mathscr{E}_1(\varphi - \varphi_2) = \mathscr{E}_1\varphi - \mathscr{E}_1\mathscr{T}_1\varphi_2 = \mathscr{E}_1\varphi.$$

We now prove (2.7). Let $\varphi \in M'$, and write $\varphi = P'\varphi + (\mathrm{id} - P')\varphi$. Decompose $P'\varphi$ and $(\mathrm{id} - P')\varphi$ with respect to P' and $\mathrm{id} - P'$ respectively:

$$P'\varphi = \varphi_1 + \varphi_2, \quad (\mathrm{id} - P')\varphi = \psi_1 + \psi_2.$$

Then

$$\begin{aligned}\mathscr{T}_1 P' \mathscr{E}_1 \varphi &= \mathscr{T}_1 P' \mathscr{E}_1(\varphi_1 + \psi_1 + \varphi_2 + \psi_2) \\ &= \mathscr{T}_1 P'(\varphi_1 + \psi_1) = \mathscr{T}_1\varphi_1 = 0.\end{aligned}$$

A similar argument gives $\mathscr{E}_1 P' \mathscr{T}_1 = 0$. □

LEMMA 4. *Let v be a minimal tripotent of P''. Then $P''v = v$.*

Proof. By [**4**: Prop. 1], $P''v = v + b$ where $b = \mathscr{T}P''v$ and $\mathscr{T}$ is defined in [**4**: Intro.]. Since $b = \mathscr{T}b$, P'' vanishes on the J^*-algebra B generated by b. Since b is orthogonal to v, the J^*-algebra $J = \mathbf{C}v \oplus B$ generated by v and b is commutative in the sense of [**3**]. By restriction P'' is a bicontractive projection on J and so has the form $P''x = \frac{1}{2}(x + \theta x)$ for $x \in J$, where θ is a J^*-automorphism of J of order 2 [**3**: Prop. 3.3]. Now $\theta = -\mathrm{id}$ on B so $\theta(B) = B$ and therefore θv is orthogonal to B. Hence $\theta v = \lambda v$ and therefore $P''v = v$. □

LEMMA 5. *Let $x \in P''(M'')$. Then $\mathscr{E}_0 x = \mathscr{E}_1 x$, and $\mathscr{T}_0 x = \mathscr{T}_1 x$.*

Proof. Since $x = \mathscr{E}_0 x + \mathscr{T}_0 x$, we have $x = P''x = P''\mathscr{E}_0 x + P''\mathscr{T}_0 x$. by Lemma 4 and (0.5), $P''\mathscr{E}_0 x = \mathscr{E}_0 x$, whence $\mathscr{T}_0 x = P''\mathscr{T}_0 x$. Let $y = \mathscr{T}_0 x$. If $\psi \in M'$ is arbitrary,

$$\begin{aligned}\langle y, \psi\rangle &= \langle P''y, \psi\rangle = \langle y, P'\psi\rangle = \langle \mathscr{T}_0 x, \mathscr{E}_0 P'\psi + \mathscr{T}_1 P'\psi\rangle \\ &= \langle \mathscr{T}_0 x, \mathscr{T}_1 P'(\mathscr{E}_1\psi + \mathscr{T}_1\psi)\rangle = \langle y, \mathscr{T}_1 P'\mathscr{T}_1\psi\rangle = \langle \mathscr{T}_1 P''\mathscr{T}_1 y, \psi\rangle\end{aligned}$$

(we have used (2.7)). Therefore $y = \mathscr{T}_1 y$. We now have $\mathscr{T}_0 x = y = \mathscr{T}_1 y = \mathscr{T}_1\mathscr{T}_0 x = \mathscr{T}_1 x$, and thus $\mathscr{E}_0 x = \mathscr{E}_1 x$. □

PROPOSITION 1. *Let P be a bicontractive projection on a J^*-algebra M. Then $P(M)$ is a J^*-subalgebra of M.*

Proof. Let $x \in P(M)$. Write $x = \mathscr{E}_0 x + \mathscr{T}_0 x$. Then

$$xx^*x = \mathscr{E}_0 x(\mathscr{E}_0 x)^*\mathscr{E}_0 x + \mathscr{T}_0 x(\mathscr{T}_0 x)^*\mathscr{T}_0 x$$

and

$$P(xx^*x) = P''(\mathscr{E}_0 x(\mathscr{E}_0 x)^*\mathscr{E}_0 x) + P''(\mathscr{T}_0 x(\mathscr{T}_0 x)^*\mathscr{T}_0 x). \tag{2.8}$$

By (0.5) and Lemma 4, $P''(\mathscr{E}_0 x(\mathscr{E}_0 x)^*\mathscr{E}_0 x) = \mathscr{E}_0 x(\mathscr{E}_0 x)^*\mathscr{E}_0 x$. Also by Lemma 5 $P''(\mathscr{T}_0 x(\mathscr{T}_0 x)^*\mathscr{T}_0 x) = \mathscr{T}_1 P''(\mathscr{T}_0)x(\mathscr{T}_0 x)^*\mathscr{T}_0(x)$. Applying $\mathscr{E}_1$ to (2.8) therefore yields

$$\mathscr{E}_1 P(xx^*x) = \mathscr{E}_0 x(\mathscr{E}_0 x)^*\mathscr{E}_0 x = \mathscr{E}_1(xx^*x)$$

by Lemma 5. Since the map $y \to \mathscr{E}_1 y$ is isometric on M we have proved that $P(xx^*x) = xx^*x$. □

For any partial isometry v in a J^*-algebra, $P = E(v) + F(v)$ is a contractive projection by [**4**: Lemma 1.1]. Here, id $- P = G(v)$ is also contractive and $\theta = 2P - \mathrm{id} = E(v) + F(v) - G(v)$ is the symmetry defined by v (cf. [**5**: Lemma 3.1]).

Formula (ii) of Theorem 1 has been obtained recently for contractive projections on JB^*-triples by W. Kaup in a preprint "Contractive Projections on Jordan C^*-algebras and generalizations", using methods different from ours. In particular, this settles the Robertson-Youngson conjecture for JB-algebras.

The following question arises naturally in connection with Theorem 2: Let P_1, P_2, P_3 be contractive projections on a J^*-algebra M and suppose $P_1 + P_2 + P_3 = \mathrm{id}$. Does there exist a J^*-automorphism θ of order 3 such that

$$(2.9)\qquad \begin{cases} P_1x = (x + \theta x + \theta^2 x)/3 \\ P_2x = (x + \omega\theta x + \omega^2\theta^2 x)/3 \\ P_3x = (x + \omega^2\theta x + \omega\theta x)/3 \end{cases}$$

where $\omega = \exp(2\pi i/3)$?

The answer is easily verified to be yes for the Peirce projections $P_1 = E(v)$, $P_2 = G(v)$, $P_3 = F(v)$ of an arbitrary partial isometry v. The answer can also be shown to be yes for commutative J^*-algebras by using [**3**]. However, the answer is no in general. To see this note that (2.9) implies

$$(2.10)\qquad \theta = P_1 + \omega P_2 + \omega^2 P_3.$$

Now let M be the J^*-algebra of 2 by 2 complex matrices and for $x = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in M$, let

$$P_1x = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}, \quad P_2x = \begin{bmatrix} 0 & \frac{1}{2}(b+c) \\ \frac{1}{2}(b+c) & 0 \end{bmatrix},$$

$$P_3x = \begin{bmatrix} 0 & \frac{1}{2}(b-c) \\ \frac{1}{2}(c-b) & 0 \end{bmatrix}.$$

By (2.10),

$$\theta x = \begin{bmatrix} a & \frac{1}{2}(b+c)\omega + \frac{1}{2}(b-c)\omega^2 \\ \frac{1}{2}(b+c)\omega^2 + \frac{1}{2}(c-b)\omega & d \end{bmatrix}$$

and it follows that θ is not a J^*-automorphism, i.e., $\theta(x)\theta(x)^*\theta(x) \neq \theta(xx^*x)$ if $x = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ for example.

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