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NONOSCILLATORY FUNCTIONAL DIFFERENTIAL EQUATIONS

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Our aim in this paper is to obtain sufficient conditions under which certain functional differential equations have a "large" number of non-oscillatory solutions. Using the characteristic equation of a "majorant" delay differential equation with constant coefficients and Schauder's fixed point theorem, we obtain conditions under which the functional differential equation in question has a nonoscillatory solution. Then a known comparison theorem is employed as a tool to demonstrate that if the functional differential equation has a nonoscillatory solution, then it really has a "large" number of such solutions.

Our aim in this paper is to obtain sufficient conditions under which the functional differential equation

$$x'(t) + \sum_{i=1}^{n} p_i(t)x(t - \tau_i(t)) = 0 \tag{1}$$

has a "large" number of nonoscillatory solutions. It is to be noted that the literature is scarce concerning conditions under which there exist nonoscillatory solutions. Using the characteristic equation of a "majorant" delay differential equation with constant coefficients and Schauder's fixed point theorem, we obtain conditions under which (1) has a nonoscillatory solution. Then we employ a known comparison theorem [see **1**, p. 224, also **4**, Ch. 6] as a tool to demonstrate that if (1) has a nonoscillatory solution then it really has a "large" number of such solutions.

As it is customary, a solution is said to be oscillatory if it has arbitrarily large zeros. A differential equation is called oscillatory if all of its solutions oscillate; otherwise, it is called nonoscillatory. In this paper we restrict our attention to real valued solutions $x(t)$.

2. Non-oscillations.

THEOREM 1. *Consider the differential equation*

$$x'(t) + \sum_{i=1}^{n} p_i(t)x(t - \tau_i(t)) = 0 \tag{1}$$

where $p_i(t)$ *and* $\tau_i(t)$ *are continuous functions such that* $|p_i(t)| \le P_i$, $|\tau_i(t)| \le T_i$, $|p_i'(t)| \le A_i$ *and* $|\tau_i'(t)| \le B_i$, $i = 1, 2, \ldots, n$, *where* P_i, T_i, A_i *and* B_i *are positive constants. Assume that*

$$\lambda = \sum_{i=1}^{n} P_i e^{\lambda T_i} \tag{2}$$

has a positive root. Then equation (1) *has a nonoscillatory solution of the form*

$$x(t) = \exp\left(- \int_{t_0}^{t} \lambda(s)\, ds \right) \tag{3}$$

where $\lambda(t)$ *is a bounded continuous function.*

Proof. Suppose that λ_0 is a positive root of (2), i.e.,

$$\lambda_0 = \sum_{i=1}^{n} P_i e^{\lambda_0 T_i}.$$

We will prove that (1) has a nonoscillatory solution of the form (3). Substituuting (3) into (1) we obtain

$$\lambda(t) = \sum_{i=1}^{n} p_i(t) \exp\left(\int_{t-\tau_i(t)}^{t} \lambda(s)\, ds \right). \tag{4}$$

It suffices to show that (4) has a bounded solution. We will employ Schauder's fixed point theorem. Define the sets

$$X = \{ \lambda(t) : \text{ bounded continuous functions mapping } \mathbf{R} \text{ into } \mathbf{R} \}$$

with sup-norm, which is a Banach space, and

$$M = \{ \lambda(t) \in X : \|\lambda(t)\| \le \lambda_0 \}$$

which is a closed and convex subset of X. Consider the mapping F on M given by

$$F\lambda(t) = \sum_{i=1}^{n} p_i(t) \exp\left(\int_{t-\tau_i(t)}^{t} \lambda(s)\, ds \right).$$

Observe that

$$\begin{aligned} \|F\lambda(t)\| &\le \sum_{i=1}^{n} |p_i(t)| \exp\left(\left| \int_{t-\tau_i(t)}^{t} \|\lambda(s)\| ds \right| \right) \\ &\le \sum_{i=1}^{n} P_i e^{\lambda_0 T_i} = \lambda_0. \end{aligned}$$

Hence $F: M \to M$.

To show that (4) has a solution it suffices to show that the mapping F has a fixed point. To this end it remains to show that F is continuous and that FM is a relatively compact subset of X.

We will show that F is continuous by showing that each of the mappings

$$F_i\lambda(t) = \exp\left(\int_{t-\tau_i}^{t} \lambda(s)\, ds\right), \qquad i = 1, 2, \ldots, n,$$

is continuous. Let $\lambda_n \to \lambda$ where $\lambda_n, \lambda \in M$. Then

$$|F_i\lambda(t) - F_i\lambda(t)| = F_i\lambda(t)\left|\frac{F_i\lambda_n(t)}{F_i\lambda(t)} - 1\right|$$

$$= F_i\lambda(t)\left|\exp\left(\int_{t-\tau_i}^{t} [\lambda_n(s) - \lambda(s)]\, ds\right) - 1\right|.$$

But

$$\left|\int_{t-\tau_i(t)}^{t} [\lambda_n(s) - \lambda(s)]\, ds\right| \le \|\lambda_n - \lambda\| \cdot T_i \to 0 \quad \text{as } n \to \infty.$$

and because $F_i\lambda(t)$ is bounded, it follows that F_i is continuous.

To prove that FM is a relatively compact subset of X it suffices to prove that if K is a positive constant and λ is a function in X such that $\|\lambda\| \le K$, then $(F\lambda(t))'$ is uniformly bounded. We have

$$(F\lambda(t))' = \sum_{i=1}^{n} p_i'(t) \exp\left(\int_{t-\tau_i(t)}^{t} \lambda(s)\, ds\right)$$

$$+ \sum_{i=1}^{n} p_i(t)[\lambda(t) - \lambda(t - \tau_i(t))(1 - \tau_i'(t))]$$

$$\cdot \exp\left(\int_{t-\tau_i(t)}^{t} \lambda(s)\, ds\right)$$

and therefore

$$|(F\lambda(t))'| \le \sum_{i=1}^{n} A_i e^{KT_i} + \sum_{i=1}^{n} P_i K B_i e^{KT_i}.$$

Therefore Schauder's fixed point theorem applies and the proof is complete.

Note that the r.h.s. of (2) is a positive convex function of λ and so (2) has either two real roots, one real root, or no real root. Except in the case that all the P_i are zero, the roots are always positive. Thus (2) really just means $T_1,\ldots,T_n$, $P_1,\ldots,P_i$n are fairly small.

For the delay differential equation

$$(1)' \qquad x'(t) + \sum_{i=1}^{n} p_i x(t - \tau_i) = 0$$

whose coefficients and delays are positive constants, it has been proved [5], see also [3], that every solution oscillates if and only if the characteristic equation

$$(2)' \qquad \lambda + \sum_{i=1}^{n} p_i e^{-\lambda\tau_i} = 0$$

has no real roots. This is equivalent to saying that (1)′ has a nonoscillatory solution if and only if (2)′ has a real root.

The following are immediate corollaries of Theorem 1.

COROLLARY 1. *Equation* (1) *is nonoscillatory provided that the "majorant" delay differential equation*

$$(5) \qquad x'(t) + \sum_{i=1}^{n} P_i x(t - T_i) = 0,$$

where P_i and T_i are as defined in Theorem 1, *is nonocillatory.*

COROLLARY 2. *The functional differential equation with constant coefficients and constant arguments*

$$(6) \qquad x'(t) + \sum_{i=1}^{n} p_i x(t - \tau_i) = 0$$

is nonoscillatory provided that the delay differential equation

$$(7) \qquad x'(t) + \sum_{i=1}^{n} |p_i| x(t - |\tau_i|) = 0$$

is nonoscillatory.

3. A comparison theorem and its applications. Next we will demonstrate how the following comparison result [see **1**, p. 224, also 4, Ch. 6] may be used as a tool to establish that if a functional differential equation has a nonoscillatory solution then it has a "large" number of such solutions in a sense that will be made clear below.

THEOREM 2. (*Comparison Theorem.*) *Consider the delay differential equation*

$$(*)\qquad x'(t) + \sum_{i=1}^{n} p_i(t)x(t - \tau_i) = 0, \qquad t \geq 0, n \geq 1,$$

where $0 = \tau_0 < \tau_1 < \cdots < \tau_n = \tau$ *are constants,* $p_0, p_1, \ldots, p_n$ *are continuous functions and* $p_1(t), p_2(t), \ldots, p_n(t)$ *positive on* $[0, \infty)$. *Let* $\theta, \tilde{\theta}$: $[-\tau, 0) \to \mathbf{R}$ *be continuous and such that*

$$(8)\qquad \theta(t) < \tilde{\theta}(t) \quad \text{on } [-\tau, 0) \qquad \text{and} \qquad \theta(0) = \tilde{\theta}(0) > 0.$$

Let x *and* $\tilde{x}$ *be the unique solutions of* $(*)$ *with initial functions* θ *and* $\tilde{\theta}$ *respectively. Assume that*

$$(9)\qquad \tilde{x}(t) > 0 \quad \text{on } [0, \infty).$$

Then

$$(10)\qquad x(t) > \tilde{x}(t) \quad \text{on } (0, \infty).$$

REMARK 1. If we denote by $x(t, t_0, \theta)$ the unique solution of $(*)$ with initial function θ at $t = t_0$, then $x(t, t_0, -\theta) = -x(t, t_0, \theta)$. From this observation we obtain a dual to the above theorem by simply reversing the signs of the inequalities in (8), (9), and (10). That is, under the hypotheses of Theorem 2 we have, on $(0, \infty)$,

$$x(t, 0, \theta) > \tilde{x}(t, 0, \tilde{\theta}) > 0 \quad \text{and} \quad x(t, 0, -\theta) < \tilde{x}(t, 0 - \tilde{\theta}) < 0.$$

Finally a close look at the proof of the comparison theorem [see **1**, p. 224] shows that the functional arguments in $(*)$ do not have to be constants. The results is true if we assume tha $\tau_i(t)$ are continuous function satisfiying the following condition

$$(11)\qquad \begin{cases} \text{(i) } \tau_0(t) \equiv 0 \quad \text{and} \quad \tau_j(t) \not\equiv 0 \quad \text{for } j = 1, 2, \ldots, n; \\ \text{(ii) } \exists \tau > 0 \quad \text{such that } 0 \leq \tau_j(t) \leq \tau, \ j = 1, 2, \ldots, n. \end{cases}$$

First we apply the comparison theorem to the delay differential equation

$$(1)'\qquad x'(t) + \sum_{i=1}^{n} p_i x(t - \tau_i) = 0$$

where p_i and τ are positive constants. As discussed above $(1)'$ has a nonoscillatory solution provided that the characteristic equation

$$(2)'\qquad f(\lambda) \equiv \lambda + \sum_{i=1}^{n} p_i e^{-\lambda\tau_i} = 0$$

has a real root. The condition, for example,

$$\left(\sum_{i=1}^{n} p_i\right)\rho \le \frac{1}{e} \quad \text{where } \tau = \max\{\tau_1, \tau_2, \dots, \tau_n\} \tag{12}$$

implies that $f(0)f(-1/\tau) \le 0$ and therefore $(2)'$ has a real (negative) root in the interval $(-1/\tau, 0)$.

Now assume that $(2)'$ has a real root λ_0. Then $(1)'$ has the nonoscillatory solution

$$\mu e^{\lambda_0 t} \quad \text{for any } \mu \in \mathbf{R},\ \mu \ne 0.$$

But then, by the comparison theorem, any solution of $(1)'$ with initial function $\phi(t)$ satisfying

$$\phi(t) < \phi(0)e^{\lambda_0 t}, \qquad -\tau \le t < 0 \quad \text{and} \quad \phi(0) > 0$$

and any solution of $(1)'$ with initial function $\psi(t)$ satisfying

$$\psi(t) > \psi(0)e^{\lambda_0 t}, \qquad -\tau \le t < 0 \quad \text{and} \quad \psi(0) < 0$$

is nonoscillatory. In particular (and also when λ is not known) we have the following result.

COROLLARY 3. *Assume that* $(2)'$ *has a real root. Then any solution of* $(1)'$ *with initial function* ϕ *or* ψ *satisfying*

$$\phi(t) < \phi(0), \qquad -\tau \le t < 0 \quad \textit{and} \quad \phi(0) > 0$$

or

$$\psi(t) > \psi(0), \qquad -\tau \le t < 0 \quad \textit{and} \quad \psi(0) < 0$$

is nonoscillatory.

EXAMPLE 1. For the delay differential equation

$$x'(t) + \tfrac{1}{2}e^{-1/3}x\left(t - \tfrac{1}{3}\right) + \tfrac{1}{2}e^{-1/2}x\left(t - \tfrac{1}{2}\right) = 0 \tag{13}$$

condition (12) is satisfied. Therefore its characteristic equation

$$\lambda + \tfrac{1}{2}e^{-1/3-\lambda/3} + \tfrac{1}{2}e^{-1/2-\lambda/2} = 0 \tag{14}$$

has a real (negative) root in the interval $(-2, \infty)$. Observe that $\lambda = -1$ is a root of (14). Thus (13) has the nonoscillatory solution μe^{-t} for any $\mu \in \mathbf{R}$, $\mu \ne 0$. Also, using the comparison theorem, any solution of (13) with initial function ϕ or ψ satisfying

$$\phi(t) < \phi(0)e^{-t}, \qquad -\tau \le t < 0 \quad \text{and} \quad \phi(0) > 0$$

or

$$\psi(t) > \psi(0)e^{-t}, \qquad -\tau \le t < 0 \quad \text{and} \quad \psi(0) < 0$$

is nonoscillatory.

In view of Theorems 1 and 2 and Remark 1, we obtain the following result equation (1).

COROLLARY 4. *Consider the differential equation* (1) *subject to the hypotheses of Theorem* 1 *and in addition assume that* $p_i(t) > 0$, $i = 1, 2, \ldots, n$, *and condition* (11) *is satisfied. Then, any solution of* (1) *with initial function* ϕ *or* ψ *satisfying*

$$\phi(t) < \phi(0), \qquad -\tau \le t < 0 \quad \textit{and} \quad \phi(0) > 0$$

or

$$\psi(t) > \psi(0), \qquad -\tau \le t < 0 \quad \textit{and} \quad \psi(0) < 0$$

is nonoscillatory.

Finally we apply the comparison theorem to the delay differential equation

$$x'(t) + p(t)x(t - \tau) = 0, \qquad t \ge t_0, \tag{15}$$

where τ is a positive constant and $p(t)$ is a τ-periodic continuous function with

$$K \equiv \int_{t-\tau}^{t} p(s)\, ds \le \frac{1}{e}. \tag{16}$$

With these hypotheses equation (15) has a nonoscillatory solution of the form

$$x(t) = \exp\left(\lambda \int_{t_0}^{t} p(s)\, ds\right) \tag{17}$$

with $\lambda < 0$. In fact, substituting (17) into (15), we obtain

$$g(\lambda) \equiv \lambda e^{K\lambda} + 1 = 0.$$

It suffices to show that $g(\lambda)$ has a negative root.

Case 1. $K < 0$. Then $g(-\infty) = -\infty$ and $g(0) = 1$. Therefore $g(\lambda)$ has a root in $(-\infty, 0)$.

Case 2. $K = 0$. Then $\lambda = -1$ is a root

Case 3. $K > 0$. Then $g(-1/K) = (Ke - 1)/Ke \le 0$ and $g(0) = 1$. Therefore $g(\lambda)$ has a root in $[-1/K, 0)$.

Thus in each case (15) has a nonoscillatory solution of the form given by (17). If in addition to (16) we assume that $p(t) > 0$ then the comparison theorem applies and we have the following result.

COROLLARY 5. *Consider the differential equation* (15) *under the assumptions that* $p(t) > 0$ *and* (16) *holds. Then the solution of* (15) *with initial function* ϕ *and* ψ *satisfying*

$$\phi(t) < \phi(t_0), \qquad t_0 - \tau \le t < t_0 \quad \textit{and} \quad \phi(t_0) > 0$$

or

$$\psi(t) > \psi(t_0), \qquad t_0 - \tau \le t < t_0 \quad \textit{and} \quad \psi(t_0) < 0$$

is nonoscillatory.

EXAMPLE 2. Consider the differential equation

$$x'(t) + (\sin t)x(t - 2\pi) = 0, \qquad t \ge 0.$$

Observe that $\sin t$ is a 2π-periodic function and condition (15) is satisfied, with $K = 0$. Note that $e^{\cos t}$ is a multiple of the nonoscillatory solution given by (17).

REMARK 2. When $p(t) > 0$ the condition $K > 1/e$ implies, see [**2**], that every solution of (15) oscillates. This is our motivation for the following

Conjecture. If $K > 1/e$ then (15) is oscillatory.

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