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WHEN THE CONTINUUM HAS COFINALITY ω_1

ARNOLD WILLIAM MILLER AND KAREL LIBOR PRIKRY

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In this paper we consider models of set theory in which the continuum has cofinality ω_1 . We show that it is consistent with $\neg\text{CH}$ that for any complete boolean algebra \mathbf{B} of cardinality less than or equal to c (continuum) there exists an ω_1 -generated ideal J in $P(\omega)$ (power set of ω) such that \mathbf{B} is isomorphic to $P(\omega) \text{ mod } J$. We also show that the existence of generalized Luzin sets for every ω_1 -saturated ideal in the Borel sets does not imply Martin's axiom.

Introduction. In §1 we prove our main result that it is consistent with $\neg\text{CH}$ that every complete boolean algebra of cardinality $\leq c$ is isomorphic to $P(\omega) \text{ mod } J$ for some J ω_1 -generated. We think of this as generalizing Kunen's theorem that it is consistent with $\neg\text{CH}$ that there is an ω_1 generated nonprincipal ultrafilter on ω .

For I an ideal in the Borel subsets of the reals we say that a set of reals X is a κ - I -Luzin set iff X has cardinality κ and for every A in I , $A \cap X$ has cardinality less than κ . If c is regular, then it follows easily from Martin-Solovay [9] that MA is equivalent to the statement "for every ω_1 -saturated σ -ideal I in the Borels there is a c - I -Luzin set". In §2 we show that the regularity of c is necessary. This answers a question of Fremlin [5].

We also show that it is consistent with $\neg\text{CH}$ that for every such I there exists an ω_1 - I -Luzin set. These results can be thought of as a weak form of the following conjecture.

Conjecture. It is consistent with $\neg\text{CH}$ that for every c.c.c partial order \mathbf{P} of cardinality $\leq c$ there exist $\langle G_\alpha: \alpha < \omega_1 \rangle$ an ω_1 -sequence of \mathbf{P} -filters such that for every dense $D \subseteq \mathbf{P}$ all but countably many G_α meet D .

Note that this is a trivial consequence of CH.

Next we give a result of Kunen that some restriction of the cardinality of \mathbf{P} (e.g. $(2^{\omega_1})^+$) is necessary in our conjecture. We also show that for every c.c.c. \mathbf{P} of cardinality $\leq \omega_2$ we can force (without adding reals) the existence of \mathbf{P} -filters $\langle G_\alpha: \alpha < \omega_1 \rangle$ eventually meeting each dense subset of \mathbf{P} .

1. ω_1 -generated ideals in $P(\omega)$. Sikorski [12] showed that every complete boolean algebra of cardinality $\leq c$ is isomorphic to $P(\omega)/J$ for some ideal J . Kunen (see [7], p. 289) showed it is consistent with $\neg\text{CH}$ that there exists a nonprincipal ω_1 -generated ultrafilter U on ω , i.e. $P(\omega)$ mod the dual of U is the two element boolean algebra.

THEOREM 1. *It is consistent with $\text{ZFC} + \neg\text{CH}$ that for every complete boolean algebra \mathbf{B} of cardinality $\leq c$ there exists an ω_1 generated nonprincipal ideal I such that \mathbf{B} is isomorphic to $P(\omega)/I$.*

Proof. We begin by describing the model which will be used here and in the next section. Let M_0 be a countable transitive model of $\text{ZFC} + \text{GCH}$. Using the usual finite support forcing do an ω_1 iteration where at step $\alpha < \omega_1$ obtain $M_{\alpha+1}$ a model of $\text{MA} + c = \aleph_{\alpha+2}$. For $\alpha < \omega_1$ a limit, M_α just models $c = \aleph_{\alpha+1}$ but not MA . Finally M_{ω_1} models that $c = \aleph_{\omega_1}$ and is an ω_1 limit of models of MA . This model (or one very similar to it) was used by Stepran [13] and Bell and Kunen [2]. A similar ω_1 -iteration (without increasing c) was done by van Douwen and Fleissner [4] and also Roitman [10].

We will need the following two lemmas of Sikorski:

LEMMA 1.1. (Sikorski [12] 33.1, p. 141). *Suppose \mathbf{B} is a complete boolean algebra, and C_0 is a subalgebra of a boolean algebra C (neither of which need be complete). Then any homomorphism from C_0 into \mathbf{B} can be extended to a homomorphism of C into \mathbf{B} .*

LEMMA 1.2. (Sikorski [12] 12.2, p. 36). *Suppose A_0 generates a Boolean algebra A and $h: A_0 \rightarrow B$ is an arbitrary map into a boolean algebra B . Then h extends to a homomorphism from A into B iff for every sequence a_1, a_2, \dots, a_n from A_0 and sequence $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ of signs $+, -$;*

$$\varepsilon_1 a_1 \wedge \dots \wedge \varepsilon_n a_n = 0 \Rightarrow \varepsilon_1 h(a_1) \wedge \varepsilon_2 h(a_2) \wedge \dots \wedge \varepsilon_n h(a_n) = 0.$$

We will be using the proof of the following lemma so we include it here.

LEMMA 1.3. (Sikorski) *If \mathbf{B} is any complete boolean algebra of cardinality $\leq c$, then there exists an ideal J in $P(\omega)$ such that \mathbf{B} is isomorphic to $P(\omega)/J$.*

Proof. Let \mathcal{F} be a family of c independent subsets of ω (see Kunen [7], p. 257). that is given any finite sequences A_1, A_2, \dots, A_n and $B_1, B_2, B_3, \dots, B_n$ of distinct elements of \mathcal{F}

$$A_1 \cap A_2 \cap \dots \cap A_n \cap (\omega \setminus B_1) \cap (\omega \setminus B_2) \cap \dots \cap (\omega \setminus B_m)$$

is infinite. Let h be any map from \mathcal{F} onto \mathbf{B} . By Lemma 1.2 h extends to the subalgebra of $P(\omega)$ generated by \mathcal{F} and by Lemma 1.1, h extends to $P(\omega)$. J is just the kernel of this extension. \square

LEMMA 1.4. (*Martin-Solovay, Sikorski*) *Every c.c.c. complete boolean algebra of cardinality $\leq c$ is isomorphic to F/I where F is a σ -subfield of the Borel subsets of 2^ω and I is a c.c.c. σ -ideal.*

Proof. Theorem 2.3, page 155 of Martin Solovay [9] states that every c.c.c. complete boolean algebra of cardinality $\leq c$ is a complete subalgebra of a countably generated complete boolean algebra satisfying the c.c.c. According to Sikorski [12], 31.6 page 136, every countably generated σ -boolean algebra is isomorphic to $\text{Borel}(2^\omega)/I$ for some σ -ideal I . So the given algebra \mathbf{B} is isomorphic to a subalgebra of $\text{Borel}(2^\omega)/I$ for some I a c.c.c. σ -ideal. Now let F be a σ -subfield of $\text{Borel}(2^\omega)$ so that \mathbf{B} is isomorphic to F/I . \square

Note that in the model M_{ω_1} , $2^{\omega_1} > c$ and so any complete boolean algebra of cardinality $\leq c$ must have the c.c.c.. For $\alpha < \omega_1$ let \Vdash_α denote the forcing which has the ground model M_α and as the generic extension M_{ω_1} . Let \mathbf{B} be any complete boolean algebra of cardinality $\leq c$ in M_{ω_1} and suppose

$$\Vdash_0 \text{“} \mathbf{B} \cong F/I \text{ where } F \text{ is a } \sigma\text{-subfield of Borel}(2^\omega) \text{ and } I \text{ is a c.c.c. } \sigma\text{-ideal in Borel}(2^\omega)\text{.”}$$

For $\alpha < \omega_1$ define (in M_α)

$$F_\alpha = \{ A \in \text{Borel}(2^\omega)^{M_\alpha} : \Vdash_\alpha \text{“} A \in F \text{”} \}$$

and

$$I_\alpha = \{ A \in \text{Borel}(2^\omega)^{M_\alpha} : \Vdash_\alpha \text{“} A \in I \text{”} \}.$$

LEMMA 1.5. (*Kunen*) *(In M_α) F_α/I_α is complete and I_α has the c.c.c.*

Proof. Clearly F_α is a σ -field and I_α is a σ -ideal so it suffices to show I_α has the c.c.c.. Suppose $\langle A_\beta: \beta < \omega_1 \rangle \in M_\alpha$ and for all $\beta \neq \gamma$, $A_\beta \cap A_\gamma \in I_\alpha$. Then since

$$\Vdash_\alpha \text{“} I \text{ is c.c.c. ”,}$$

$I_\alpha \subseteq I$, and \Vdash_α is c.c.c. forcing, for all but countably many β

$$\Vdash_\alpha \text{“} A_\beta \in I \text{”}$$

and thus $A_\beta \in I_\alpha$. □

Working in M_{ω_1} we build a sequence of functions

$$h_\alpha: P(\omega) \cap M_\alpha \rightarrow F_\alpha$$

with $h_\alpha \in M_\alpha$ and such that for $\alpha < \beta$, h_β is an extension of h_α . They also have the following properties:

(i) the map $\hat{h}_\alpha: P(\omega) \cap M_\alpha \rightarrow F_\alpha/I_\alpha$ defined $\hat{h}_\alpha(A) = [h_\alpha(A)]_{I_\alpha}$ is a homomorphism;

(ii) for successor ordinals, $\alpha + 1$, $h_{\alpha+1}$ is onto $F_{\alpha+1}$; and

(iii) for successor ordinals, $\alpha + 1$, there exists X_α in the kernel of $\hat{h}_{\alpha+1}$ such that for all $A \in \text{kernel}(\hat{h}_\alpha)$ $A \subseteq {}^* X_\alpha$ (i.e. $A \setminus X_\alpha$ is finite).

Now suppose we already had the h_α and X_α as above and let us finish the proof of Theorem 1. Working in M_{ω_1} define

$$h: P(\omega) \rightarrow F$$

by $h = \bigcup_{\alpha < \omega_1} h_\alpha$ and let $\hat{h}: P(\omega) \rightarrow F/I$ be defined by $\hat{h}(A) = [h(A)]_I$. It follows from (i) that \hat{h} is a homomorphism. Since $F = \bigcup_{\alpha < \omega_1} F_\alpha$ it follows from (ii) that \hat{h} is onto. Since $\text{kernel } \hat{h} = \bigcup_{\alpha < \omega_1} \text{kernel}(h_\alpha)$ and by (iii) it is ω_1 -generated. Hence F/I is isomorphic to $P(\omega)/J$ for J ω_1 -generated. Now we indicate how to construct the h_α . Let h_0 be any map from $P(\omega) \cap M_0$ into F_0 such that \hat{h}_0 is a homomorphism from $P(\omega) \cap M_0$ into F_0/I_0 and the kernel of \hat{h}_0 contains the finite sets. E. g. let U be any nonprincipal ultrafilter in M_0 and define $h_0(A) = 2^\omega$ if $A \in U$ and $h_0(A) = \emptyset$ if $A \notin U$. Suppose $\alpha < \omega_1$ is a limit ordinal and $\langle h_\beta: \beta < \alpha \rangle \in M_\alpha$. Let $Q = \bigcup_{\beta < \alpha} (P(\omega) \cap M_\beta)$ and let $h: Q \rightarrow F_\alpha$ be defined by $h = \bigcup_{\beta < \alpha} h_\beta$. Let $\hat{h}: Q \rightarrow F_\alpha/I_\alpha$ be defined by $\hat{h}(A) = [A]_{I_\alpha}$. Since $\bigcup_{\beta < \alpha} I_\beta \subseteq I_\alpha$, the I_β are increasing, and the \hat{h}_β are homomorphism, \hat{h} is also a homomorphism.

Working in M_α we see that by Lemma 1.5, F_α/I_α is complete, and by Lemma 1.1 there exists (in M_α) a homomorphism $\hat{h}_\alpha: P(\omega) \cap M_\alpha \rightarrow F_\alpha/I_\alpha$ which extends \hat{h} . Since the I_α equivalence classes are bigger than the I_β for $\beta < \alpha$ we can pick a representing function $h_\alpha: P(\omega) \cap M_\alpha \rightarrow F_\alpha$ for \hat{h}_α

which extends h . This does the limit case. Now we do the successor case. Suppose we have $h_\alpha: P(\omega) \cap M_\alpha \rightarrow F_\alpha$ with $h_\alpha \in M_\alpha$. Note that $|P(\omega) \cap M_\alpha| = \aleph_{\alpha+1}$ and $M_{\alpha+1}$ is a model of MA and $c = \aleph_{\alpha+2}$. Let P be the kernel of h_α , i.e. $P = h_\alpha^{-1}(I_\alpha)$. Let $Q = (P(\omega) \cap M_\alpha) \setminus P$. Note that for any $B \in Q$ and finite $K \subseteq P$ we have that $B \cap (\cup K)$ is infinite. By Solovay's almost disjoint forcing (see Rudin [11]) in $M_{\alpha+1}$ there exists $Y \subseteq \omega$ such that for all $B \in Q$, $B \cap Y$ is infinite and for all $B \in P$, $B \cap Y$ is finite. Let $X_\alpha = \omega \setminus Y$. Note that for all A in the kernel of \hat{h}_α , $A \subseteq {}^* X_\alpha$ and for all B not in the kernel $B \setminus X_\alpha$ is infinite. Define $\hat{h}: (P(\omega) \cap M_\alpha) \cup \{X_\alpha\} \rightarrow F_\alpha/I_\alpha$ by extending \hat{h}_α and letting $\hat{h}(X_\alpha) = 0$. Let us use Lemma 1.2 to check that \hat{h} extends to a homomorphism from $\mathcal{B} =$ smallest boolean algebra generated by $(P(\omega) \cap M_\alpha) \cup \{X_\alpha\}$ into F_α/I_α . Since $P(\omega) \cap M_\alpha$ is a boolean algebra it is enough to check for each $A \in P(\omega) \cap M_\alpha$:

- (i) if $A \cap X_\alpha = \emptyset$, then $\hat{h}_\alpha(A) \wedge 0 = 0$; and
- (ii) if $A \cap (\omega \setminus X_\alpha) = \emptyset$, then $\hat{h}_\alpha(A) \wedge \mathbf{1} = 0$.

But (i) is trivial and (ii) follows from the choice of X_α . Next we use the method of independent sets to make $\hat{h}_{\alpha+1}$ onto $F_{\alpha+1}/I_{\alpha+1}$. Using MA, in $M_{\alpha+1}$ there exists a family \mathcal{F} of cardinality $c = \aleph_{\alpha+2}$ of independent mod Q subsets of ω . That is, for any infinite $A \in Q$ and distinct $Y_1, Y_2, \dots, Y_n, Z_1, Z_2, \dots, Z_m$ from \mathcal{F}

$$A \cap Y_1 \cap Y_2 \cap \dots \cap Y_n \cap (\omega \setminus Z_1) \cap (\omega \setminus Z_2) \cap \dots \cap (\omega \setminus Z_m)$$

is infinite. Construct the family \mathcal{F} by induction using the easy consequence of MA that for any family H of infinite subsets of ω with $|H| < c$ there exists $X \subseteq \omega$ such that for all $A \in H$, $A \cap X$ and $A \setminus X$ are both infinite. Choose $h: \mathcal{B} \rightarrow F_\alpha$ so that $\hat{h}(A) = [h(A)]_{I_\alpha}$ and h extends h_α . Let $k: \mathcal{B} \cup \mathcal{F} \rightarrow F_{\alpha+1}$ extend h and take \mathcal{F} onto $F_{\alpha+1}$. by Lemmas 1.1 and 1.2 there exists $h_{\alpha+1}: P(\omega) \cap M_{\alpha+1} \rightarrow F_{\alpha+1}$ which extends k and $\hat{h}_{\alpha+1}: P(\omega) \cap M_{\alpha+1} \rightarrow F_{\alpha+1}/I_{\alpha+1}$ is a homomorphism. This concludes the construction of the h_α 's and thus the proof of Theorem 1. □

One question we were unable to answer with this method is the following: Is it consistent with ZFC that there exists an ω_1 -generated ideal J in $P(\omega)$ such that $P(\omega_1)$ is isomorphic to $P(\omega)/J$?

Finally, we remark that in M_{ω_1} the measure algebra (the Borel sets modulo the sets of Lebesgue measure zero) has density ω_1 , i.e. there is a collection D of ω_1 sets of positive measure such that every set of positive measure contains one from D . This follows from the fact that under MA

given any collection F of sets of positive measure such that $|F| < c$, there exists a countable collection $\{C_n: n < \omega\}$ of sets of positive measure such that for every $A \in F$ there exists $n < \omega$ such that $C_n \subseteq A$. Thus in the model M_{ω_1} there exists an atomless finitely additive measure μ on $P(\omega)$ and a family $F \subseteq P(\omega)$ of cardinality ω_1 such that for all $X \subseteq \omega$ and $\varepsilon > 0$ there exists $X_0, X_1 \in F$ such that $X_0 \subseteq X \subseteq X_1$ and

$$\mu(X_1 \setminus X_0) < \varepsilon.$$

2. Luzin sets. Recall that for an ideal I in the Borel sets and a cardinal κ , a set of reals X is called κ - I -Luzin iff X has cardinality κ and every set in I meets X in a set of cardinality strictly less than κ .

THEOREM 2.1. *In the model of ZFC, M_{ω_1} of §1 (in which the continuum, c , is \aleph_{ω_1}), for any nontrivial c.c.c. σ -ideal I in the Borel sets there are both c - I -Luzin sets and ω_1 - I -Luzin sets.*

Proof. Letting $B_\alpha = \text{Borel}^{M_\alpha}$ and as before $I_\alpha = \{A \in B_\alpha: \Vdash_\alpha \text{“}A \in I\text{”}\}$ we know by Lemma 1.5 that in M_α , B_α/I_α has the c.c.c.. Since each $M_{\alpha+1}$ is a model of MA, by Martin-Solovay [9] the union $\bigcup I_\alpha$ cannot cover the reals of $M_{\alpha+1}$ (since $I_\alpha \subseteq I_{\alpha+1}$ a c.c.c. ideal of $M_{\alpha+1}$). Consequently we can find X_α of cardinality $\aleph_{\alpha+2}$ in $M_{\alpha+1}$ with

$$X_\alpha \cap \bigcup I_\alpha = \emptyset.$$

Then $X = \bigcup_{\alpha < \omega_1} X_\alpha$ is c - I -Luzin in M_{ω_1} . □

For \mathbf{P} a partial order define $\langle G_\alpha: \alpha < \omega_1 \rangle$ a sequence of \mathbf{P} -filters to be an ω_1 -generic sequence for \mathbf{P} iff for all dense $D \subseteq \mathbf{P}$ all but countably many G_α meet D . This is motivated by van Douwen and Fleissner [4].

Question. In the model M_{ω_1} is it true that for all \mathbf{P} c.c.c. of cardinality $\leq c$ there exists an ω_1 -generic sequence for \mathbf{P} ?

Note that by a result of Martin and Solovay [9] it is enough to prove the above for \mathbf{P} of the form Borel/I for I a c.c.c. σ -ideal. The difficulty with the above arguments is that filters on B_α/I_α may not lift to filters on B/I since elements of B_α/I_α may turn out to be in I . When we were first considering this question it was not clear to us that any restriction of the cardinality of \mathbf{P} is necessary. The following theorem of Kunen shows that there is. $\text{FIN}(\kappa)$ is the partial order of functions whose domain is a finite subset of κ and whose range is $\{0, 1\}$.

THEOREM 2.2. (*Kunen*) *If $\kappa \geq (2^{\omega_1})^+$ there is no ω_1 -generic sequence for $\text{FIN}(\kappa)$.*

Proof. Suppose $\langle G_\alpha: \alpha < \omega_1 \rangle$ is given with each $G_\alpha: \kappa \rightarrow 2$. For each $\lambda < \kappa$ define $H_\lambda: \omega_1 \rightarrow 2$ by $H_\lambda(\alpha) = G_\alpha(\lambda)$. Since $\kappa \geq (2^{\omega_1})^+$ there exists an infinite $\Sigma \subseteq \kappa$ such that $H_\lambda = H_{\lambda'}$ for each $\lambda, \lambda' \in \Sigma$. Now choose $i \in \{0, 1\}$ and an uncountable $\Gamma \subseteq \omega_1$ so that for each $\lambda \in \Sigma$, $H_\lambda \upharpoonright \Gamma$ is constantly i . But this means that for each $\alpha \in \Gamma$, $G_\alpha \upharpoonright \Sigma$ is constant. But then the G_α do not eventually meet the dense set $D = \{ p \in \text{FIN}(\kappa): \exists \lambda \in \Sigma p(\lambda) \neq i \}$.

It is clear from the above proof that all we need is that

$$\binom{\kappa}{\omega_1} \rightarrow \binom{\omega}{\omega_1}_2^{1,1}$$

to see that there are no ω_1 -generic sequences for $\text{FIN}(\kappa)$. This partition relation is known to be consistent with GCH for $\kappa = \omega_2$ (see Laver [8]) assuming the consistency of a huge cardinal.

Theorem 2.2 can be strengthened to: if $\kappa > 2^{\omega_1}$ there is no family $\langle G_\alpha: \alpha < \omega_1 \rangle$ of filters in $\text{FIN}(\kappa)$ such that every dense set meets some G_α . This was pointed out by D. H. Fremlin. To see how to prove this let

$$G = \bigcup_{\alpha < \omega_1} G_\alpha.$$

If for every $p \in \text{FIN}(\kappa)$ there is some $q \leq p$ with $q \neq G$, then G misses some dense set. Consequently there is some condition in $\text{FIN}(\kappa)$ such that G contains everything beneath it. But it is well known that the compact space 2^κ has density $> \omega_1$ (see Juhasz [6] 6.8, p.68).

Our next theorem goes in the other direction.

THEOREM 2.3. *Assume CH. Suppose that \mathbf{P} is a c.c.c. partial order of cardinality ω_2 . Then there exists \mathbf{Q} an ω_2 c.c. order which is countably closed and has the same cardinality as \mathbf{P} which adds an ω_1 -generic sequence for \mathbf{P} .*

Proof. We can assume without loss of generality that \mathbf{P} is a boolean algebra. Elements of \mathbf{Q} have the form

$$\langle \langle H_\alpha: \alpha < \beta \rangle, \mathcal{D} \rangle$$

where $\beta < \omega_1$, \mathcal{D} is a countable family of dense subsets of \mathbf{P} , and each H_α

is a countably generated **P** filter. The order on **Q** is defined as follows:

$$\langle \langle \hat{H}_\alpha : \alpha < \hat{\beta} \rangle, \hat{\mathcal{D}} \rangle \leq \langle \langle H_\alpha : \alpha < \beta \rangle, \mathcal{D} \rangle$$

iff $\hat{\beta} \geq \beta$, $\hat{\mathcal{D}} \supset \mathcal{D}$, $\hat{H}_\alpha \supset H_\alpha$ for all $\alpha < \beta$, and for all γ with $\beta \leq \gamma < \hat{\beta}$ and $D \in \mathcal{D}$, $D \cap \hat{H}_\gamma \neq \emptyset$.

It is easy to check that Q is countably closed and that Q adds an ω_1 -generic sequence for **P**. Now let us see that it has the ω_2 chain condition. By an obvious argument it is enough to show that Q^* has ω_2 -c.c. where

$$Q^* = \{ \langle H_n : n < \omega \rangle : \text{each } H_n \text{ is a countably generated } \mathbf{P}\text{-filter} \}$$

ordered by

$$\langle \hat{H}_n : n < \omega \rangle \leq \langle H_n : n < \omega \rangle \quad \text{iff for all } n, \hat{H}_n \supset H_n.$$

Suppose $\{ P_\alpha = \langle H_n^\alpha : n < \omega \rangle : \alpha < \omega_2 \}$ are pairwise incompatible. And let for each n and α

$$H_n^\alpha = \{ q \in \mathbf{P} : \exists m < \omega \ q < p_m^{\alpha,n} \}$$

where $p_{m+1}^{\alpha,n} \leq p_m^{\alpha,n}$ is a descending sequence.

Thus for each $\alpha \neq \beta$ there exists n and m such that

$$p_m^{\alpha,n} \wedge p_m^{\beta,n} = 0.$$

by the Erdos-Rado Theorem ($\omega_2 \rightarrow (\omega_1)_\omega^2$) (see Kunen [7], p. 290) there exists n_0 and m_0 and $X \in [\omega_2]^{\omega_1}$ such that for all $\alpha \neq \beta \in X$ $p_{m_0}^{\alpha,n_0} \wedge p_{m_0}^{\beta,n_0} = 0$. This contradicts the countable chain condition for **P**. \square

This forcing can probably be iterated to take care of all c.c.c. **P** of cardinality ω_2 . Unfortunately this would blow up 2^{ω_1} to ω_3 . A better way would be to try to deduce the existence of ω_1 -generic sequences from morasses with built-in \diamond . The “black box” theorems of Velleman and Shelah-Stanley do not seem to apply because of the lack of homogeneity.

Question. In L does every c.c.c. partial order of cardinality ω_2 have an ω_1 -generic sequence?

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