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S. S. Khare

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Conner and Floyd proved that if $\mathbf{Z}_2^k$ acts on a closed manifold M differentiably and without any fixed point, then M is a boundary. Stong gave a stronger result proving that if (M, θ) is a closed $\mathbf{Z}_2^k$-differential manifold with no stationary point, then (M, θ) is a $\mathbf{Z}_2^k$-boundary. In the present note, we discuss this problem for a finite group in detail. Let G be a finite group. By the 2-central component $G_2(C)$ of G, we will mean the subgroup of G consisting of the identity element and all the elements of order 2 in the center of G. We prove in this note that the fixed data of the 2-central component $G_2(C)$ of G determines G-bordism.

1. Preliminaries. Throughout the note we will take G to be a finite group. By a G-manifold we will mean a differential compact manifold with a differential action of G on it. A family $\mathscr{F}$ in G is a collection of subgroups of G such that if $H \in \mathscr{F}$, then all the subgroups of H and all the conjugates of H are in $\mathscr{F}$. Let $\mathscr{F}' \subset \mathscr{F}$ be families in G such that $\exists$ a central element a in G of order 2 such that

(i) $a \notin H, \forall H \in \mathscr{F} - \mathscr{F}'$

(ii) $H \in \mathscr{F}' \Rightarrow [H \cup \{a\}] \in \mathscr{F}'$

(iii) The intersection S of all members of $\mathscr{F} - \mathscr{F}'$ is in $\mathscr{F} - \mathscr{F}'$. We call such a pair $(\mathscr{F}, \mathscr{F}')$ of families an admissible pair of families in G with respect to $a \in G$.

EXAMPLE 2.1. Let G be a finite group. We can write the 2-central component $G_2(C)$ as $\mathbf{Z}_2^r = [t_1, \dots, t_r]$, where $t_1, \dots, t_r$ are generators of $\mathbf{Z}_2^r$ with $t_i^2 =$ the identity element and $t_i t_j = t_j t_i$. Let $\mathscr{F}_k$ be the family of all subgroups of G not containing $\mathbf{Z}_2^k$, $0 < k \leq r$, where $\mathbf{Z}_2^k$ denotes the subgroup of G generated by the first k generators $t_1, \dots, t_k$. Then $(\mathscr{F}_{k+1}, \mathscr{F}_k)$ is an admissible pair with respect to t_{k+1}, $0 < k < r$.

2. Stationary point free action of $G_2(C)$ and G-bordism. The object of this section is to show that if (M, θ) is a G-manifold with the stationary point free action of $G_2(C)$ then (M, θ) is G-boundary. Following the notation of Stong [2], let $\mathfrak{N}_*(G; \mathscr{F}, \mathscr{F}')$ denote the $(\mathscr{F}, \mathscr{F}')$-free G-bordism group for a pair $(\mathscr{F}, \mathscr{F}')$ of families in G. For a given family $\mathscr{F}$

in G and an element g in G, let $\mathscr{F}_g$ denote the smallest family in G consisting of all subgroups $[H \cup \{g\}]$, $H \in \mathscr{F}$.

THEOREM 3.1. *If* $(\mathscr{F}, \mathscr{F}')$ *is an admissible pair of families in* G *with respect to* a *in* G, *then an* $(\mathscr{F}, \mathscr{F}')$*-free element in* $\mathfrak{N}_*(G, \mathscr{F}, \mathscr{F}')$ *is zero in* $\mathfrak{N}_*(G; \mathscr{F}_a, \mathscr{F}_a')$.

Proof. Let $[M, \theta]$ be in $\mathfrak{N}_*(G, \mathscr{F}, \mathscr{F}')$. Let F denote the fixed points set of S in M, S being the intersection of all the members of $\mathscr{F} - \mathscr{F}'$. Since $\mathscr{F} - \mathscr{F}'$ is invariant under conjugation, S is normal in G and hence the action θ on M induces an action on F which we denote once again by θ. Let ν be the normal bundle of the imbedding of F in the interior of M and $D(\nu)$ be its disc bundle with the action θ^* of G on $D(\nu)$ induced by the real vector bundle maps covering the action θ on F. Since F is fixed point set of S, $a \notin H$, $\forall H \in \mathscr{F} - \mathscr{F}'$ and no point of F is fixed by the subgroup $[S \cup \{a\}]$ generated by $S \cup \{a\}$, a will act freely on F and hence on $D(\nu)$. Let $F' = F/[a]$ and $D'(\nu) = D(\nu)/[a]$. Since a is central the actions θ and θ^* on F and $D(\nu)$ induce actions θ' and $\theta^{*\prime}$ on F' and $D'(\nu)$ respectively. Let C_1 and C_2 be the mapping cylinders of the equivariant double covers q_1: $F \to F'$ and q_2: $D(\nu) \to D'(\nu)$ respectively and ψ_1 and ψ_2 be the induced actions on C_1 and C_2 respectively. We have the following commutative diagram

$$\begin{array}{ccc} C_2 & \to & D'(\nu) \\ \downarrow \alpha & & \downarrow \nu' \\ C_1 & \to & F' \end{array},$$

where α: $C_2 \to C_1$ is the map induced from ν': $D'(\nu) \to F'$ by going to mapping cylinders. Clearly ∂C_1 is homeomorphic to F, $\alpha^{-1}(\partial C_1)$ is homeomorphic to $D(\nu)$ and the action ψ_1 on $\alpha^{-1}(\partial C_1)$ is isomorphic to the action θ^* on $D(\nu)$. Consider

$$W = (M \times [0,1]) \cup C_2/\sim ,$$

where $\sim$ is the equivalence relation in W obtained by identifying $D(\nu) \times \{1\}$ with $\alpha^{-1}(\partial C_1)$. Let the action ϕ of G on W be given by $\phi \mid M \times [0,1] = \theta \times 1$ and $\phi \mid C_2 = \psi_1$. Take V to be $(\partial M \times [0,1]) \cup (M \times \{1\} - (D(\nu) \times \{1\})^\circ) \cup (\partial C_2 - (\alpha^{-1}(\partial C_1))^\circ)$, where $^\circ$ denotes the interior operator. Since S is the intersection of all the members of $\mathscr{F} - \mathscr{F}'$, V will be $(\mathscr{F}_a', \mathscr{F}_a')$-free. Also W is $(\mathscr{F}_a', \mathscr{F}_a')$-free and ∂W is homeomorphic to $M \cup V$ by identifying ∂V with ∂M. This shows that $[M, \theta]$ is zero in $\mathfrak{N}_*(G; \mathscr{F}_a, \mathscr{F}_a')$. □

Let $\mathfrak{A}$ denote the family of all subgroups of G and $\mathscr{F}_0$ denote the empty family. Then following the notations of Example 2.1 and using the above Theorem, one immediately gets the following.

COROLLARY 3.2. *For every* k, $0 \leq k < r$, *the homomorphism* $\mathfrak{N}_*(G; \mathscr{F}_{k+1}, \mathscr{F}_k) \to \mathfrak{N}_*(G; \mathfrak{A}, \mathscr{F}_k)$ *induced from the inclusion map* $(\mathscr{F}_{k+1}, \mathscr{F}_k) \to (\mathfrak{A}, \mathscr{F}_k)$ *is zero.*

Proof. Since $(\mathscr{F}_{k+1}, \mathscr{F}_k)$ is admissible pair of families with respect to t_{k+1} for $0 \leq k < r$ and no point of the submanifold V in the above construction is fixed by $\mathbf{Z}_2^k$. Theorem 3.1 gives the Corollary immediately. □

COROLLARY 3.3. *Let* $\mathbf{P}$ *be the family of all subgroups of* G *which do not contain* $G_2(C)$. *Then the homomorphism* $\mathfrak{N}_*(G; \mathbf{P}) \to \mathfrak{N}_*(G; \mathfrak{A})$ *induced from the inclusion map* $\mathbf{P} \to \mathfrak{A}$ *is the zero homomorphism.*

Proof. By Corollary 3.2, one gets that

$$\mathfrak{N}_*(G; \mathscr{F}_{k+1}, \mathscr{F}_k) \xrightarrow{i_*} \mathfrak{N}_*(G; \mathfrak{A}, \mathscr{F}_k)$$

is the zero homomorphism, $0 \leq k < r$. Consider the exact bordism sequence for the triple

$$(\mathfrak{A}, \mathscr{F}_{k+1}, \mathscr{F}_k) \to \cdots \mathfrak{N}_*(G; \mathscr{F}_{k+1}, \mathscr{F}_k) \xrightarrow{i_*} \mathfrak{N}_*(G; \mathfrak{A}, \mathscr{F}_k)$$

$$\xrightarrow{i_*} \mathfrak{N}_*(G; \mathfrak{A}, \mathscr{F}_{k+1}) \to \cdots$$

where j_* is the homomorphism induced from the inclusion j: $(\mathfrak{A}, \mathscr{F}_k) \to (\mathfrak{A}, \mathscr{F}_{k+1})$. Since i_* is the zero homomorphism, j_* will be a monomorphism. Therefore the composite

$$\mathfrak{N}_*(G; \mathfrak{A}, \mathscr{F}_o) \to \mathfrak{N}_*(G; \mathfrak{A}, \mathscr{F}_1) \to \cdots \to \mathfrak{N}_*(G; \mathfrak{A}, \mathscr{F}_r)$$

is a monomorphism and hence by the exact bordism sequence of the triple $(\mathfrak{A}, \mathscr{F}_r, \mathscr{F}_0)$, one get that $\mathfrak{N}_*(G; \mathscr{F}_r, \mathscr{F}_0) \to \mathfrak{N}_*(G; \mathfrak{A}, \mathscr{F}_0)$ is the zero homomorphism. This completes the proof since $\mathscr{F}_r = \mathbf{P}$ and $\mathscr{F}_0 = \varnothing$. □

COROLLARY 3.4. *If* $G_2(C)$ *acts on* M *under* θ *without any stationary point then* (M, θ) *is a* G-*boundary.*

3. The stationary points set $F_{G_2(C)}$ and the normal bundle. In the last section we dealt with the case when $F_{G_2(C)}$ is empty. In this section we consider the case when $F_{G_2(C)} \neq \varnothing$. For this we introduce the concept of equivariant trivial normal bundle and use this concept to settle the case $F_{G_2(C)} \neq \varnothing$ in the form of Theorem 4.2.

Let (M^n, θ) be a closed G-manifold. Consider the decomposition of $F = F_{G_2(C)}(M^n)$ as $F = \bigcup_{l=0}^n F^l$, where F^l denotes the l-dimensional component of F. Let $\mathscr{D}(\nu_l)$ be the normal disc bundle of F^l in M^n with the induced action θ_l of G on $\mathscr{D}(\nu_l)$.

DEFINITION 4.1. F is said to have an equivariant trivial normal bundle in M^n, if $G/G_2(C)$ acts trivially on F and $\exists$ some positive dimensional G-representations (W_l, ϕ_l), $0 \leq l \leq n$, such that in $\mathfrak{N}_*(G; \mathfrak{A}, \mathbf{P})$

$$[D(\nu_l), \theta_l] = [F^l][D(W_l), \phi_l],$$

$D(W_l)$ being the unit disc of W_l.

Let $\{V_k, \psi_k\}_{1 \leq k \leq m}$ be the finite set of all irreducible representations of G. Let $\mathbf{Z}^+$ be the set of all non-negative integers. Then any G-representation can be written as $(V(f), \psi(f))$ for some map f: $\{1, \ldots, m\} \to \mathbf{Z}^+$ where $V(f) = \oplus_{k=1}^m (V_k, \psi_k)^{f(k)}$, $(V_k, \psi_k)^{f(k)}$ being the direct sum of $f(k)$ copies of (V_k, ψ_k). Let us denote the unit disc and the unit sphere of $V(f)$ by $D(f)$ and $S(f)$.

THEOREM 4.2. *If F has an equivariant trivial normal bundle in M^n, then F is a boundary and (M^n, θ) is a G-boundary.*

Proof. Since F has an equivariant trivial normal bundle in M^n, we have

$$[\mathscr{D}(\nu_l), \theta_l] = [F^l][D(W_l), \theta_l]$$

for some positive dimensional G-representations (W_l, ϕ_l), $0 \leq l \leq n$. Also $(W_l, \phi_l) = (V(f_l), \psi(f_l))$ for some map f_l: $\{1, \ldots, m\} \to \mathbf{Z}^+$. Therefore

$$[\mathscr{D}(\nu_l), \theta_l] = [F^l][D(f_l), \psi(f_l)].$$

Let i_*: $\mathfrak{N}_*(G; \mathfrak{A}) \to \mathfrak{N}_*(G; \mathfrak{A}, \mathbf{P})$ be the homomorphism induced by the inclusion map i: $(\mathfrak{A}, \phi) \to (\mathfrak{A}, \mathbf{P})$. Then

$$i_*[M^n, \theta] = \sum_{l=0}^n [\mathscr{D}(\nu_l), \theta_l] = \sum_{l=0}^n [F^l][D(f_l), \psi(f_l)].$$

Therefore

$$\partial_* i_*[M^n, \theta] = \sum_{l=0}^n [F^l][S(f_l), \psi(f_l)] = 0$$

in $\mathfrak{N}_*(G; \mathbf{P})$, where $\partial_*: \mathfrak{N}_*(G; \mathfrak{A}, \mathbf{P}) \to \mathfrak{N}_*(G; \mathbf{P})$ is the boundary homomorphism. Therefore $\exists$ a $\mathbf{P}$-free G-manifold (D, η) such that

$$(1) \qquad (\partial D, \eta) = \bigcup_{l=0}^{n} \left(F^l \times (S(f_l), \psi(f_l))\right).$$

Since (W_l, ϕ_l) is positive dimensional G-representation, $\forall l$, $\exists$ a member $k(l)$ in the set $\{1,\dots,m\}$ such that $f_l(k(l)) \neq 0$. Consider the irreducible G-representation $(V_{k(l)}, \psi_{k(l)})$. Let $(\tilde{V}_{k(l)}, \tilde{\psi}_{k(l)})$ be an irreducible component of the $G_2(C)$-representation induced by the G-representation $(V_{k(l)}, \psi_{k(l)})$. Then $\exists$ a subgroup $H_{k(l)}$ of G isomorphic to $\mathbf{Z}_2^{r-1}$ which fixes $\tilde{V}_{k(l)}$, $G_2(C)$ being $\mathbf{Z}_2^r$. Let us fix some $\beta, 0 \leq \beta \leq n$.

From the equation (1), we get

$$F_{H_{k(\beta)}}(\partial D, \eta) = F_{H_{k(\beta)}}\left(\bigcup_{l=0}^{n} \left(F^l \times (S(f_l), \psi(f_l))\right)\right).$$

Let $F_{H_{k(\beta)}}(D) = F^*$ and $\mathbf{Z}_{2,\beta} \approx \mathbf{Z}_2$ be the complement of $H_{k(\beta)}$ in $G_2(C) = \mathbf{Z}_2^r$. Then one gets

$$\left(\partial F^*, \eta \,|\, \mathbf{Z}_{2,\beta}\right) = \bigcup_{l=0}^{n} \left(F^l \times (S^{\Delta(l,\beta)-1}, a)\right),$$

where a is the antipodal involution and the integer $\Delta(l, \beta)$ is the nonnegative integer depending on l and β. Since $H_{k(\beta)}$ fixed $\tilde{V}_{k(\beta)}$ and $f_\beta(k(\beta)) \neq 0$, one infers that $\Delta(\beta, \beta) \geq 1$. Since D is $\mathbf{P}$-free, $\mathbf{Z}_{2,\beta}$ will act freely on F^* and therefore $[\partial F^*, \eta \,|\, \mathbf{Z}_{2,\beta}]$ is zero in $\mathfrak{N}_*(\mathbf{Z}_{2,\beta}; \mathscr{F}_1)$, $\mathscr{F}_1$ being the family consisting of only trivial subgroup of $\mathbf{Z}_{2,\beta}$. This gives

$$\sum_{l=0}^{n} [F^l][S^{\Delta(l,\beta)-1}, a] = 0$$

in $\mathfrak{N}_*(\mathbf{Z}_{2,\beta}; \mathscr{F}_1)$. But $\mathfrak{N}_*(\mathbf{Z}_{2,\beta}; \mathscr{F}_1)$ is free $\mathfrak{N}_*$-module with a set $\{[S^n, a], n \in \mathbf{Z}^+\}$ of generators. This together with the fact that $\Delta(\beta, \beta) \geq 1$ gives $[F^\beta] = 0$ in $\mathfrak{N}_*$. By varying β, one gets $[F^\beta] = 0$, $\forall \beta = 0,\dots,n$. Hence $[F] = 0$ in $\mathfrak{N}_*$. Therefore

$$i_*[M^n, \theta] = \sum_{l=0}^{n} [F^l][D(f_l), \psi(f_l)] = 0 \quad \text{in } \mathfrak{N}_*(G; \mathfrak{A}, \mathbf{P}).$$

But from Corollary 3.3, one infers that $i_*: \mathfrak{N}_*(G, \mathfrak{A}) \to \mathfrak{N}_*(G; \mathfrak{A}, \mathbf{P})$ is an injection. Therefore $[M^n, \theta]$ is zero in $\mathfrak{N}_*(G; \mathfrak{A})$. □

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Department of Mathematics
North-Eastern Hill University
Bijni Campus, Shillong 793003
Meghalaya, India

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