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We prove that every Banach space with a 1-unconditional basis has the fixed point property for nonexpansive mappings. In fact the argument works if the unconditional constant is $< (\sqrt{33} - 3)/2$.

1. Introduction. Let K be a weakly compact convex subset of a Banach space X. We say K has the *fixed point property* if every *nonexpansive* map $T: K \to K$ (i.e. $\|Tx - Ty\| \leq \|x - y\|$ for $x, y \in K$) has a fixed point. We say X has the fixed point property if every weakly compact convex subset of X has the fixed point property.

It is known that L_1 fails the fixed point property [**A**]. On the other hand, Kirk [**Ki 1**] proved that every Banach space with normal structure (for the definition see [**D**]) has the fixed point property. Karlovitz (see [**Ka 1**] and [**Ka 2**]) extended Kirk's work. Let us explain what Karlovitz did.

Suppose K is weakly compact convex and $T: K \to K$ is nonexpansive. K contains a weakly compact convex subset K_0 which is *minimal* for T. This means $T(K_0) \subseteq K_0$ and no strictly smaller weakly compact convex subset of K_0 is invariant under T. If K_0 contains only one point, then T has a fixed point. Hence, we may assume that $\operatorname{diam} K_0 = \sup\{\|x - y\|: x, y \in K_0\} > 0$. It is easy to see that K_0 contains a sequence (x_n) with $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. We call such a sequence an *approximate fixed point sequence* for T. Indeed, fixed $y \in K_0$, one can choose x_n to be the fixed point of the strict contraction, $T_n: K_0 \to K_0$, given by $T_n x = (1 - n^{-1})Tx + n^{-1}$. Note we only need that K_0 is closed, bounded and convex for this argument. Karlovitz proved the following theorem.

THEOREM A. *Let K be a minimal weakly compact convex set for a nonexpansive map T, and let (x_n) be an approximate fixed point sequence. Then for all $x \in K$*

$$\lim_{n\to\infty} \|x - x_n\| = \operatorname{diam} K.$$

Maurey [**M**] used the ultraproduct techniques to prove that c_0 and every reflexive subspace of L_1 have the fixed point property. Odell and the author [E-L-O-S] used Maurey's technique to prove that T_s (the Tsireleson space of Figiel and Johnson [F-J]) and T_s^* have the fixed point property.

In §II we give some examples of Banach spaces with an unconditional basis and discuss the fixed point property on those spaces.

In §III we introduce the ultraproduct technique and rewrite the Karlovitz Theorem in the ultraproduct language.

In §IV we prove that every Banach space with a 1-unconditional basis has the fixed point property. Indeed, our argument shows that if X has an unconditional basis with unconditional constant (for definition see §II) $\lambda < 1.37$, then X has the fixed point property. Also we prove the every superreflexive space (by Enflo [En] this is a space isomorphic to a uniformly convex space) with a suppression unconditional basis has the fixed point property.

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2. Examples of spaces with an unconditional basis. Let X be a Banach space. A sequence $\{e_n\}_{n=1}^{\infty}$ in X is called a *Schauder basis* of X if for every $x \in X$ there is a unique sequence of scalars $\{a_n\}_{n=1}^{\infty}$ so that $x = \sum_{n=1}^{\infty} a_n e_n$. A Schauder basis $\{e_n\}_{n=1}^{\infty}$ is called an *unconditional basis* if for every choice of signs ε_n (i.e. $\varepsilon_n = \pm 1$), $\sum_{n=1}^{\infty} \varepsilon_n a_n e_n$ converges whenever $\sum_{n=1}^{\infty} a_n e_n$ converges. If $\{e_n\}$ is an unconditional basis, then the number

$$\sup\left\{\left\|\sum_1^n \varepsilon_i a_i e_i\right\| : \left\|\sum_1^n a_i e_i\right\| = 1; \varepsilon_i = \pm 1\right\}$$

is called the *unconditional constant* of $\{e_n\}_{n=1}^{\infty}$. If $\{e_n\}_{n=1}^{\infty}$ is an unconditional basis and F is a subset of $\mathbf{N}$, then the projection

$$P\left(\sum_{n=1}^{\infty} a_n e_n\right) = \sum_{n \in F} a_n e_n$$

is called the *natural projection* associated with F to the unconditional basis $\{e_n\}_{n=1}^{\infty}$. It is clear that the norm of any natural projection is smaller than the unconditional constant of the basis. We say an unconditional basis is *suppression unconditional* if every natural projection associated to the basis has norm 1.

EXAMPLE 1. The natural basis $e_n = \{0, 0, 0, \ldots, \overset{n}{1}, 0, \ldots\}$ is an unconditional basis in each of the spaces c_0 and l_p, $1 \le p < \infty$. Browder [Br] proved that every uniformly convex space has the fixed point property. Since l_p, $1 < p < \infty$, are uniformly convex [C], they have the fixed point property. Lim [**Lm**] proved that every weak* compact convex subset of l_1 has weak* normal structure. Hence, every nonexpansive mapping on weak* compact convex subsets of l_1 has a fixed point. Maurey proved c_0 has the fixed point property.

EXAMPLE 2. Let X_M be l_2 with the new norm

$$\|x\| = \max\{\|x\|_\infty, M^{-1}\|x\|_2\}.$$

Then the natural basis is an unconditional basis with unconditional constant $\lambda = 1$. It is known that X_M fail to have normal structure whenever $M \geq \sqrt{2}$. But X_M still have the fixed point property ([**Ka 1**], [**B-S**] and [**E-L-O-S**]).

EXAMPLE 3. The norm on the sequence space T_s is given implicitly by

$$\|x\|_s = \sup\left\{\|x\|_\infty, \frac{1}{2}\sum_{k=1}^{n} \|E_k x\|_s\right\}$$

where the "sup" is taken over all admissible set $(E_k)_{k=1}^n$ and $(Ex)(i)$ equals $x(i)$ for $i \in E$ and 0 otherwise. $(E_k)_{k=1}^n$ is *admissible* if the E_k's are finite subsets $\mathbf{N}$ with $n < \min E_1 \leq \max E_1 < \min E_2 \leq \max E_2 < \cdots < \min E_n$. T_s is a reflexive Banach space with a 1-unconditional basis. Hence, T_s has the fixed point property ([**E-L-O-S**]).

EXAMPLE 4. $(l_1, |\cdot|)$ is l_1 with norm

$$|x| = \max(\|x^+\|_1, \|x^-\|_1)$$

where x^+ and x^- are the positive and negative parts of x. Then $(l_1, |\cdot|)$ is isometrically isomorphic to the dual of $(c_0, \|\cdot\|)$ where the norm is given by

$$\|x\| = \|x^+\|_\infty + \|x^-\|_\infty.$$

The natural basis is a suppression unconditional basis of $(l_1, |\cdot|)$, and the unconditional constant of this basis is 2. Lim [**Lm**] showed that there is a weak* compact subset K of l_1 and an isometry $T\colon K \to K$ such that T has no fixed points. But every weakly compact subset of l_1 is compact. Hence, $(l_1, |\cdot|)$ has the fixed point property.

EXAMPLE 5. An *Orlicz function* M is a continuous non-decreasing and convex function defined for $t \geq 0$ such that $M(0) = 0$ and $\lim_{t\to\infty} M(t) = \infty$. To any Orlicz function M we associate the space l_M of all sequences of scalars $x = (a_1, a_2, \ldots)$ such that $\sum_{n=1}^\infty M(|a_n|/\rho) < \infty$ for some $\rho > 0$.

The space l_M equipped with the norm

$$\|x\| = \inf\left\{\rho > 0 : \sum_{n=1}^{\infty} M\left(\frac{|a_n|}{\rho}\right) \leq 1\right\}$$

is a Banach space called an *Orlicz sequence space*. If

$$\lim_{t\to 0} \sup M(2t)/M(t) < \infty,$$

then l_M has a 1-unconditional basis. In this case, l_M has the fixed point property.

EXAMPLE 6. Let $(T_s, |\cdot|_s)$ be the T_s with the norm

$$|x|_s = \max\{\|x^+\|_s, \|x^-\|_s\}.$$

Then $(T_s, |\cdot|_s)$ has a suppression unconditional basis. It is still open whether $(T_s, |\cdot|_s)$ has the fixed point property or not. (Note: T_s is not superreflexive.)

3. Ultraproducts. Let $\mathcal{U}$ be a free ultrafilter on $\mathbf{N}$, and let X be a Banach space. The ultraproduct space $\tilde{X}$ of X is the quotient space of

$$l_\infty(X) = \left\{(x_n) : x_n \in X \text{ for all } n \in \mathbf{N} \text{ and } \|(x_n)\| = \sup_n \|x_n\| < \infty\right\}$$

by $\mathcal{N} = \{(x_n) \in l_\infty(X): \lim_{n\to\mathcal{U}}\|x_n\| = 0\}$. (Note $\lim_{n\to\mathcal{U}}\|x_n\|$ is the limit of $\|x_n\|$ over the ultrafilter $\mathcal{U}$.) We shall not distinguish between (x_n) and the coset $(x_n) + \mathcal{N} \in X$. Clearly,

$$\|(x_n)\|_{\tilde{X}} = \lim_{n\to\mathcal{U}} \|x_n\|.$$

It is also clear that X is isometric to a subspace of $\tilde{X}$ by the mapping $x \to (x, x, \ldots)$. So we may assume that X is a subspace of $\tilde{X}$. We will write $\tilde{y}$, $\tilde{z}$, $\tilde{w}$ for the general elements of $\tilde{X}$ and $\tilde{f}$, $\tilde{g}$ for the elements of the dual $\tilde{X}^*$. If S_n's are uniformly bounded operators (projections) on X, then $\tilde{S} = (S_n)$ which is given by $\tilde{S}(x_n) = (S_n x_n)$ is a bounded operator (projection) on $\tilde{X}$, and $\|\tilde{S}\| \leq \sup_n\|S_n\|$. Suppose X has an unconditional basis (e_n). We say $\tilde{P}$ is a *natural projection* with respect to (e_n) if there exist natural projections P_n on X associated to (e_n) such that $\tilde{P} = (P_n)$. We say $\tilde{x}$, $\tilde{y} \in \tilde{X}$ are *disjoint* if there exist two natural projections $\tilde{P}$, $\tilde{Q}$ on $\tilde{X}$ such that $\tilde{P}\tilde{x} = \tilde{x}$, $\tilde{Q}\tilde{y} = \tilde{y}$ and $\tilde{P}\tilde{Q} = \tilde{Q}\tilde{P} = 0$. In other words, $\tilde{x}$ and $\tilde{y}$ are disjoint if they have the representations (x_n) and (y_n) such that x_n and y_n are disjoint in X for all n.

Now let us translate Theorem A into ultraproduct language. Let K be a weakly compact convex subset of X which is minimal for nonexpansive map T. Let $\tilde{K} = \{(x_n): x_n \in K \text{ for all } n\}$ and define $\tilde{T}: \tilde{K} \to \tilde{K}$ by $\tilde{T}(x_n) = (Tx_n)$. Clearly, $\tilde{K}$ is closed bounded and convex and $\tilde{T}$ is nonexpansive on $\tilde{K}$. Furthermore, $\tilde{T}$ has fixed points in $\tilde{K}$. Indeed, if $(x_n)_{n=1}^{\infty}$ is an approximate fixed point sequence for T in K, then for $\tilde{y} = (x_n)$

$$\|\tilde{T}\tilde{y} - \tilde{y}\| = \lim_{n \to \mathcal{U}} \|Tx_n - x_n\| = \lim_{n \to \infty} \|Tx_n - x_n\| = 0,$$

and hence $\tilde{T}\tilde{y} = \tilde{y}$. On the other hand, $\tilde{T}\tilde{y} = \tilde{y}$ for $\tilde{y} = (x_n)$ then some subsequence of $(x_n)_{n=1}^{\infty}$ is an approximate fixed point sequence for T. In ultraproduct language, Theorem A becomes

THEOREM A′. *Let K be a minimal weakly compact convex set for a nonexpansive map T. If $\tilde{y}$ is a fixed point of $\tilde{T}$ in $\tilde{K}$ and $x \in K$, then $\|\tilde{y} - x\| = \operatorname{diam}(K)$. Moreover, suppose* $\operatorname{diam} K = 1$ *and* $0 \in K$. *Then for any $\varepsilon > 0$ there is $\delta > 0$ such that $\|\tilde{y}\| > 1 - \varepsilon$ whenever $\|\tilde{T}\tilde{y} - \tilde{y}\| < \delta$.*

4. The main result.

THEOREM 1. *Every Banach space X with* 1-*unconditional basis* (e_n) *has the fixed point property.*

Proof. Suppose it were not true. Then there is a weakly compact convex subset K which is minimal for a nonexpansive map T. Moreover, we may assume $\operatorname{diam} K = 1$. By translation of K, then passing to subsequences, we may suppose that $0 \in K$ and there exist an approximate fixed point sequence $(x_n)_{n=1}^{\infty}$ for T and natural projections P_n on X (with respect to (e_n)) such that $P_n P_m \neq 0$ if $n \neq m$ and

$$\lim_{n \to \infty} \|P_n x_n\| = \lim_{n \to \infty} \|x_n\| = 1 \quad \text{and} \quad \lim_{n \to \infty} \|(I - P_n)x_n\| = 0.$$

Let $\tilde{h} = (x_n)$ and $\tilde{z} = (z_n)$ with $z_n = x_{n+1}$. Then $\tilde{y}$ and $\tilde{z}$ are fixed points of $\tilde{T}$ with $\|\tilde{y} - \tilde{z}\| = 1$. For any $x \in K$, x, $\tilde{y}$ and $\tilde{z}$ are disjoint. Indeed, let $\tilde{P} = (P_n)$ and $\tilde{Q} = (Q_n)$ with $Q_n = P_{n+1}$. Then $\tilde{P}\tilde{y} = \tilde{y}$ and $\tilde{Q}\tilde{z} = \tilde{z}$ and for any $x \in K$,

$$\tilde{P}x = \tilde{Q}x = \tilde{P}\tilde{z} = 0 = \tilde{Q}\tilde{y}.$$

Also since (e_n) is 1-unconditional, $\|\tilde{y} - \tilde{z}\| = 1 = \|\tilde{y} + \tilde{z}\|$. Let

$$\tilde{W} = \{\tilde{w} : \tilde{w} \in \tilde{K} \text{ such that there exists } x \in K$$
$$\text{(depending on } \tilde{w}) \text{ with } \max(\|\tilde{w} - x\|, \|\tilde{w} - \tilde{y}\|, \|\tilde{w} - \tilde{z}\|) \le 1/2\}.$$

Clearly, $\tilde{W}$ is a nonempty bounded closed convex set. (Note $\|(\tilde{y} + \tilde{z})/2 - 0\| = \|(\tilde{y} + \tilde{z})/2\| = \|(\tilde{y} - \tilde{z})/2\| = 1/2$. So $(\tilde{y} + \tilde{z})/2 \in \tilde{W}$.) Since

$\tilde{y}$, $\tilde{z}$ are fixed points of $\tilde{T}$ and T is a nonexpansive mapping, if $\tilde{w} \in \tilde{W}$,

$$\max(\|\tilde{T}\tilde{w} - Tx\|, \|\tilde{T}\tilde{w} - \tilde{y}\|, \|\tilde{T}\tilde{w} - \tilde{z}\|) \leq \max(\|\tilde{w} - x\|, \|\tilde{w} - \tilde{y}\|, \|\tilde{w} - \tilde{z}\|) \leq 1/2.$$

Thus $\tilde{W}$ is invariant under $\tilde{T}$; hence, it contains an approximate fixed point sequence for $\tilde{T}$. On the other hand, for any $\tilde{w} \in \tilde{W}$ there exists $x \in K$ so that $\|\tilde{w} - x\| \leq 1/2$. Hence if $\tilde{I}$ is the identity map in $\tilde{X}$,

$$\begin{aligned}\|\tilde{w}\| &= \frac{1}{2}\|(\tilde{P} + \tilde{Q})\tilde{w} + (\tilde{I} - \tilde{P})\tilde{w} + (\tilde{I} - \tilde{Q})\tilde{w}\| \\ &\leq \frac{1}{2}[\|(\tilde{P} + \tilde{Q})\tilde{w}\| + \|(\tilde{I} - \tilde{P})\tilde{w}\| + \|(\tilde{I} - \tilde{Q})\tilde{w}\|] \\ &= \frac{1}{2}[\|(\tilde{P} + \tilde{Q})(\tilde{w} - x)\| + \|(\tilde{I} - \tilde{P})(\tilde{w} - \tilde{y})\| + \|(\tilde{I} - \tilde{Q})(\tilde{w} - \tilde{z})\|] \\ &\leq \frac{1}{2}\left[\frac{1}{2} + \frac{1}{2} + \frac{1}{2}\right] = \frac{3}{4}.\end{aligned}$$

By Theorem A′, $\tilde{W}$ cannot contain any approximate fixed point sequences for $\tilde{T}$. We have a contradiction. □

We note that the proof of the above Theorem has some leeway. More precisely we have the following more general result.

THEOREM 2. *If X has an unconditional basis with unconditional constant $\lambda < (\sqrt{33} - 3)/2$, then X has the fixed point property.*

Proof. Let $\tilde{y}$, $\tilde{z}$, $\tilde{P}$ and $\tilde{Q}$ be as in Theorem 1, and let

$$\tilde{W} = \{\tilde{w} : \tilde{w} \in \tilde{K} \text{ such that there exists } x \in K \text{ with } \|\tilde{w} - x\| \leq \lambda/2 \text{ and } \max(\|\tilde{w} - \tilde{y}\|, \|\tilde{w} - \tilde{z}\|) \leq 1/2\}.$$

Since $\|(\tilde{y} + \tilde{z})/2\| \leq \lambda\|(\tilde{y} - \tilde{z})/2\| = \lambda/2$, $\tilde{W}$ is a nonempty bounded closed convex set invariant under $\tilde{T}$. Hence, $\tilde{W}$ contains an approximate fixed point sequence for $\tilde{T}$. For easy calculaiton, we assume that $\tilde{W}$ has an element $\tilde{w}$ with $\|\tilde{w}\| = 1$. Let $x \in K$ with $\|x - \tilde{w}\| \leq \lambda/2$ and let $\tilde{f} \in X^*$ with $\tilde{f}(\tilde{w}) = 1 = \|\tilde{f}\|$. Hence, $1 - \tilde{f}(\tilde{y}) = \tilde{f}(\tilde{w} - \tilde{y}) \leq \|\tilde{w} - \tilde{y}\| \leq 1/2$, and so $\tilde{f}(\tilde{y}) \geq 1/2$. Similarly, we also have the inequalities $\tilde{f}(\tilde{z}) \geq 1/2$ and $\tilde{f}(x) \geq 1 - \lambda/2$. Let $\alpha = \tilde{f}((\tilde{I} - \tilde{P} - \tilde{Q})\tilde{w})$. Then

$$\begin{aligned}1 - \alpha &= \tilde{f}(\tilde{w}) - \tilde{f}((\tilde{I} - \tilde{P} - \tilde{Q})\tilde{w}) \\ &= \tilde{f}((\tilde{P} + \tilde{Q})\tilde{w}) = \tilde{f}(\tilde{P}\tilde{w}) + \tilde{f}(\tilde{Q}\tilde{w}),\end{aligned}$$

and so either $\tilde{f}(\tilde{P}\tilde{w}) \leq (1-\alpha)/2$ or $\tilde{f}(\tilde{Q}\tilde{w}) \leq (1-\alpha)/2$, say $\tilde{f}(\tilde{P}\tilde{w}) \leq (1-\alpha)/2$. Since $\tilde{I} - 2\tilde{P}$ and $\tilde{I} - 2\tilde{P} - 2\tilde{Q}$ are reflections, $\|\tilde{I} - 2\tilde{P}\| \leq \lambda$ and $\|\mathrm{w}\tilde{P} + 2\tilde{Q} - \tilde{I}\| \leq \lambda$. Hence, we have

$$\begin{aligned}(2-2\alpha) - \lambda/2 &\leq 2\tilde{f}((\tilde{P}+\tilde{Q})\tilde{w}) - \tilde{f}(\tilde{w}-x)\\ &= \tilde{f}((2\tilde{P}+2\tilde{Q})\tilde{w}) - \tilde{f}(\tilde{w}-x)\\ &= \tilde{f}((2\tilde{P}+2\tilde{Q})(\tilde{w}-x)) - \tilde{f}(\tilde{w}-x)\\ &= \tilde{f}((2\tilde{P}+2\tilde{Q}-\tilde{I})(\tilde{w}-x))\\ &\leq \|\tilde{f}\|\,\|2\tilde{P}+2\tilde{Q}-\tilde{I}\|\,\|\tilde{w}-x\| \leq \lambda^2/2,\end{aligned}$$

and

$$\begin{aligned}\alpha + \frac{1}{2} = \frac{1}{2} + 1 - (1-\alpha) &\leq \tilde{f}(\tilde{y}) + \tilde{f}(\tilde{w}) - 2\tilde{f}(\tilde{P}\tilde{w})\\ &= \tilde{f}(\tilde{w}-\tilde{y}) + 2\tilde{f}(\tilde{y}) - 2\tilde{f}(\tilde{P}\tilde{w})\\ &= \tilde{f}(\tilde{w}-\tilde{y}) + 2\tilde{f}(\tilde{P}\tilde{y}) - 2\tilde{f}(\tilde{P}\tilde{w})\\ &= \tilde{f}(\tilde{w}-\tilde{y}) + 2\tilde{f}(\tilde{P}(\tilde{y}-\tilde{w})) = \tilde{f}((\tilde{I}-2\tilde{P})(\tilde{w}-\tilde{y}))\\ &\leq \|\tilde{f}\|\,\|\tilde{I}-2\tilde{P}\|\,\|\tilde{w}-\tilde{y}\| \leq \lambda/2.\end{aligned}$$

Therefore, $3 - 3\lambda/2 \leq \lambda^2/2$ and $\lambda \geq (\sqrt{33} - 3)/2$. □

If X has a suppression unconditional basis, we have the following strong result.

THEOREM 3. *Suppose X has a suppression unconditional basis (e_i). Then X has the fixed point property whenever X is superreflexive.*

Proof. Suppose not and, as usual, let K be a minimal set of diameter 1 for a nonexpansive map T. Let $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n$ be disjoint fixed points for $\tilde{T}$ in $\tilde{K}$. We shall prove $(\tilde{x}_i)_1^n$ is 2-equivalent to the unit basis of l_1^n. Indeed, if $\sum_{i=1}^n \alpha_i = 1$, $\alpha_i \geq 0$ and $0 < c < 1$, then the same argument as given in the proof of Theorem 1 shows that every element in

$$\tilde{W} = \{\tilde{w} : \tilde{w} \in \tilde{K} \text{ such that } x \in K \text{ with } \|x - \tilde{w}\| \leq c \text{ and } \|\tilde{w} - \tilde{x}_i\| \leq 1 - \alpha_i \text{ for } i = 1, 2, \ldots, n\}$$

has norm less than or equal to $1 - (1-c)/n$. $\tilde{W}$ is a closed convex set which is invariant under $\tilde{T}$; hence, $\tilde{W}$ is empty. But

$$\left\|\tilde{x}_j - \sum_{i=1}^n \alpha_i \tilde{x}_i\right\| = \left\|\sum_{i \neq j} \alpha_i (\tilde{x}_j - \tilde{x}_i)\right\| \leq 1 - \alpha_j,$$

for $j = 1, 2, \ldots, n$. So $\|\sum_{i=1}^n \alpha_i \tilde{x}_i\| > c$ and so $\|\sum_{i=1}^n \alpha_i \tilde{x}_i\| = 1$. □

REMARK 1. The disjoint fixed point sequence $(\tilde{x}_i)$ for $\tilde{T}$ as given in the proof of Theorem 3 is 1-equivalent to the unit vector basis of $(l_1^n, |\cdot|)$. Indeed, let $\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_{2n-1}$ be disjoint fixed points of $\tilde{T}$ and $\tilde{y}_{2n} = 0$. Then for $\sum_{i=1}^n \alpha_i = 1$ and $\alpha_i \geq 0$

$$1 = \left\| \sum_{i=1}^n \alpha_i \tilde{y}_{2i-1} \right\| \leq \left\| \sum_{i=1}^n \alpha_i (\tilde{y}_{2i-1} - \tilde{y}_{2i}) \right\| \leq \sum_{i=1}^n \alpha_i = 1.$$

Hence, $\|\sum_{i=1}^n \alpha_i (\tilde{y}_{2i-1} - \tilde{y}_{2i})\| = 1$. In general, we have that

$$\left\| \sum_{i=1}^n \beta_i \tilde{x}_i \right\| = \max\left(\|(\beta_i)^+\|_1, \|(\beta_i)^-\|_1 \right) = |(\beta_i)|.$$

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