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By means of a Moore-Postnikov decomposition we compute the first homology groups of some spaces of sections of projective bundles associated to complex vector bundles.

1. Introduction. Let $P\xi$: $P(V) \to X$ be the projective bundle associated to a complex $(n+1)$-dimensional vector bundle ξ: $V \to X$, $n \geq 1$, over a connected CW-complex X. Suppose that $P\xi$ admits a section u: $X \to P(V)$ and consider the space Γ_u of all sections vertically homotopic to u. In this paper we discuss the (co)-homology of Γ_u using the construction by Thom-Haefliger [**1**] of Γ_u as an inverse limit derived from the Moore-Postnikov factorization of $P\xi$. Explicit formulas for some (co-)homology groups of Γ_u are obtained provided $X = T$ is a closed, orientable surface, $X = P^m$, $1 \leq m \leq n$, is a complex projective space, or $X = L^{2m+1}(p)$, $1 \leq m < n$, p odd, is a lens space.

If ξ is trivial, then Γ_u is a (path-)component of the space $M(X, P^n)$ of maps of X into P^n, so in particular we obtain formulas for some homology groups of the components of $M(X, P^n)$. In fact, sufficient information is obtained to show that two components of $M(T, P^n)$ or $M(P^m, P^n)$, $1 \leq m \leq n$, are homotopy equivalent if and only if their associated degrees have the same absolute value.

The work presented here was inspired by the paper [**4**], in turn inspired by [**2**], in which Larmore and Thomas computed the fundamental group of some spaces of sections of real projective bundles associated to real vector bundles. In contrast to [**4**] we avoid, however, the use of twisted coefficients, for $P\xi$ is orientable, and the focus will be on homology groups rather than homotopy groups.

2. Moore-Postnikov factorizations of projective bundles. Since the projective bundle $P\xi$: $P(V) \to X$, having a connected structure group, is orientable, it admits a Moore-Postnikov factorization of the following type

$$\begin{array}{ccccc} & & P(V) & & \\ & & q\downarrow & & \\ K(\mathbf{Z}/2, 2n+2) & \to & E_3 & & \\ & & p_3\downarrow & & \\ K(\mathbf{Z}, 2n+1) & \to & E_2 & \overset{k^{2n+3}}{\to} & K(\mathbf{Z}/2, 2n+3) \\ & & p_2\downarrow & & \\ K(\mathbf{Z}, 2) & \to & E_1 & \overset{k^{2n+2}}{\to} & K(\mathbf{Z}, 2n+2) \\ & & p_1\downarrow & & \\ & & X & \overset{k^3}{\to} & K(\mathbf{Z}, 3) \end{array}$$

where the k-invariants k^3 and k^{2n+2} are given in

LEMMA 2.1. $k^3 = 0$, $E_1 = X \times K(\mathbf{Z}, 2)$ *and*

$$k^{2n+2} = \sum_{i=0}^{n+1} (-1)^i c_i(\xi) \otimes a^{n+1-i}$$

where a is a generator of $H^2(\mathbf{Z}, 2; \mathbf{Z})$ *and* $c_i(\xi) \in H^{2i}(X; \mathbf{Z})$, $1 \le i \le n+1$, *are the Chern classes of* ξ.

Proof. Choose an imbedding $i\colon V \to X \times \mathbf{C}^\infty$ of ξ into the trivial infinite dimensional vector bundle over X. The induced map $P(i)\colon P(V) \to P(X \times C^\infty) = X \times P^\infty$ is then an imbedding of $P\xi$ into the trivial infinite-dimensional projective bundle over X and $P(i)^*(\lambda) = \lambda_\xi$, where λ_ξ and λ are the canonical line bundles ([3], p. 233) over $P(V)$ and $X \times P^\infty$ respectively.

Since the restriction of $P(i)$ to the fiber is the usual imbedding of P^n into P^∞, we may take $E_1 = X \times P^\infty$ and $p_1 = \mathrm{pr}_1\colon E_1 = X \times P^\infty \to X$ as the first stage in the Moore-Postnikov factorization of $P\xi$.

By the defining relation ([3], Definition 2.6, p. 234) for the Chern classes of ξ, we have

$$P(i)^*\left(\sum_{i=0}^{n+1} (-1)^i c_i(\xi) \otimes a^{n+1-i}\right) = \sum_{i=0}^{n+1} (-1)^i c_i(\xi) c_1(\lambda_\xi)^{n+1-i} = 0$$

for $P(i)^*$ is an $H^*(X)$-module homomorphism and $P(i)^*(1 \otimes a) = P(i)^* c_1(\lambda) = c_1(\lambda_\xi)$. Since $H^*(P(V))$ is free $H^*(X)$-module by the

Leray-Hirsch Theorem ([3], Theorem 1.1, p. 231), it follows in fact that

$$k^{2n+2} = \sum_{i=0}^{n+1} (-1)^i c_i(\xi) \otimes a^{n+1-i}$$

generates $H^{2n+2}(X \times K(\mathbf{Z}, 2); \mathbf{Z}) \cap \operatorname{kern} P(i)^*$. □

Assume for the rest of this section that $\dim X < 2n + 1$. For $i = 1, 2, 3$, let Γ_i be the space of sections of $E_i \to X$ vertically homotopic to u_i, where $u_3 = qu$, $u_2 = p_3 u_3$ and $u_1 = p_2 u_2$. According to ([1], §2) there is then an induced tower of fibrations

$$\begin{array}{ccccc}
 & & \Gamma_u & & \\
 & & q\downarrow & & \\
K(\mathbf{Z}/2, 2n+2)^X & \to & \Gamma_3 & & \\
 & & p_3\downarrow & & \\
K(\mathbf{Z}, 2n+1)^X & \to & \Gamma_2 & \overset{\underline{k}^{2n+3}}{\to} & K(\mathbf{Z}/2, 2n+3)^X \\
 & & p_2\downarrow & & \\
 & & \Gamma_1 & \overset{\underline{k}^{2n+2}}{\to} & K(\mathbf{Z}, 2n+2)^X
\end{array}$$

where $\underline{k}^{2n+i}$ denotes the map defined by composition with k^{2n+i}, $i = 2, 3$. Moreover, p_3: $\Gamma_3 \to \Gamma_2$ is the pull-back along $\underline{k}^{2n+3}$ of the path space fibration over $K(\mathbf{Z}/2, 2n+3)^X$ and p_2: $\Gamma_2 \to \Gamma_1$ is the pull-back along $\underline{k}^{2n+2}$ of the path space fibration over $K(\mathbf{Z}, 2n+2)^X$.

There is a homotopy equivalence

$$h\colon K(\mathbf{Z}, 2) \times F_0(X, K(\mathbf{Z}, 2)) \to \Gamma_1$$

where $F_0(X, K(\mathbf{Z}, 2)) \subset K(\mathbf{Z}, 2)^X$ denotes the space of based, null-homotopic maps of X into $K(\mathbf{Z}, 2)$. Note that $F_0(X, K(\mathbf{Z}, 2)) = K(H^1(X; \mathbf{Z}), 1)$; see e.g. ([1], §1). For $y \in K(\mathbf{Z}, 2)$, $\alpha \in F_0(X, K(\mathbf{Z}, 2))$ and $x \in X$, the homotopy equivalence h is given by $h(y, \alpha)(x) = (x, y \cdot \alpha(x) \cdot \mu(x))$, where the multiplication refers to the H-space structure of $K(\mathbf{Z}, 2)$ and where μ: $X \to K(\mathbf{Z}, 2)$ is the second component of the section u_1: $X \to E_1 = X \times K(\mathbf{Z}, 2)$.

Via h, the adjoint of $\underline{k}^{2n+2}$ may be identified with a map

$$f^{2n+2}\colon K(\mathbf{Z}, 2) \times K\big(H^1(X; \mathbf{Z}), 1\big) \times X \to K(\mathbf{Z}, 2n+2).$$

In order to identify f^{2n+2} as a cohomology class, let $c_1(u) \in H^2(X; \mathbf{Z})$ denote $c_1(u^*(\lambda_\xi)) = \mu^*(a)$, let $\{x_j\}$ be a free basis of $H^1(X; \mathbf{Z})$, and let $\{x_j'\}$ be the dual basis of $H^1(H^1(X; \mathbf{Z}), 1; \mathbf{Z}) = \operatorname{Hom}(H^1(X; \mathbf{Z}), \mathbf{Z})$.

LEMMA 2.2. *The homotopy class of f^{2n+2} is given by*

$$f^{2n+2} = \sum_{i=0}^{n+1} (-1)^i (1 \otimes 1 \otimes c_i(\xi))$$
$$\cup \left(a \otimes 1 \otimes 1 + 1 \otimes 1 \otimes c_1(u) + \sum_j 1 \otimes x_j' \otimes x_j \right)^{n+1-i}.$$

Proof. Let $g\colon K(\mathbf{Z},2) \times F_0(X, K(\mathbf{Z},2)) \times X \to X \times K(\mathbf{Z},2)$ be the map given by $g(y, \alpha, x) = (x, y \cdot \alpha(x) \cdot \mu(x))$. Then

$$g^*(c_i(\xi) \otimes 1) = 1 \otimes 1 \otimes c_i(\xi),$$
$$g^*((1 \otimes a)) = a \otimes 1 \otimes 1 + 1 \otimes 1 \otimes c_1(u) + \sum_j 1 \otimes x_j' \otimes x_j$$

for it follows from ([1], §1) that $e = \sum_j x_j' \otimes x_j$, where $e\colon F_0(X, K(\mathbf{Z},2)) \times X \to K(\mathbf{Z},2)$ is the evaluation map $e(\alpha, x) = \alpha(x)$.

As $f^{2n+2} = g^*(k^{2n+2})$, Lemma 2.2 is now a consequence of Lemma 2.1. □

As a special case of the above result we emphasize

COROLLARY 2.3. *Suppose $H^1(X; \mathbf{Z}) = 0$. Then*

$$f^{2n+2} = \sum_{i=0}^{n+1} \sum_{j=0}^{n+1-i} (-1)^i \binom{n+1-i}{j} a^{n+1-i-j} \otimes c_i(\xi) c_1(u)^j.$$

Let Γ denote the space of all sections of $P\xi$. By associating to each section $u \in \Gamma$ the cohomology class $c_1(u) = c_1(u^*(\lambda_\xi))$ of the induced line bundle over X we get a map $c_1\colon \pi_0(\Gamma) \to H^2(X; \mathbf{Z})$ from the set $\pi_0(\Gamma)$ of (path-)components of Γ to $H^2(X; \mathbf{Z})$. Since $c_1(u) = \mu^*(a)$, an easy application of obstruction theory shows

PROPOSITION 2.4. *The map*

$$c_1\colon \pi_0(\Gamma) \to H^2(X; \mathbf{Z})$$

is bijective when $\dim X < 2n + 1$.

With this classification of the set of vertical homotopy classes of sections of $P\xi$, we conclude §2. The following sections contain examples of applications of the above results to the computation of the homology of Γ_u.

3. Sections of projective bundles over surfaces. Suppose $X = T$ is a closed, orientable surface of genus $g \geq 0$. By Proposition 2.4, the space Γ of sections of the projective bundle $P\xi\colon P(V) \to T$ has a countably infinite number of components classified by $H^2(T; \mathbf{Z})$. The component $\Gamma_u \subset \Gamma$ containing the section $u\colon T \to P(V)$ determines as in §2 a sequence of fibrations

$$\begin{array}{lll} & \Gamma_u & \\ & q\downarrow & \\ \prod_{i=0}^{2} K(H^{2-i}(T; \mathbf{Z}/2), 2n+i) & \to \Gamma_3 & \\ & p_3\downarrow & \\ \prod_{i=0}^{2} K(H^{2-i}(T; \mathbf{Z}), 2n+i-1) & \to \Gamma_2 & \\ & p_2\downarrow & \\ K(\mathbf{Z},2) \times K(H^1(T; \mathbf{Z}),1) & = \Gamma_1 & \overset{\underline{k}^{2n+2}}{\to} \prod_{i=0}^{2} K(H^{2-i}(T; \mathbf{Z}), 2n+i) \end{array}$$

where we have identified the fibers as well as the space $K(\mathbf{Z}, 2n+2)^T$ with products of Eilenberg-MacLane spaces ([**1**], §1). For $i = 0, 1, 2$, let

$$\underline{k}_i^{2n+2} \in H^{2n+i}(\Gamma_1; \mathbf{Z}) \otimes H^{2-i}(T; \mathbf{Z})$$

be the components of $\underline{k}^{2n+2}$ corresponding to the splitting of

$$K(\mathbf{Z}, 2n+2)^T.$$

Choose generators $A_j, B_j \in H^1(T; \mathbf{Z})$, $1 \leq j \leq g$, such that $A_iA_j = B_iB_j = 0$ and $A_iB_j = \delta_{ij}U$, where U generates $H^2(T; \mathbf{Z})$, and let as before A'_j, B'_j be the dual generators of $H^1(H^1(T; \mathbf{Z}), 1; \mathbf{Z})$.

LEMMA 3.1. *The components $\underline{k}_i^{2n+2}$ of $\underline{k}^{2n+2}$ are*

$$\underline{k}_0^{2n+2} = (n+1)a^n \otimes 1 \otimes c_1(u) - a^n \otimes 1 \otimes c_1(\xi)$$
$$-n(n+1)a^{n-1} \otimes \sum_{j=1}^{g} A'_jB'_j \otimes U$$

$$\underline{k}_1^{2n+2} = (n+1)\sum_{j=1}^{g}\left(a^n \otimes A'_j \otimes A_j + a^n \otimes B'_j \otimes B_j\right)$$

$$\underline{k}_2^{2n+2} = a^{n+1} \otimes 1 \otimes 1.$$

Proof. Let

$$e = \sum_{j=1}^{g}\left(A'_j \otimes A_j + B'_j \otimes B_j\right) \in H^2(F_0(T, K(\mathbf{Z},2)) \times T; \mathbf{Z})$$

be the evaluation map $e\colon F_0(T, K(\mathbf{Z}, 2)) \times T \to K(\mathbf{Z}, 2)$. Then

$$e^2 = -2 \sum_{j=1}^{g} A'_j B'_j \otimes U$$

while $e^3 = 0$. Lemma 2.2 then shows that the adjoint f^{2n+2} of $\underline{k}^{2n+2}$ is

$$\begin{aligned} f^{2n+2} &= \left(a \otimes 1 \otimes 1 + 1 \otimes 1 \otimes c_1(u) + 1 \otimes e\right)^{n+1} \\ &\quad -(1 \otimes 1 \otimes c_1(\xi))\left(a \otimes 1 \otimes 1 + 1 \otimes 1 \otimes c_1(u) + 1 \otimes e\right)^n \\ &= a^{n+1} \otimes 1 \otimes 1 + (n+1)a^n \otimes 1 \otimes c_1(u) - a^n \otimes 1 \otimes c_1(\xi) \\ &\quad + (n+1)a^n \otimes \sum_{j=1}^{g} \left(A'_j \otimes A_j + B'_j \otimes B_j\right) \\ &\quad - n(n+1)a^{n-1} \otimes \sum_{j=1}^{g} A'_j B'_j \otimes U \end{aligned}$$

and from this formula we can read off ([**1**], §1) the expressions for $\underline{k}_i^{2n+2}$, $i = 0, 1, 2$. □

Let $t(u) \in \mathbf{Z}$ be the integer determined by the equation

$$t(u)U = (n+1)c_1(u) - c_1(\xi)$$

in $H^2(T; \mathbf{Z})$. Then we may write

$$\underline{k}_0^{2n+2} = t(u)a^n \otimes 1 \otimes U - n(n+1)a^{n-1} \otimes \sum_{j=1}^{g} A'_j B'_j \otimes U.$$

As the main result of this section now follows

THEOREM 3.2. *The first* $2n - 1$ *integral homology groups of* Γ_u *are given by*

$$H_r(\Gamma_u; \mathbf{Z}) = \begin{cases} H_r(\Gamma_1; \mathbf{Z}), & 0 \le r < 2n-1, \\ H_{2n-1}(\Gamma_1; \mathbf{Z}) \oplus Z_u, & r = 2n-1, \end{cases}$$

where $Z_u = \mathbf{Z}/|t(u)| \otimes \mathbf{Z}/n(n+1)$ *if* $g > 0$ *and* $Z_u = \mathbf{Z}/|t(u)|$ *if* $T = S^2$ *is the* 2*-sphere.*

Proof. Since the fibres of both $q\colon \Gamma_u \to \Gamma_3$ and $p_3\colon \Gamma_3 \to \Gamma_2$ have vanishing reduced integral cohomology groups in dimension $\le 2n$, it follows that $H^r(\Gamma_u; \mathbf{Z}) = H^r(\Gamma_2; \mathbf{Z})$ for $r \le 2n$.

To compute $H^r(\Gamma_2; \mathbf{Z})$ we consider the Leray-Serre cohomology spectral sequence $\{E_s^{pq}\}$ with integral coefficients associated to p_2: $\Gamma_2 \to \Gamma_1$. We have $E_2^{pq} = 0$ for $0 < q < 2n - 1$ while $E_2^{0,2n-1} \cong \mathbf{Z}$ and $E_2^{1,2n-1} \cong H^1(T; \mathbf{Z}) \cong E_2^{0,2n}$. Since p_2: $\Gamma_2 \to \Gamma_1$ is induced by $\underline{k}^{2n+2}$ from the path space fibration over $K(\mathbf{Z}, 2n+2)^T$, the first non-trivial differential d_{2n}: $E_2^{0,2n-1} \to E_2^{2n,0}$ is determined by $\underline{k}_0^{2n+2}$. Assuming that $g > 0$, we conclude that $E_\infty^{0,2n-1} = 0$ while

$$E_\infty^{2n,0} = H^{2n}(\Gamma_1; \mathbf{Z})/\mathbf{Z} \cdot \left(t(u)a^n \otimes 1 - n(n+1)a^{n-1} \otimes \sum A_j' B_j'\right).$$

It follows that $H^r(\Gamma_u; \mathbf{Z}) = H^r(\Gamma_1; \mathbf{Z})$ for $0 \leq r \leq 2n - 1$. Moreover, since $E_2^{1,2n-1}$ and $E_2^{0,2n}$ are free abelian groups, the torsion subgroup of $H^{2n}(\Gamma_u; \mathbf{Z})$ equals the torsion subgroup, $\mathbf{Z}/|t(u)| \otimes \mathbf{Z}/n(n+1)$, of $E_\infty^{2n,0}$. □

Using the extra information contained in Lemma 3.1, the reader may carry the analysis of the spectral sequence a little further and compute $H^{2n}(\Gamma_u; \mathbf{Z})$.

Let R_u be the subring of $H^*(\Gamma_u; \mathbf{Z})$ generated by all cohomology classes of degree ≤ 2 and let T_u be R_u truncated above degree $2n$. The proof of Theorem 3.2 shows that

$$T_u = \mathbf{Z}[a] \otimes \Lambda\left(A_1', B_1', \ldots, A_g', B_g'\right)/I_u$$

where $I_u \subset \mathbf{Z}[a] \otimes \Lambda(A_1', B_1', \ldots, A_g', B_g')$ is the ideal generated by

$$t(u)a^n \otimes 1 - n(n+1)a^{n-1} \otimes \sum A_j' B_j'$$

together with all elements of degree $> 2n$. T_u is a homotopy invariant of Γ_u, so from the fact (pointed out to me by A. Thorup) that

$$T_u^{2n}/\left\langle T_u^1 \right\rangle^{2n} = \mathbf{Z}/|t(u)|,$$

where $\left\langle T_u^1 \right\rangle$ is the ideal of T_u generated by all elements of degree 1, we obtain

Corollary 3.3. *Let v: $T \to P(V)$ be a section that is not vertically homotopic to u. If Γ_u is homotopy equivalent to Γ_v, then*

$$(n+1)(c_1(u) + c_1(v)) = 2c_1(\xi).$$

As a special case we take as ξ the trivial $(n+1)$-dimensional vector bundle over T. Then $\Gamma = M(T, P^n)$ is the space of maps of T into P^n and $\Gamma_u = M_k(T, P^n)$ is the component consisting of maps of degree $k = c_1(u)$.

Corollary 3.4. *Two components of $M(T, P^n)$ are homotopy equivalent if and only if their associated degrees have the same absolute value.*

This result was also obtained for $n = 1$ in [2] and for $g = 0$ in [5].

In the nonorientable case we get

PROPOSITION 3.5. *Suppose that the base space $X = U_h$ is a closed, nonorientable surface of genus $h > 1$. Then there is a $(2n-1)$-connected map*

$$p_1\colon \Gamma_u \to \Gamma_1 = K(\mathbf{Z},2) \times K(\mathbf{Z}^{h-1},1)$$

and

$$H_{2n-1}(\Gamma_u; \mathbf{Z}/2) \cong H_{2n-1}(\Gamma_1; \mathbf{Z}/2) \oplus Z_u$$

where

$$Z_u = \begin{cases} \mathbf{Z}/2 & \text{if } (n+1)c_1(u) = c_1(\xi), \\ 0 & \text{if } (n+1)c_1(u) \neq c_1(\xi). \end{cases}$$

In particular the two components of Γ_u are not homotopy equivalent when n is even.

Depending on knowledge of the cup square e^2, the above method actually makes possible the computation of the first $2n-1$ homology groups of Γ_u when the base space X is any 2-dimensional CW-complex.

4. Sections of projective bundles over projective spaces. In this section we assume that the base space $X = P^m$ is the complex projective m-space, $1 \leq m \leq n$. The space of sections Γ then has a countably infinite number of components classified by $H^2(P^m; \mathbf{Z})$. The component $\Gamma_u \subset \Gamma$, containing the section $u\colon P^m \to P(V)$, determines a tower of fibrations

$$\begin{array}{ccccc}
 & & \Gamma_u & & \\
 & & \downarrow & & \\
\prod_{i=0}^{m} K(\mathbf{Z}/2, 2n-2m+2+2i) & \to & \Gamma_3 & & \\
 & & \downarrow & & \\
\prod_{i=0}^{m} K(\mathbf{Z}, 2n-2m+1+2i) & \to & \Gamma_2 & & \\
 & & \downarrow & & \\
K(\mathbf{Z},2) & = & \Gamma_1 & \xrightarrow{k^{2n+2}} & \prod_{i=0}^{m} K(\mathbf{Z}, 2n-2m+2+2i)
\end{array}$$

where the fibers are products of Eilenberg-MacLane spaces.

Let $a_m \in H^2(P^m; \mathbf{Z})$ be a generator and let $t(u) \in \mathbf{Z}$ be the integer determined by the equation

$$t(u)a_m^m = \sum_{i=0}^{m} (-1)^i \binom{n+1-i}{n+1-m} c_i(\xi) c_1(u)^{m-i}$$

in $H^{2m}(P^m; \mathbf{Z})$. Then by Corollary 2.3, the first component $\underline{k}_0^{2n+2} \in H^{2n-2m+2}(\mathbf{Z}, 2; \mathbf{Z}) \otimes H^{2m}(P^m; \mathbf{Z})$ of the map $\underline{k}^{2n+2}$ is

$$\underline{k}_0^{2n+2} = t(u) a^{n-m+1} \otimes a_m^m.$$

By a spectral sequence argument similar to that of §3 we get

THEOREM 4.1. *For $0 \le r < 2n - 2m + 1$, $H^r(\Gamma_u; \mathbf{Z}) = H^r(\mathbf{Z}, 2; \mathbf{Z})$, while the $(2n - 2m + 1)$- and $(2n - 2m + 2)$-dimensional integral cohomology groups of Γ_u are determined by the exact sequence*

$$0 \to H^{2n-2m+1}(\Gamma_u) \to \mathbf{Z} \xrightarrow{d_u} \mathbf{Z} \to H^{2n-2m+2}(\Gamma_u) \to 0$$

where d_u is multiplication by $t(u)$.

With the trivial $(n + 1)$-plane bundle as ξ, this yields

COROLLARY 4.2. *For $k \in \mathbf{Z}$, let $M_k(P^m, P^n)$ be the space of maps of degree k of P^m into P^n, $1 \le m \le n$. Then*

$$H_{2n-2m+1}(M_k(P^m, P^n); \mathbf{Z}) = \mathbf{Z}\Big/\binom{n+1}{m}|k|^m.$$

This result, which also was obtained in [5] by a different method, shows that two components of $M(P^m, P^n)$ are homotopy equivalent if and only if their associated degrees have the same absolute value.

5. Sections of projective bundles over lens spaces. As an example where the space Γ has a finite number of components we shall here consider $X = L^{2m+1}(p)$, the lens space obtained by letting $\mathbf{Z}/p$ act on S^{2m+1} in the usual way. Throughout this section we assume that $1 \le m < n$ and that p is odd. By Proposition 2.4, the components of Γ are classified by $H^2(L^{2m+1}(p); \mathbf{Z}) \cong \mathbf{Z}/p$.

As above, let $\Gamma_u \subset \Gamma$ be the component containing the section u: $L^{2m+1}(p) \to P(V)$ of $P\xi$. After inserting an extra stage in the Moore-Postnikov decomposition of $P\xi$ and noting that

$$H^r(L^{2m+1}(p); \mathbf{Z}/2) = \begin{cases} \mathbf{Z}/2, & r = 0, 2m+1, \\ 0, & \text{otherwise}, \end{cases}$$

we obtain the following tower of fibrations

$$\begin{array}{ccl}
 & & \Gamma_u \\
 & & \downarrow \\
K(\mathbf{Z}/2, t+2) \times K(\mathbf{Z}/2, 2n+3) & \to & \Gamma_4 \\
 & & \downarrow \\
K(\mathbf{Z}/2, t+1) \times K(\mathbf{Z}/2, 2n+2) & \to & \Gamma_3 \\
 & & \downarrow \\
K(\mathbf{Z}, t) \times \prod_{i=0}^{m} K(\mathbf{Z}/p, t+1+2i) & \to & \Gamma_2 \\
 & & \downarrow \\
K(\mathbf{Z}, 2) & = & \Gamma_1 \xrightarrow{\underline{k}^{2n+2}} K(\mathbf{Z}, t+1) \times \prod_{i=0}^{m} K(\mathbf{Z}/p, t+2+2i)
\end{array}$$

where $t = 2n - 2m$. Since

$$H^r(K(\mathbf{Z}/2, t+1) \times K(\mathbf{Z}/2, 2n+1+i); \mathbf{Z}/p) = 0$$

for $0 \leq r \leq t+2$, $i = 1, 2$, this implies

LEMMA 5.1. *$H^r(\Gamma_u; \mathbf{Z}/p) = H^r(\Gamma_2; \mathbf{Z}/p)$ for $0 \leq r \leq 2n - 2m + 2$.*

Let $a_m \in H^2(L^{2m+1}(p); \mathbf{Z})$ be a generator and choose $t(u) \in \mathbf{Z}$ such that $0 \leq t(u) < p$ and

$$t(u) a_m^m = \sum_{i=0}^{m} (-1)^i \binom{n+1-i}{n+1-m} c_i(\xi) c_1(u)^{m-i}$$

in $H^{2m}(L^{2m+1}(p); \mathbf{Z}) \cong \mathbf{Z}/p$. The first non-trivial component

$$k_0^{2n+2}\colon \Gamma_1 \to K\big(H^{2m}(L^{2m+1}(p); \mathbf{Z}), 2n - 2m + 2\big)$$

of $\underline{k}^{2n+2}$ is then, by Corollary 2.3, given by

$$\underline{k}_0^{2n+2} = t(u) a^{n-m+1} \otimes a_m^m.$$

Combining this with Lemma 5.1, we can prove

THEOREM 5.2. *When the base space $X = L^{2m+1}(p)$, $1 \leq m < n$, p odd, is a lens space, we have:*

(i) *$H^r(\Gamma_u; \mathbf{Z}) = H^r(\mathbf{Z}, 2; \mathbf{Z})$ for $0 \leq r < 2n - 2m$.*
(ii) *$H^{2n-2m}(\Gamma_u; \mathbf{Z}) = \mathbf{Z} \oplus \mathbf{Z}$.*
(iii) *$H^{2n-2m+1}(\Gamma_u; \mathbf{Z}/p) = \mathbf{Z}/t(u) \otimes \mathbf{Z}/p$.*
(iv) *There is a filtration*

$$\mathbf{Z}/t(u) \otimes \mathbf{Z}/p = J^0 \subset J^1 \subset J^2 = H^{2n-2m+2}(\Gamma_u; \mathbf{Z}/p)$$

where $J^1/J^0 \cong J^2/J^1 \cong \mathbf{Z}/p$.

Proof. The first two assertions follows easily from the constructed tower of fibrations. To prove the remaining cases we can, according to Lemma 5.1, use the cohomology spectral sequence $\{E_r^{st}\}$ with $\mathbf{Z}/p$-coefficients associated to $p_2\colon \Gamma_2 \to \Gamma_1$. Note that $E_2^{st} = 0$ when $0 < t < 2n - 2m$ or s is odd. The differentials $d_2\colon E_2^{0,2n-2m+2} \to E_2^{2,2n-2m+1}$ and $d_2\colon E_2^{0,2n-2m+1} \to E_2^{2,2n-2m}$ are trivial for so are the corresponding differentials in the spectral sequence for the path space fibration from which $p_2\colon \Gamma_2 \to \Gamma_1$ is induced. The only non-trivial relevant differential is thus $d_{2n-2m+2}\colon E_2^{0,2n-2m+1} \to E_2^{2n-2m+2,0}$ which is determined by $\underline{k}_0^{2n+2}$; i.e. there is an exact sequence

$$0 \to E_\infty^{0,2n-2m+1} \to \mathbf{Z}/p \xrightarrow{d_u} \mathbf{Z}/p \to E_\infty^{2n-2m+2,0} \to 0$$

where d_u is multiplication by $t(u)$. This proves (iii). To prove (iv), we note that $E_\infty^{0,2n-2m+2} = E_2^{0,2n-2m+2} = \mathbf{Z}/p$ and $E_\infty^{2,2n-2m} = E_2^{2,2n-2m} = \mathbf{Z}/p$. □

For any $k \in \mathbf{Z}$, let $M_k(L^{2m+1}(p), P^n)$ be the space of maps of degree k mod p of $L^{2m+1}(p)$ into P^n.

COROLLARY 5.3. *If* $M_k(L^{2m+1}(p), P^n)$ *is homotopy equivalent to* $M_l(L^{2m+1}(p), P^n)$, *then*

$$\gcd\left(\binom{n+1}{m}|k|^m, p\right) = \gcd\left(\binom{n+1}{m}|l|^m, p\right).$$

The above necessary condition only provides a partial solution to the homotopy classification problem for the components of the space $M(L^{2m+1}(p), P^n)$ of maps of $L^{2m+1}(p)$ into P^n. The complete solution is unknown.

After finishing this manuscript I learned that the results stated in Corollary 3.4 and in the remark immediately after Corollary 4.2 also have been obtained by M. C. Crabb and W. A. Sutherland.

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