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Our paper contains three theorems on regressive isols that are hereditarily odd-even. Two are characterizations of hereditarily odd-even isols in terms of a parity property of the isol and a property on the comparability of summands of the isol. In the third theorem, we show that if a regressive isol has a special comparability of summands property, then it has a predecessor that is hereditarily odd-even.

1. Introduction. The results presented in the paper developed from an interest in regressive isols that are hereditarily odd-even and in a special property about the comparability of summands that such isols are known to possess. The term *hereditarily odd-even* isol was introduced by T. G. McLaughlin in [3]. These are isols that are infinite, and each predecessor of the isol is either even or odd. It is among the regressive isols that these isols are especially interesting, for in that setting it is known that the hereditarily odd-even isols are the same as the hyper-torre isols (cf. [3]). E. Ellentuck studied hyper-torre isols in [2], and by using them it was shown that certain natural collections in the isols are models of the universal properties of arithmetic.

In this paper we are interested in regressive isols. We shall assume that the reader is familiar with topics in the monograph [3] on regressive sets and the theory of isols. In particular we use the metatheorem of A. Nerode that states that universal Horn sentences which are true in ω extend to statements which are true in the isols. This result is discussed in [3, Chapter 12]. The main concepts that we need are contained in the following two definitions.

DEFINITION D1. An isol is said to have *parity* if it is even or odd. An isol is said to have *4-parity* if it can expressed in one of the forms $4y$, $4y+1$, $4y+2$, or $4y+3$.

DEFINITION D2. An infinite regressive isol Y is said to have *comparability of summands* if whenever $Y = A + B$, either $A \leqq^* B$ or $B \leqq^* A$. Y

has *hereditary comparability of summands* if every $A \leqq Y$ has comparability of summands.

We shall use the following notations: (CS) for comparability of summands; (HCS) for hereditary comparability of summands. Much of the work presented here was motivated by the following result, which may be obtained from D2 and [**3**, Theorem 20.20]:

THEOREM T1. *Let Y be an infinite regressive isol. If Y is hereditarily odd-even, then Y has HCS, and hence also CS.*

Our paper contains three main results. Assume Y is an infinite regressive isol. We show that if Y has parity, then Y is hereditarily odd-even if and only if Y has HCS. Further if Y has 4-parity, then the following three properties for Y are equivalent: hereditarily odd-even; CS; HCS. Lastly, we show that if Y has CS, then Y has a predecessor that is hereditarily odd-even.

2. On comparability of summands. Let us assume throughout what follows that Y is an infinite regressive isol. It is easy to verify that if Y has CS, then $Y + 2$ does also. For assume Y has CS, and let $Y + 2 = A + B$. Let us assume both A and B are infinite, for otherwise their comparability is clear. Then $Y = (A - 1) + (B - 1)$, and therefore either $A - 1 \leqq {}^* B - 1$ or $B - 1 \leqq {}^* A - 1$. Then it follows that either $A \leqq {}^* B$ or $B \leqq {}^* A$. We see, that, $Y + 2$ also has CS. In contrast, we do not know if Y having CS implies that $Y + 1$ has CS.

It is easy to see that 4-parity implies parity, and that HCS implies CS. We do not know if HCS is equivalent to CS. We shall illustrate with an example to show an isol that has parity but not 4-parity. Let U be a universal regressive isol, and consider the even isol $2U$. Suppose $2U$ has 4-parity; then for some isol V and some i with $0 \leqq i < 4$, we have $2U = 4V + i$. Since an isol cannot be both even and odd, the only possibilities are $2U = 4V$ and $2U = 4V + 2$. If $2U = 4V + 2$, then $2(U - 1) = 4V$. But, it is easy to verify that if U is universal, so is $U - 1$; hence our problem reduces to showing that we cannot have U universal and $2U = 4V$. To see this, simply note $2x = 4y \rightarrow x = 2y$ is valid in ordinary arithmetic, so that if $2U = 4V$, then $U = 2V$ holds by virtue of the basic Nerode Metatheorem. But, no universal isol has parity. Thus, we can conclude that $2U$ does not have 4-parity.

We note that a regressive isol that is hereditarily odd-even may be characterized by the property that it is infinite, and each of its predecessors has parity. It follows that such an isol will have 4-parity and hence parity. Parity plays a significant role in our work, and the following lemma is fundamental for the results in our paper. For the lemma we assume the reader to be familiar with the relation $\leqq *$, between functions and between sets.

LEMMA L1. *If $2Y + 1$ has comparability of summands then Y has parity.*

Proof. Assume $2Y + 1$ has CS. Let $y_0, y_1, \ldots$ and $y_0^*, y_1^*, \ldots$ be (one-to-one) regressive enumerations of separated sets that belong to Y. Let t be any number that does not belong to one of these sets. We define:

$$\alpha = (t, y_1, y_1^*, y_3, y_3^*, y_5, y_5^*, \ldots);$$
$$\beta = (y_0, y_0^*, y_2, y_2^*, y_4, y_4^*, y_6, \ldots).$$

We note that the sets α and β are separated, and also that the enumerations given in each of their representations above is regressive. Let A and B be the regressive isols defined by $\alpha \in A$ and $\beta \in B$. It is easy to see that $2Y + 1 = A + B$. Therefore, as $2Y + 1$ has CS, it follows that $A \leqq * B$ or $B \leqq * A$.

Case 1. Assume $A \leqq * B$. Then also $\alpha \leqq * \beta$. And from that fact, it also follows that the mapping

$$\begin{array}{ccc} y_1^* & \to & y_2 \\ y_3^* & \to & y_4 \\ y_5^* & \to & y_6 \\ & \vdots & \end{array}$$

has a partial recursive extension. And from this we may conclude that the mapping, for $n \in \omega$, of y_{2n+1} to y_{2n+2} also has a partial recursive extension. Since y_n is a regressive function, we can conclude that $(y_1, y_3, y_5, \ldots)$ and $(y_2, y_4, y_6, \ldots)$ are recursively equivalent sets, and hence that they are in the same isol. If we now observe that $(y_0) \cup (y_1, y_3, \ldots) \cup (y_2, y_4, \ldots)$ is a set in Y, we may conclude from the previous fact that Y is an odd isol.

Case 2. Assume $B \leqq * A$. By reasoning similar to that in the previous case one may verify here that Y is an even isol. The details will be omitted. The proof of the lemma is complete.

PROPOSITION P1. *The following conditions are equivalent:*
(1) *Y is hereditarily odd-even;*
(2) *$2Y$ is hereditarily odd-even;*
(3) *$2Y + 1$ is hereditarily odd-even.*

Proof. It is easy to see that the hereditarily odd-even isols are closed under finite sums, and under sums with a finite number. From that fact we may obtain the implications (1) $\rightarrow$ (2) $\rightarrow$ (3). The other directions each follow because every infinite predecessor of an isol that is hereditarily odd-even is also hereditarily odd-even.

PROPOSITION P2. *Each of the following properties is valid*:
(1) *If* $2Y$ *has CS then* Y *has HCS*;
(2) *If* $2Y + 1$ *has CS then* Y *has HCS*.

Proof. We shall prove (1) first. Assume $2Y$ has CS. Let X be any predecessor of Y. We want to show that X has CS. We may assume X is infinite. Let $X + E = Y$ and let $X = A + B$. Then $2Y = (2A + E) + (2B + E)$. Hence, either $(2A + E) \leqq {}^*(2B + E)$ or $(2B + E) \leqq {}^*(2A + E)$. Recall that among regressive isols the relation $\leqq {}^*$ corresponds to the extension to the isols of the familiar relation $\leqq$ among numbers in ω. In the domain of ω, we know that the statement

$$(2a + e) \leqq (2b + e) \rightarrow a \leqq b$$

is valid. By the Nerode metatheorem it follows that the extension to the isols of that statement is also valid. But from $(2A + E) \leqq {}^*(2B + E)$, we obtain $A \leqq {}^* B$; and, similarly, from $(2B + E) \leqq {}^*(2A + E)$ we obtain $B \leqq {}^* A$. We therefore have the comparability of the two summands of X, and it follows that Y has HCS.

In the case of (2), we may argue in the same way, but using the number theoretic identity $(2a + e) \leqq (2b + e) + 1 \rightarrow a \leqq b$ in addition to $(2a + e) \leqq (2b + e) \rightarrow a \leqq b$. This completes our proof.

PROPOSITION P3. *The isol* $2Y + 1$ *has HCS if and only if* Y *is hereditarily odd-even*.

Proof. Assume first $2Y + 1$ has HCS. Let $A \leqq Y$. We wish to show A has parity. Since $2A + 1 \leqq 2Y + 1$, it follows that $2A + 1$ has CS. Clearly, if A is finite it has parity. If A is infinite then its parity follows from L1. Hence Y is hereditarily odd-even.

Assume now Y is hereditarily odd-even. Then, by P1, $2Y + 1$ is hereditarily odd-even. From T1 it then follows that $2Y + 1$ has HCS.

COROLLARY C1. *The isol* $2Y + 1$ *has HCS if and only if* $2Y + 1$ *is hereditarily odd-even*.

Proof. From P1 we know that $2Y + 1$ is hereditarily odd-even if and only if Y is hereditarily odd-even. If we now combine that fact with P3, the desired result is obtained.

THEOREM T2. *Let Y have parity. Then Y is hereditarily odd-even if and only if Y has HCS.*

Proof. From T1 it follows that if Y is hereditarily odd-even then Y has HCS. For the converse, assume Y has HCS. We shall consider two cases, based on the parity of Y. If Y is odd, then, by C1, it follows that Y is hereditarily odd-even. Assume now Y is even. Then $Y - 1$ is odd, and will also have HCS. Hence, by C1, $Y - 1$ is hereditarily odd-even. It follows easily from that fact that Y is also hereditarily odd-even. This completes our proof.

3. A characterization with 4-parity. In this section we wish to prove the second of our main results, that for regressive isols with 4-parity, all of the notions CS, HCS, and hereditarily odd-even are equivalent. We shall again assume throughout the section that Y is an infinite regressive isol. The next result is fundamental to our main theorem.

PROPOSITION P4. *If $2Y + 1$ has CS then $2Y + 1$ has HCS.*

Proof. Assume $2Y + 1$ has CS. Then, by L1 and P2, Y has parity and has HCS. By T2, Y is therefore hereditarily odd-even. From P1 and T1 respectively, it then follows that $2Y + 1$ is hereditarily odd-even and has HCS. The desired result follows.

COROLLARY C2. *If Y is odd, then the following conditions are equivalent*:

(1) *Y has CS*;
(2) *Y has HCS*;
(3) *Y is hereditarily odd-even.*

Proof. Assume Y is odd. Then the implications (1) $\rightarrow$ (2) $\rightarrow$ (3) follow from P4 and T2. The implication (3) $\rightarrow$ (2) follows from T1, and (2) $\rightarrow$ (1) is clear.

REMARK R1. We note two straightforward properties of the pair of isols Y and $Y + 1$. If Y has parity, then one of Y or $Y + 1$ must be odd. In addition, it is easy to see that if one of Y and $Y + 1$ is hereditarily odd-even, then the other is also. If we combine these with C2 we obtain the following result: if Y has parity, and both Y and $Y+1$ have CS, then Y is hereditarily odd-even.

PROPOSITION P5. *If any one of* $4Y$, $4Y+1$, $4Y+2$, $4Y+3$ *has CS, then all have HCS and all are hereditarily odd-even.*

Proof. Let $i \in (0, 1, 2, 3)$ and assume $4Y+i$ has CS. If $i=1$ or $i=3$ then $4Y+i$ is odd. In that event, it follows from C2 that $4Y+i$ is hereditarily odd-even. But then Y is also hereditarily odd-even. Let us now consider separately the two cases $i=2$ and $i=0$. In each case, we should like to prove that Y is hereditarily odd-even.

Case 1. Assume $i=2$. Then $4Y+2=2(2Y+1)$ has CS. By P2, then, $2Y+1$ has HCS. By P3 it follows that Y is hereditarily odd-even.

Case 2. Assume $i=0$. Then $4Y$ has CS. By the comment at the beginning of §2, it follows that $4Y+2$ also has CS. We may now apply Case 1 to conclude that Y is hereditarily odd-even.

From our assumption that $4Y+i$ has CS, it thus follows that Y is hereditarily odd-even. It is easy to verify that when Y is hereditarily odd-even each of $4Y+i$, for $i \in (0, 1, 2, 3)$, is also hereditarily odd-even. And then from T1 it follows that each of these isols also has HCS. This gives the desired result and completes our proof.

THEOREM T3. *If Y has 4-parity, then the following conditions are equivalent*:

(1) *Y has CS*;
(2) *Y has HCS*;
(3) *Y is hereditarily odd-even.*

Proof. Let us assume Y has 4-parity. The implication (1) $\to$ (2) follows from P5, and the implication (2) $\to$ (3) follows from P5 and the fact HCS implies CS. From the latter fact we also obtain (2) $\to$ (1), and (3) $\to$ (2) follows from T1. This completes our proof.

4. On the CS property alone. Assume Y is an infinite regressive isol. In this section we prove that if Y has CS, then Y has a predecessor that is hereditarily odd-even.

LEMMA L2. *Let Y have CS. Then there is an infinite isol A such that $2A \leqq Y$ and $2A$ has CS.*

Proof. Let y be a regressive function that ranges over a set belonging to Y. We define:

$$\alpha_0 = (y_0, y_3, y_4, y_7, y_8, \ldots);$$

$$\alpha_1 = (y_1, y_2, y_5, y_6, y_9, y_{10}, \ldots).$$

Then α_0 and α_1 are separated sets, and the enumerations given in each of their representations is regressive. Also, their union is the range of y, and hence either $\alpha_0 \leqq * \alpha_1$ or $\alpha_1 \leqq * \alpha_0$. We first consider separately these two possibilities.

Case 1. Assume $\alpha_0 \leqq * \alpha_1$. Then the mapping

$$\begin{array}{ccc} y_0 & \to & y_1 \\ y_4 & \to & y_5 \\ y_8 & \to & y_9 \\ & \vdots & \end{array}$$

has a partial recursive extension. Because y is a regressive function, it follows that the sets $(y_0, y_4, y_8, \ldots)$ and $(y_1, y_5, y_9, \ldots)$ are recursively equivalent. Let A be the isol that contains either one (and hence both) of these sets. Then $2A \leqq Y$.

Case 2. Assume $\alpha_1 \leqq * \alpha_0$. Then the mapping

$$\begin{array}{ccc} y_2 & \to & y_3 \\ y_6 & \to & y_7 \\ y_{10} & \to & y_{11} \\ & \vdots & \end{array}$$

has a partial recursive extension. As in the previous case, we see here that the sets $(y_2, y_6, y_{10}, \ldots)$ and $(y_3, y_7, y_{11}, \ldots)$ are recursively equivalent. Let A be the isol that contains one (and hence both) of these sets. We note $2A \leqq Y$.

In each of the cases considered above we have defined a particular infinite (regressive) isol $2A$ with $2A \leqq Y$. We now verify that $2A$ has CS. Consider the following two enumerations of the range of y; each corresponds to one of the cases, and, for that case, the associated members belonging to 2A are enclosed within blocks; from Case 1,

$$\left(\boxed{y_0, y_1}, y_2, y_3, \boxed{y_4, y_5}, y_6, y_7, \boxed{y_8, y_9}, \ldots\right),$$

and, from Case 2,

$$\left(y_0, y_1, \boxed{y_2, y_3}, y_4, y_5, \boxed{y_6, y_7}, y_8, y_9, \dots\right).$$

If Case 1 occurs, let

$$(y_2, y_6, y_{10}, \dots) \in B_0, \quad \text{and}$$
$$(y_3, y_7, y_{11}, \dots) \in B_1;$$

and, if Case 2 occurs, let

$$(y_0, y_4, y_8, \dots) \in B_0, \quad \text{and}$$
$$(y_1, y_5, y_9, \dots) \in B_1.$$

We note, that, independent of which case occurs, $Y = 2A + B_0 + B_1$, $B_1 \leqq {}^* B_0$, and $B_0 \leqq {}^* B_1 + 1$. To prove $2A$ has CS, assume $2A = U + V$. Then $Y = (U + B_0) + (V + B_1)$. Since Y has CS, either

$$\text{(a)} \quad U + B_0 \leqq {}^* V + B_1, \quad \text{or}$$
$$\text{(b)} \quad V + B_1 \leqq {}^* U + B_0.$$

We wish to show that the summands U and V of $2A$ are comparable under the relation $\leqq {}^*$. Note that in ω the following statements are valid:

$$(\mathrm{S_a}) \quad (u + b_0 \leqq v + b_1 \wedge b_1 \leqq b_0) \rightarrow u \leqq v,$$
$$(\mathrm{S_b}) \quad (v + b_1 \leqq u + b_0 \wedge b_0 \leqq b_1 + 1 \wedge u + v = 2a) \rightarrow v \leqq u.$$

Therefore, by the Nerode metatheorem, the extensions to the isols of these statements are also valid. If we combine the extension of $(\mathrm{S_a})$ with (a) and the property $B_1 \leqq {}^* B_0$, then it follows $U \leqq {}^* V$. If we combine the extension of $(\mathrm{S_b})$ with (b) and the properties $B_0 \leqq {}^* B_1 + 1$ and $U + V = 2A$, then it follows that $V \leqq {}^* U$. Hence the summands U and V of $2A$ shall be comparable under the relation $\leqq {}^*$. Thus we see that $2A$ has CS, and this completes our proof.

Remark R2. Let U be an infinite isol. Then U is said to be *multiple-free* if whenever $2A \leqq U$ then A is a finite isol. Isols that are regressive and multiple-free are studied in [3]; such isols have a special interest for they are known to be universal. We note from L2 that no isol that is regressive and multiple-free can have CS.

Theorem T4. *Let Y have CS. Then Y has a predecessor that is hereditarily odd-even.*

Proof. By L2 we can first obtain an infinite even predecessor $2A$ or Y with $2A$ having CS. From P2 it follows that A has HCS. Applying L2 to A

will give an infinite even predecessor $2B$ of A. Clearly $2B$ also has HCS, and, therefore, by T2, $2B$ is hereditarily odd-even. Since $2B \leqq Y$, the theorem follows.

5. Concluding remarks.

REMARK R3. We should like to discuss briefly one property of isols with CS or HCS that was not taken up earlier in the paper. If $f\colon \omega \to \omega$ is a recursive function then f_Λ denotes the *Myhill–Nerode extension* of f to the isols. Let f be an increasing recursive function and let Y be hereditarily odd-even (and regressive). It is proved in [**3**, Theorem 20.18] that the value of $f_\Lambda(Y)$ is either finite or hereditarily odd-even. In view of this result and the earlier theorems in the paper, we see that there are some different settings where properties of $f_\Lambda(Y)$ having CS or HCS may be obtained.

REMARK R4. We should like to close the paper with some open problems. Assume Y is an infinite regressive isol. It was noted, in §2, that if Y has CS then $Y+2$ has CS. The following question is open: (1) If Y has CS, will $Y+1$ have CS? The result L1 was important for the new theorems in the paper, because it related a CS property to a conclusion about parity. L1 concerns odd isols that have CS. Whether or not the corresponding result for even isols is valid is open. Stated completely, it is the following question: (2) If $2Y$ has CS, does Y have parity? The following two questions are also open; the second one was also posed in [**3**]: (3) If both Y and $Y+1$ have CS, does it follow that Y has parity? (4) Does there exist a regressive isol that is both hereditarily odd-even and cosimple?

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