

# Pacific Journal of Mathematics

**NOT EVERY LODATO PROXIMITY IS COVERED**

K. C. CHATTOPADHYAY

# NOT EVERY LODATO PROXIMITY IS COVERED

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In a recent paper Reed wrote, "In fact it may be that all Lodato proximities are covered. I was unable to find a counterexample". (Remark 1.10)

The purpose of this note is to show that, in general, Lodato proximities are not covered.

**1. Preliminaries.** A closed filter  $\mathcal{F}$  on a topological space  $(X, c)$  is a proper filter (that is, a filter which does not contain the empty set) which has a base consisting of only closed sets. Maximal (with respect to set inclusion) closed filters are all called *ultraclosed filters*. For more information on the concept of ultraclosed filters see Thron [3].

*Ultrafilters* are maximal proper filters on a set and *grills* are exactly the unions of ultrafilters. For a detailed discussion on ultrafilters and grills, see Thron [2].

A basic proximity  $\pi$  on a set  $X$  is a symmetric binary relation on the power set  $\mathcal{P}(X)$  of  $X$  satisfying the conditions:

$$\begin{aligned}(A, B \cup C) \in \pi &\Leftrightarrow (A, B) \in \pi \text{ or } (A, C) \in \pi, \\ A \cap B \neq \emptyset &\Rightarrow (A, B) \in \pi, \\ (A, \emptyset) &\notin \pi, \quad \forall A \subset X.\end{aligned}$$

The pair  $(X, \pi)$  is called a *basic proximity space* provided  $\pi$  is a basic proximity on  $X$ .

For a basic proximity  $\pi$  on  $X$ , we define

$$c_\pi(A) = \{x \in X: (\{x\}, A) \in \pi\} \quad \text{for all } A \subset X.$$

It is easily verified that  $c_\pi$  is a symmetric (Čech) closure operator. For a basic proximity  $\pi$ ,  $c_\pi$  need not be a Kuratowski closure operator.

A basic proximity  $\pi$  on  $X$  is called a *Lodato proximity* if the following condition is satisfied:

$$(c_\pi(A), c_\pi(B)) \in \pi \Rightarrow (A, B) \in \pi.$$

If  $\pi$  is a Lodato proximity on  $X$  then  $c_\pi$  is a Kuratowski closure operator on  $X$  and hence  $(X, c_\pi)$  is a topological space.

Let  $(X, \pi)$  be a basic proximity space and  $\mathcal{G}$  be a grill on  $X$ . Then  $\mathcal{G}$  is called a  $\pi$ -clan if

$$(A, B) \in \pi \quad \text{for all } A, B \text{ in } \mathcal{G}.$$

For more detailed information on the concepts discussed above, see Thron [2].

Let  $\pi$  be a Lodato proximity on  $X$ . Following Reed [1] we define the following concepts:

A *Wallman  $\pi$ -clan* is a  $\pi$ -clan which contains some ultraclosed filter. The proximity  $\pi$  is said to be *covered* if for each  $(A, B) \in \pi$  there exists a Wallman  $\pi$ -clan  $\mathcal{G}$  such that  $\{A, B\} \subset \mathcal{G}$ .

We conclude this section by proving the following results which will be used to make the final conclusion.

**1.1. PROPOSITION.** *Let  $\mathcal{U}$  be an ultraclosed filter on  $(X, c)$  and  $\mathcal{A}$  a base of  $\mathcal{U}$  consisting of closed sets. If  $F$  is a closed set and  $F \cap A \neq \emptyset$  for all  $A$  in  $\mathcal{A}$  then  $F \in \mathcal{U}$ .*

*Proof.* Let  $\mathcal{B}$  be the collection of all finite intersections of members of the family  $\mathcal{A} \cup \{F\}$ . Then  $\mathcal{B}$  is a filter base consisting of closed sets. Let  $\mathcal{U}_0$  be the filter generated by  $\mathcal{B}$  as a base. Then  $\mathcal{U}_0$  is a closed filter and  $\mathcal{U}_0 \supset \mathcal{U} \cup \{F\}$ . By the maximality of  $\mathcal{U}$  it follows that  $F \in \mathcal{U}$ .

**1.2. COROLLARY.** *Let  $\mathcal{U}$  be an ultraclosed filter on  $(X, c)$  and  $V$  an open set such that  $V \cap F \neq \emptyset$  for all  $F$  in  $\mathcal{U}$ . Then  $V \in \mathcal{U}$ .*

*Proof.* If possible suppose that  $V \notin \mathcal{U}$ . Let  $\mathcal{A}$  be a base of  $\mathcal{U}$  consisting of closed sets. Then  $V \not\supset A$  for all  $A \in \mathcal{A}$ . Thus  $(X - V) \cap A \neq \emptyset$  for all  $A \in \mathcal{A}$ . Since  $X - V$  is closed, by the above result it follows that  $X - V \in \mathcal{U}$  and hence  $V \cap (X - V) \neq \emptyset$  — a contradiction.

**1.3. PROPOSITION.** *On a compact topological space  $(X, c)$  every ultraclosed filter converges.*

*Proof.* Let  $\mathcal{U}$  be an ultraclosed filter on  $(X, c)$ . Since the space is compact it follows that there exists an  $x$  in  $X$  such that  $x \in c(F)$  for all  $F \in \mathcal{U}$ . Let  $V$  be an open neighbourhood of  $x$ . Then  $V \cap F \neq \emptyset$  for all  $F \in \mathcal{U}$ . Thus by the above corollary,  $V \in \mathcal{U}$ . Hence  $\mathcal{U}$  converges to  $x$ .

**1.4. PROPOSITION.** *On a  $T_1$ -space  $(X, c)$ , every convergent ultraclosed filter has the form  $\mathcal{U}(x)$ , for some  $x \in X$ , where  $\mathcal{U}(x) = \{A \subset X: x \in A\}$ .*

*Proof.* Let  $\mathcal{U}$  be an ultraclosed filter on  $(X, c)$  such that it converges to a point  $x \in X$ . Obviously  $x \in c(F)$  for all  $F \in \mathcal{U}$ . Hence, in particular,

$x$  belongs to each member of a base of  $\mathcal{U}$  consisting of closed sets. Since  $\{x\}$  is a closed set it follows by Proposition 1.1, that  $\{x\} \in \mathcal{U}$ . Thus  $\mathcal{U} = \mathcal{U}(x)$ .

**1.5. THEOREM.** *Let  $(X, c)$  be a compact  $T_1$ -space such that it has two infinite components. Then*

$$\pi = \{(E, F): c(E) \cap c(F) \neq \emptyset \text{ or } E \text{ and } F \text{ are both infinite}\}$$

*is a Lodato proximity on  $X$  such that  $c_\pi = c$  and  $\pi$  is not covered.*

*Proof.* It is easy to verify that  $\pi$  is indeed a Lodato proximity on  $X$  such that  $c_\pi = c$ .

Let  $A, B$  be two infinite components of  $(X, c)$ . Obviously  $(A, B) \in \pi$ . However, no Wallman  $\pi$ -clan can contain both  $A$  and  $B$ . For suppose  $\mathcal{G}$  is such a Wallman  $\pi$ -clan. Let  $\mathcal{U}$  be an ultraclosed filter such that  $\mathcal{U} \subset \mathcal{G}$ . Then since  $(X, c)$  is a compact  $T_1$ -space it follows, by Propositions 1.3 and 1.4, that  $\mathcal{U} = \mathcal{U}(x)$  for some  $x \in X$ . Thus  $\{x\}, A$  and  $B$  are all in  $\mathcal{G}$ . From this it follows that

$$x \in c_\pi(A) \cap c_\pi(B) = c(A) \cap c(B) = A \cap B = \emptyset.$$

Clearly this is impossible.

**2.** Many examples of compact  $T_1$ -spaces with two infinite components can easily be constructed. Two such examples are given below.

**2.1. EXAMPLE.** Let  $X$  be the union of closed intervals  $[1, 2]$  and  $[3, 4]$ . Then  $X$  with the topology induced by the usual topology of real line is an example of a compact  $T_1$ -space with two infinite components.

**2.2. EXAMPLE.** Let  $X = A \cup B$  such that  $A, B$  are both infinite sets and  $A \cap B = \emptyset$ . Define  $c: \mathcal{P}(X) \rightarrow \mathcal{P}(X)$  by

$$\begin{aligned} c(D) &= D && \text{if } D \text{ is a finite subset of } X, \\ &= A \cup D && \text{if } A \cap D \text{ is infinite and } B \cap D \text{ is finite,} \\ &= B \cup D && \text{if } B \cap D \text{ is infinite and } A \cap D \text{ is finite,} \\ &= X && \text{otherwise.} \end{aligned}$$

Then  $(X, c)$  is a  $T_1$ -topological space with two infinite components  $A$  and  $B$ .

Set

$$\begin{aligned} \mathcal{A}_1 &= \{A - F: F \text{ is a finite subset of } A\}, \\ \mathcal{A}_2 &= \{B - F: F \text{ is a finite subset of } B\}, \\ \mathcal{A}_3 &= \{X - F: F \text{ is a finite subset of } X\}. \end{aligned}$$

Then  $\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3$  is the collection of all nonempty open sets in  $(X, c)$ . Let  $\mathcal{A}$  be an open cover of  $X$ . If  $\mathcal{A} \cap \mathcal{A}_3 \neq \emptyset$  then obviously  $\mathcal{A}$  has a finite subcover. If  $\mathcal{A} \cap \mathcal{A}_3 = \emptyset$  then, since  $\mathcal{A}$  covers  $X$ ,  $\mathcal{A} \cap \mathcal{A}_1 \neq \emptyset$  and  $\mathcal{A} \cap \mathcal{A}_2 \neq \emptyset$  and hence in this case also  $\mathcal{A}$  has a finite subcover. Thus the space is compact.

**2.3. REMARK.** By Theorem 1.5 and Examples 2.1 and 2.2 it follows that there are Lodato proximities which are not covered.

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