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We consider the following problem: let $x \in R^n$, $t \in R^+$, and let σ: $R^n \to R^+$ be a given lipschitz continuous surface with lipschitz constant 1:

$$(1) \qquad |\nabla\sigma(x)| \leq 1, \quad \text{a.e. on } R^n.$$

Let $f \in H^1_{\text{loc}}(R^n)$ and $g \in L^2_{\text{loc}}(R^n)$; then we prove that there exists a unique solution of the following system of equations:

$$(2) \qquad \text{Supp}\Box u \subset \{(x,t): t = \sigma(x), t > 0\};$$

$$(3) \qquad u(x,0) = f(x); \quad u_t(x,0) = g(x);$$

$$(4) \qquad \frac{\partial u}{\partial t}(x, \sigma(x) + 0) = -\frac{\partial u}{\partial t}(x, \sigma(x) - 0)$$

$$\text{on } \{x: \sigma(x) > 0 \,\&|\nabla\sigma(x)| \leq 1\},$$

where $\Box = \partial^2/\partial t^2 - \Delta$ is the wave operator in $\mathbf{R}^n \times R^+$. The one-dimensional case has been studied by M. Schatzman, who used it in the problem of a string compelled to remain above an obstacle.

The difficulty in solving the problem lies in the fact that as $\sigma 2$ may be characteristic, one has to show that (4) makes sense. More generally, we show that, if u is a solution of finite energy of the wave equation, one may take traces of $\partial u/\partial t$ on either side of the non-characteristic parts of a non-time-like surface. We make use of techniques from harmonic analysis, such as maximal functions on thin sets, and Fourier integral operators.

Once this is done, we show that if v is the solution of the free wave equation

$$(5) \qquad \Box v = 0, \quad v(x,0) = f(x), \quad v_t(x,0) = g(x);$$

and if a measure $\mu(v)$ is defined on test functions by

$$(6) \quad \langle \mu(v), \psi \rangle = -2\int_{x:\,\sigma(x)>0} \psi(x, \sigma(x))\, v_t(x, \sigma(x))\left(1 - |\nabla\sigma(x)|^2\right) dx,$$

then the unique solution of (2)–(4) is given by

$$(7) \qquad u = v + \mathscr{E} * \mu(v),$$

where $\mathscr{E}$ is the elementary solution supported in $t > |x|$ of the wave equation in $\mathbf{R}^n \times R^+$.

Our result represents a trend towards some kind of "hyperbolic capacity" theory; it is known that one take traces of solutions of the Laplace and heat equations on sets of elliptic (respectively, parabolic) positive Λ hyperbolic capacity. If one defines a characteristic surface as a set of zero hyperbolic capacity, then we have proved that one can take traces on subsets of positive hyperbolic capacity of time-like surfaces.

1. For a general space-like and sufficiently smooth hypersurface S in $\mathbf{R}^{n+1}$, the solution to the Cauchy problem

$$\Box u = 0, \quad u|_S = f, \quad \left.\frac{\partial u}{\partial n}\right|_S = g,$$

where $\Box = \partial^2/\partial t^2 - \partial^2/\partial x_1^2 - \cdots - \partial^2/\partial x_n^2$ and $\partial/\partial n$ is the normal derivative to S, is given by the integral representation

$$(*) \quad u(P) = \frac{1}{H_n(\alpha)} \int_{S^P} \Big(g(Q)[P-Q]^{(\alpha-n-1)/2} - f(Q) \frac{\partial}{\partial n} [P-Q]^{(\alpha-n-1)/2} \Big)\, ds$$

for $\alpha = 2$ in the sense of the analytic continuation of the integral as a function of α, and

$$[P] = t^2 - x_1^2 - \cdots - x_n^2, \qquad P = (t, x_1, \ldots, x_n),$$
$$H_n(\alpha) = \pi^{(n-1)/2} 2^{\alpha-1} \Gamma\Big(\frac{\alpha}{2}\Big) \Gamma\Big(\frac{\alpha+1-n}{2}\Big),$$
$$S^P = \{Q \in S\colon [P-Q] \geq 0, (P-Q)\cdot \mathbf{1} \leq 0\},$$
$$\mathbf{1} = (1, 0, \ldots, 0) \in \mathbf{R}^{n+1}.$$

Formula $(*)$ is an application of Green's theorem with respect to the Lorentz metric (for details, see [3]). In the sequel we need the following:

LEMMA 1. *For* $S = \{(t, x_1, \ldots, x_n)\colon t = 0\}$,
(i)

$$g \mapsto M_1 g(x) = \sup_{t>0} \left| \frac{t^{1-\alpha}}{H_n(\alpha)} \int_{\mathbf{R}^n} g(y) \left[t^2 - |x-y|^2\right]^{(\alpha-n-1)/2} dy \right|$$

is a bounded operator from $L^2(\mathbf{R}^n)$ *to* $L^2(\mathbf{R}^n)$ *for all* $\alpha > 1$, *and*
(ii)

$$f \mapsto M_2 f(x) = \sup_{t>2} \left| \frac{t^{2-\alpha}}{H_n(\alpha)} \int_{\mathbf{R}^n} f(y) \frac{\partial}{\partial t} \left[t^2 - |x-y|^2\right]^{(\alpha-n-1)/2} dy \right|$$

is a bounded operator from $H^1(\mathbf{R}^n)$ *to* $L^2(\mathbf{R}^n)$ *for all* $\alpha > 1$.

Proof. (ii) is an easy consequence of (i) via the Fourier transform, and (i) is a result of Stein [5].

We now let $t = \sigma(x)$ be a Lipschitz continuous function with Lipschitz constant 1, i.e., the graph of $\sigma(x)$ is a non-time-like hypersurface and T is the following operator:

$$Tf(x) = \int_{\mathbf{R}^n} e^{i(x\cdot\xi + \sigma(x)|\xi|)}\hat{f}(\xi)\, d\xi.$$

For this operator we have the following estimate:

LEMMA 2. *T: $L^2_{\text{comp}}(\mathbf{R}^n) \to L^2_{\text{loc}}(\Omega)$, where $\Omega = \{x \in \mathbf{R}^n, |\nabla\sigma(x)| < 1\}$.*

Proof. Let K be a compact set contained in Ω and φ a non-negative smooth function with compact support in Ω, where $\varphi \equiv 1$ on K. Then

$$\int_K |Tf(x)|^2\, dx \le \int_{\mathbf{R}^n} |Tf(x)|^2 \varphi(x)\, dx$$
$$= \iint \hat{f}(\xi)\overline{\hat{f}(\eta)} d\xi\, d\eta \int e^{ix(\xi-\eta)} e^{i\sigma(x)(|\xi|-|\eta|)}\varphi(x)\, dx.$$

We now let

$$\phi(\xi, \eta) = \int e^{ix(\xi-\eta)} e^{i\sigma(x)(|\xi|-|\eta|)}\varphi(x)\, dx.$$

Using the change of variable

$$u = x + \sigma(x)\frac{\xi - \eta}{|\xi - \eta|^2}(|\xi| - |\eta|)$$

and integrating by parts, one sees easily that

$$|\varphi(\xi, \eta)| \le C_N / \left(1 + |\xi - \eta|^N\right).$$

We use the Cauchy-Schwarz inequality to finish the proof.

LEMMA 3. *We let S be a space-like hypersurface in $\mathbf{R}^{n+1}$,*

$$v_1^\alpha(P) = \frac{1}{H_n(\alpha)} \int_{S^P} g(Q)[P - Q]^{(\alpha-n-1)/2}\, dS,$$

$$v_2^\alpha(P) = \frac{1}{H_n(\alpha)} \int_{S^P} f(Q)\frac{\partial}{\partial n}[P - Q]^{(\alpha-n-1)/2}\, dS,$$

and $P = R + \nu N$, where $R \in S$ and N is the normal to S at R. Then

(i) $$g \to M_1^\alpha g(R) = \sup_{k\ge 0} 2^{k(\alpha-1)}\left|v_1^\alpha(R + 2^{-k}N)\right|$$

is a bounded operator from L^2_{comp} to L^2_{loc} for all $\alpha > 1$ and

(ii) $$f \to M_2^\alpha f(R) = \sup_{k\ge 0} 2^{k(\alpha-2)}\left|v_2^\alpha(R + 2^{-k}N)\right|$$

is a bounded operator from H^1_{comp} to L^2_{loc} for all $\alpha > 1$.

Proof. This result is essentially known and has been proved in collaboration with other authors (see *On operators of harmonic analysis which are not convolution* by R. R. Coifman in [7].) The proof is lengthy but straightforward. We omit the details.

An immediate consequence is the following:

COROLLARY. *Let $\sigma(x)$ be a Lipschitz continuous function with Lipschitz constant* 1 *and $S = \{(t, x): t = \sigma(x)\}$ (S is a non-time-like hypersurface in $\mathbf{R}^{n+1}$), then for every compact $K \subset \{(t, x): \sigma(x) > 0,\ t = \sigma(x)$ and $|\nabla\sigma(x)| < 1\}$ there exists an integer $l = l(K)$ such that the conclusions of Lemma* 3 *hold for the maximal functions*

$$M_1^{l,\alpha}g(R) = \sup_{k \geq l} 2^{k(\alpha-1)}\left|v_1^\alpha(R + 2^{-k}N)\right|,$$

$$M_2^{l,\alpha}f(R) = \sup_{k \geq l} 2^{k(\alpha-2)}\left|v_2^\alpha(R + 2^{-k}N)\right|$$

on K.

From this corollary we deduce that

$$g(R) = \lim_{k\to\infty} \frac{1}{2^{-k}} v_1^\alpha(R + 2^{-k}N)\big|_{\alpha=2}$$

and

$$f(R) = \lim_{k\to\infty} v_2^\alpha(R + 2^{-k}N)\big|_{\alpha=2}$$

a.e. on the noncharacteristic part of S.

2. We now state our main result.

THEOREM. *The initial value problem*

(i)

$$u \in W^{1,\infty}_{\mathrm{loc}}\left(\mathbf{R}^+; \left(\mathbf{R}^+; L^2_{\mathrm{loc}}(\mathbf{R}^n)\right)\right) \cap L^\infty_{\mathrm{loc}}\left(\mathbf{R}^+, H^1_{\mathrm{loc}}(\mathbf{R}^n)\right),$$

(ii)

$$\operatorname{Supp} \Box u \subset \{(t, x): t = \sigma(x), t > 0\},$$

(iii)

$$u(0, x) = f(x), \quad f \in H^1_{\mathrm{loc}}(\mathbf{R}^n),$$
$$u_t(0, x) = g(x), \quad g \in L^2_{\mathrm{loc}}(\mathbf{R}^n),$$

(iv)

$$u_t^+(\sigma(x), x) = -u_t^-(\sigma(x), x) \quad \text{on } \{x: \sigma(x) > 0, |\nabla\sigma(x)| < 1\}$$

has a unique solution given by $u = v + \mathscr{E} * \mu(v)$, where v is the free solution to the wave equation with initial data f and g, $\mathscr{E}$ is the elementary solution to the wave equation, and $\mu(v)$ is the measure given by

$$\langle \psi, \mu(v) \rangle = -2 \int_{x:\, \sigma(x) > 0} \psi(\sigma(x), x) v_t(\sigma(x), x)\left(1 - |\nabla \sigma(x)|^2\right) dx.$$

The proof of this theorem will be a consequence of the following lemmas. We let $S = \{(t, x) : t = \sigma(x)\}$.

LEMMA. *For u satisfying conditions* (i)–(iii) *of Theorem* 1, $u_t^+|_S$ *and* $u_t^-|_S$ *are defined a.e. on the set* $\{x : \sigma(x) > 0, |\nabla \sigma(x)| < 1\}$.

Proof. For $0 < t < \sigma(x)$,

$$u(t, x) = v(t, x) = \int_{\mathbf{R}^n} e^{ix\cdot\xi} \cos t|\xi| \hat{f}(\xi)\, d\xi + \int_{\mathbf{R}^n} e^{ix\cdot\xi} \frac{\sin t|\xi|}{|\xi|} \hat{g}(\xi)\, d\xi$$

and

$$u_t(x, x) = -\int_{\mathbf{R}^n} e^{ix\cdot\xi} \sin t|\xi|\, |\xi| \hat{f}(\xi)\, d\xi + \int_{\mathbf{R}^n} e^{ix\cdot\xi} \cos t|\xi| \hat{g}(\xi)\, d\xi,$$

which shows that $u_t^-(\sigma(x), x)$ is a linear combination of integrals similar to the one given in Lemma 2. Using Lemma 2, we then have the desired conclusion for $u_t^-|_S$. To show the same conclusion for $u^+t|_S$, we write

$$u_t^+ = u_t^- + \left(\frac{\partial^+ u}{\partial n} - \frac{\partial^- u}{\partial n}\right)\left(1 - |\nabla \sigma|^2\right)^{-1/2} \quad \text{on } S$$

(see the proof of Lemma 5). The proof is then reduced to showing the existence a.e. of $\partial^+ n/\partial n$ and $\partial^- n/\partial n$ on the same set. It is easily seen that $\partial^- u/\partial n|s$ is a linear combination of integrals similar to the one given in Lemma 2. We thus have the desired conclusion in this case. For $\partial^+ u/\partial n$ we use condition (i), the integral representation of u, and the corollary to Lemma 3(i) to obtain the desired boundary values. We notice that in this case we have a Fatour type theorem. Other results of this type can be found in **[2]**, **[6]**.

LEMMA 5. *For u as in Theorem* 1, $\Box u = \mu(v)$ *is a measure given by*

$$\langle \psi, \mu(v) \rangle = -2 \int_{x:\, \sigma(x) > 0} \psi(\sigma(x), x) v_t(\sigma(x), x)\left(1 - |\nabla \sigma(x)|^2\right) dx.$$

Proof. We first recall the following Green formula: for D a domain in $\mathbf{R} \times \mathbf{R}^n$, we have

$$\int_D (\psi \Box \varphi - \varphi \Box \psi)\, dt\, dx = -\int_{\partial D} \left(\psi \frac{\partial \varphi}{\partial n} - \varphi \frac{\partial \psi}{\partial n}\right) dS,$$

where n (resp. dS) is the inner normal to ∂D (resp. the area element) with respect to the Lorentz metric. The proof of this formula is given in [3].

We now let D be a component of $\{(t, x)\colon 0 < t < \sigma(x)\}$, ψ a C^∞ function with compact support in the upper half-space $t > 0$, η a C^∞ function equal to 1 near the set $\{(\sigma(x), x)\colon (\sigma(x), x) \in \partial D\}$ and equal to zero outside a tubular neighborhood of the same set, and ρ_ε an approximation of unity. Then

$$\begin{aligned}\iint_D \psi\Box u &= \lim_{\varepsilon\to 0}\iint_D \psi\Box(u\eta * \rho_\varepsilon)\\ &= \lim_{\substack{\varepsilon\to 0\\ \eta}} u\eta * \rho_\varepsilon\Box\psi - \lim_{\varepsilon\to 0}\int_{\partial D}\psi\frac{\partial u\eta}{\partial n} * \rho_\varepsilon - u\eta * \rho_\varepsilon\frac{\partial\psi}{\partial n}\\ &= \lim_{\eta}\iint_D u\eta\Box\psi - \lim_{\eta}\psi\frac{\partial u\eta}{\partial n} - u\frac{\partial\psi}{\partial n}.\end{aligned}$$

But

$$\frac{\partial u\eta}{\partial n} = \frac{\partial u}{\partial n}\eta = \frac{\partial u}{\partial n} \quad \text{on } \{(\sigma(x), x)\colon (\sigma(x), x) \in \partial D\}.$$

Therefore:

$$\iint_D \psi\Box u = -\int_{\partial D}\left(\psi\frac{\partial^- u}{\partial n} - \frac{\partial\psi}{\partial n}\right) dS.$$

By summing over all the components we have

$$\iint_{t>0}\psi\Box u = \int_{\sigma(x)>0}\psi\left(\frac{\partial^+ u}{\partial n} - \frac{\partial^- u}{\partial n}\right) dS.$$

Now, for any function w,

$$\frac{\partial w}{\partial n} = \frac{1}{\left(1 - |\nabla\sigma|^2\right)^{1/2}}(w_t + \nabla_x w\cdot\nabla\sigma)$$

and

$$(\nabla w|_S)\cdot\nabla\sigma = w_t|\nabla\sigma|^2 + \nabla_x w\cdot\nabla\sigma \quad \text{on } S.$$

For $w = u - v$, $w|_S = 0$; thus

$$w_t|\nabla\sigma|^2 + \nabla_x w\cdot\nabla\sigma = 0 \quad \text{on } S.$$

Hence, on S,

$$\frac{\partial^+ u}{\partial n} - \frac{\partial^- u}{\partial n} = \frac{\partial^+ u}{\partial n} - \frac{\partial u}{\partial n} = \frac{1}{\left(1 - |\nabla\sigma|^2\right)^{1/2}}\left[(u_t^+ - v_t) - (u_t^+ - v_t)|\nabla\sigma|^2\right]$$

$$= (u_t^+ - v_t)\left(1 - |\nabla\sigma|^2\right)^{1/2} = -2v_t\left(1 - |\nabla\sigma|^2\right)^{1/2}$$

by condition (iv) of Theorem 1. This establishes the desired formula. Notice that $dS = (1 - |\nabla\sigma(x)|^2)^{1/2}\,dx$. To finish the proof of Lemma 5, we need to show that the distribution

$$\mu(v)\colon \psi \to -2\int_{x:\,\sigma(x)>0} \psi(\sigma(x), x) v_t(\sigma(x), x)\left(1 - |\nabla\sigma(x)|^2\right) dx$$

extends to a measure. It is then enough to show that $v_t(1 - |\nabla\sigma|^2)$ is locally integrable. By Lemma 2, $v_t(1 - |\nabla\sigma|^2)^{1/2}$ is locally square integrable on the set $\{x\colon |\nabla\ \sigma(x)| < 1\}$, which implies that $v_t(1 - |\nabla\sigma|^2)$ is locally integrable on the same set.

LEMMA 6. *For $u = v + \mathscr{E} * \nu(v)$, u_t^+ exists a.e. on the set $\{x\colon \sigma(x) > 0, |\nabla\sigma(x)| < 1\}$, and, for $\sigma(x) > 0$, $u_t^+(\sigma(x), x) = -v_t(\sigma(x), x)$.*

Proof. For (t, x) sufficiently close to the set $\{(\sigma(y), y)\colon \sigma(y) > 0, |\nabla\sigma(y)| < 1\}$, we may write

$$u = v - \frac{2}{H_n(\alpha)}\int_{\sigma(y)>0} \left[(t - \sigma(y))^2 - |x - y|^2\right]_+^{(\alpha-n-1)/2}$$

$$\cdot v_t(\sigma(y), y)\left(1 - |\nabla\sigma(y)|^2\right) dy$$

for $\alpha = 2$ in the sense of the analytic continuation of the integral as a function of α. We let $P = (t, x)$, $Q = (\sigma(y), y)$, $w(Q) = (1 - |\nabla\sigma(y)|^2)^{1/2}$, and $r_{PQ}^+ = [(t - \sigma(y))^2 - |x - y|^2]_+$. We then have

$$u = v - \frac{2}{H_n(\alpha)}\int r_{PQ}^{+\,(\alpha-n-1)/2} v_t w\, dS|_{\alpha=2}.$$

The distribution

$$\lambda(P, dS) = \frac{1}{H_n(\alpha)} r_{PQ}^{+\,(\alpha-n-1)/2}\, dS$$

in the sense of the analytic continuation as a function of α, for $\alpha = 2$, is supported by S^P, and as a function of P its restriction to S is zero. We then have

$$\frac{\partial\lambda}{\partial n} = \frac{\partial\lambda}{\partial t}\left(1 - |\nabla\sigma|^2\right)^{1/2} \quad \text{on } S.$$

Thus

$$u_t^+ - v_t = -\frac{2}{H_n(\alpha)}\int \frac{\partial}{\partial t} r_{PQ}^{+\,(\alpha-n-1)/2} v_t w\, dS$$

$$= -2\frac{1}{H_n(\alpha)}\int \frac{\partial}{\partial n_P} r_{PQ}^{+\,(\alpha-n-1)/2} v_t\, dS$$

$$= 2\frac{1}{H_n(\alpha)}\int \frac{\partial}{\partial n_Q} r_{PQ}^{+\,(\alpha-n-)/2} v_t\, dS.$$

By the construction of the solution to the Cauchy problem (see §1), the restriction of the integral

$$\frac{1}{H_n(\alpha)} \int \frac{\partial}{\partial n} r_{PQ}^{+\,(\alpha-n-1)/2} v_t \, dS$$

to the part of S corresponding to the set $\{x\colon \sigma(x) > 0, |\nabla\sigma(x)| < 1\}$ is equal to $-v_t(\sigma(x), x)$, which shows that $u_t^+(\sigma(x), x) = -v_t(\sigma(x), x)$ on the same set. To finish the proof of Lemma 6 we need to show that such a restriction defines $-v_t(\sigma(x), x)$ a.e. But this is an immediate consequence of the corollary to Lemma 3.

LEMMA 7. *(The energy condition.) For* $S_u = (|\nabla_x u|^2 + u_t^2, -2u_{x_1}u_t, \ldots, -2u_{x_n}u_t)$ *we have* $\operatorname{div} S_u = 2u_t \Box u = 0$ *in the sense of distributions.*

Proof. For ψ a test function,

$$\int \psi u_t \Box u = \int \psi(v_t + \mathscr{E}_t * \mu(v)) \Box u = \int \psi v_t \Box u + \int \psi \mathscr{E}_t * \mu(v) \Box u.$$

By Lemma 5

$$\int \psi v_t \Box u = -2 \int_{x\colon \sigma(x)>0} \psi(\sigma(x), x) v_t^2(\sigma(x), x)\left(1 - |\nabla\sigma(x)|^2\right) dx.$$

Also, in the proof of Lemma 6, we showed that $\mathscr{E}_t * \mu(v)$ has a restriction to the part of S corresponding to the set $\{x\colon \sigma(x) > 0, |\nabla\sigma(x)| < 1\}$ equals to $-v_t(\sigma(x), x)$ a.e.. Thus

$$\int \psi \mathscr{E}_t * \mu(v) \Box u = 2 \int_{x\colon \sigma(x)>0} \psi(\sigma(x), x) v_t^2(\sigma(x), x)\left(1 - |\nabla\sigma(x)|^2\right) dx,$$

which finishes the proof of Lemma 7.

By a standard argument, one deduces the following estimate from the energy condition:

$$\int_{|x-x_0|\le t_0 - T} \left(|\nabla_x u|^2 + |u_t|^2\right)_{t=T} dx \le \int_{|x-x_0|\le t_0} \left(|\nabla_x u|^2 + |u_t|^2\right)_{t=0} dx$$

(for details see **[1]**), which implies condition (i) and the uniqueness part of the theorem.

3. A refinement of Lemma 2 is given by the following estimate:

THEOREM 2. *With the same notation as before, we have*

$$(**) \qquad \int_{\mathbf{R}^n} |Tf(x)|^2\left(1 - |\nabla\sigma(x)|^2\right) dx \le C \int_{\mathbf{R}^n} |f(x)|^2 \, dx,$$

where C is an absolute constant.

Proof. To show $(**)$ we let

$$T_1 f(x) = \int_{\mathbf{R}^n} e^{ix\cdot\xi} \cos\sigma(x)|\xi|\hat{f}(\xi)\, d\xi$$

$$T_2 f(x) = \int_{\mathbf{R}^n} e^{ix\cdot\xi} \sin\sigma(x)|\xi|\hat{f}(\xi)\, d\xi$$

and show $(**)$ separable for T_1, T_2.

Let $v(t, x)$ be the solution to the initial value problem:

$$\Box v = 0,$$
$$v(0) = 0,$$
$$v_t(0) = f,$$

and let $S_v = (|\nabla_x v|^2 + |v_t|^2, -2v_{x_1}v_t, \dots, -2v_{x_n}v_t)$. We then have $\operatorname{div} S_v = 0$, and, by Gauss' theorem (given in [3]),

$$\int_{\partial D} S_v \cdot n\, dS = 0,$$

where D is any bounded domain in the half-space $t > 0$, and n (resp. dS) is the outer normal to ∂D (resp. the area element) with respect to the Lorentz metric. If, for D, we take a lens-shaped domain bounded below by the hyperplane $t = 0$ and above by S, then

$$(S_v \cdot n)\, dS = \begin{cases} -|v_t(0, x)|^2\, dx & \text{on } \partial D \cap \{(t, x), t = 0\}, \\ \left(|\nabla_x v|^2 + |v_t|^2 - 2\nabla_x v \cdot \nabla\omega v_t\right) dx & \text{otherwise,} \end{cases}$$

where ω is a defining function for ∂D; $\omega \equiv 0$ for $\partial D \cap \{(t, x); t = 0\}$, $\omega = \sigma$ for $\partial D \cap S$, and $\omega = t_0 - |x - x_0|$ for the remaining part of ∂D, where (t_0, x_0) is some point in the upper half-space $t > 0$.

Since

$$|\nabla_x v|^2 + |v_t|^2 - 2\nabla_x v \cdot \nabla\omega v_t = |\nabla_x v - v_t \nabla\omega|^2 + \left(1 - |\nabla\omega|^2\right) v_t^2,$$

we have

$$\begin{aligned}\int_{\partial D} S_v \cdot n\, dS = &-\int_{|x - x_0| \le t_0} v^2_{t_{|t=0}}\, dx + \int_{t_0 - \sigma(x) \ge |x - x_0|} |\nabla_x v - v_t \nabla\sigma|^2\, dx \\ &+ \int_{t_0 - \sigma(x) \ge |x - x_0|} v^2_{t_{|t=\sigma(x)}}\left(1 - |\nabla\sigma|^2\right) dx \\ &+ \int_{t_0 - \sigma(x) \le |x - x_0| \le t_0} \left|\nabla_x v + v_t \nabla_x |x - x_0|\right|^2 dx,\end{aligned}$$

which shows that

$$-\int_{|x-x_0|\le t_0} v_{t_{|t=0}}^2\,dx + \int_{|x-x_0|\le t_0-\sigma(x)} v_{t_{|t=\sigma(x)}}^2\left(1-|\nabla\sigma|^2\right)dx \le 0.$$

But

$$v(t,x) = \int_{\mathbf{R}^n} e^{ix\cdot\xi}\frac{\sin t|\xi|}{|\xi|}\hat{f}(\xi)\,d\xi.$$

Hence $v_t(0,x) = f(x)$ and $v_t(\xi(x),x) = T_1 f(x)$.Therefore,

$$\int_{|x-x_0|\le t_0-\sigma(x)} |T_1 f(x)|^2\left(1-|\nabla\sigma(x)|^2\right)dx \le \int_{|x-x_0|\le t_0} |f(x)|^2\,dx,$$

from which one easily deduces $(**)$ for T_1.

To show the same estimate for T_2, we consider the following initial value problem:

$$\begin{aligned}\Box w &= 0,\\ w(0) &= f,\\ w_t(0) &= 0,\end{aligned}$$

where f is a smooth and rapidly decreasing function such that $\hat{f}(\xi)\equiv 0$ for $|\xi|\le 1$ (a simple application of the Cauchy-Schwarz inequality shows that we can always reduce the problem to this case). We then can write $\hat{f}(\xi) = \hat{g}(\xi)/|\xi|$; g then has the same properties as f.

By using the same energy condition and the same domain as before, we have

$$\int_{|x-x_0|\le t_0-\sigma(x)} |w_t|_{t=\sigma(x)}^2\left(1-|\nabla\sigma|^2\right)dx \le \int_{|x-x_0|\le t_0} |\nabla_x w|_{t=0}^2\,dx.$$

But

$$\begin{aligned} w(t,x) &= \int_{\mathbf{R}^n} e^{ix\cdot\xi}\cos t|\xi|\hat{f}(\xi)\,d\xi,\\ w_t(0,x) &= 0,\\ w_t(\sigma(x),x) &= T_2 g(x),\end{aligned}$$

and

$$|\nabla_x w|^2(0,x) = \sum_{j=1}^{n} |R_j(g)|^2,$$

where for each $j = 1,2,\ldots,n$, R_j is the Riesz transform $R_j(g)\hat{\ }(\xi) = (\xi_j/|\xi|)\cdot\hat{g}(\xi)$. Since R_j is bounded on $L^2(\mathbf{R}^n)$ with norm 1, for each

$j = 1, 2, \ldots, n$ we have

$$\int_{|x-x_0|\le t_0-\sigma(x)} |T_2 g(x)|^2 \left(1 - |\nabla\sigma(x)|^2\right) dx \le n \int_{\mathbf{R}^n} |g(x)|^2\, dx.$$

The estimate $(**)$ for T_2 is now an easy consequence.

As a final remark, we notice that T is a Fourier integral operator (Egorov's operator) with a degenerate phase function: $\phi(x, \xi) = x \cdot \xi + \sigma(x)|\xi|$. It is then desirable to have a direct proof for $(**)$. For $n = 1$ such a proof is immediate. In this case it is easily seen that, up to a constant, the kernel of T is

$$K(x, x - y) = \frac{1}{\sigma(x) - (x - y)} - \frac{1}{\sigma(x) + (x - y)},$$

and, hence, up to a constant, we have

$$Tf(x) = Hf(x - \sigma(x)) - Hf(x + \sigma(x)),$$

where H is the Hilbert transform

$$Hf(x) = \text{p.v.} \int_{-\infty}^{\infty} f(x - y) \frac{dy}{y}.$$

Thus

$$\begin{aligned} \int_{-\infty}^{\infty} |Tf(x)|^2 \left(1 - \sigma'(x)^2\right) dx &\le 2\int_{-\infty}^{\infty} |Hf(x - \sigma(x))|^2 \left(1 - \sigma'(x)^2\right) dx \\ &\quad + 2\int_{-\infty}^{\infty} |Hf(x + \sigma(x))|^2 \left(1 - \sigma'(x)^2\right) dx \\ &\le 4\int_{-\infty} |Hf(x - \sigma(x))|^2 (1 - \sigma'(x))\, dx \\ &\quad + 4\int_{-\infty}^{\infty} |Hf(x + \sigma(x))|^2 (1 + \sigma'(x))\, dx. \end{aligned}$$

The obvious changes of variables and the well-known estimate for H show the desired estimate for T. This case shows also that $(**)$ is the best possible. In the case $n = 1$ we, in fact, have

$$\int_{-\infty}^{\infty} |Tf(x)|^p \left(1 - \sigma'(x)^2\right) dx \le C_p \int_{-\infty}^{\infty} |f(x)|^p\, dx$$

for all $1 < p < \infty$. This suggests that we might have some L^p-estimate $(p \ne 2)$ in the general case.

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