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ON THE WALLMAN ORDER COMPACTIFICATION

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The Wallman order compactification w_0X of a topological ordered space X has been constructed by Choe and Park. This paper establishes necessary and sufficient conditions for their compactification to be T_2 -ordered, in which case it coincides with the Nachbin (or Stone-Čech order) compactification.

Introduction. Let (X, \leq) be a poset. For $x \in X$, let $i(x) = \{y \in X: x \leq y\}$ and let $d(x) = \{y \in X: y \leq x\}$. If $A \subseteq X$, let $i(A) = \cup\{i(x): x \in A\}$, and $d(A) = \cup\{d(x): x \in A\}$. If $A = iA$ (respectively, $A = d(A)$), then A is called an *increasing* (respectively, *decreasing*) set; a set which is either increasing or decreasing is said to be *monotone*.

A *topological ordered space* (X, \leq, τ) consists of a poset (X, \leq) equipped with a topology τ . If τ has an open subbase consisting of monotone sets, then the topological ordered space is said to be *convex*. Since only convex topological ordered spaces can have order compactifications which are T_2 -ordered (see below), we shall henceforth consider only spaces of this type. For brevity, a convex topological ordered space (X, \leq, τ) will be simply called a *space* and designated by " X ".

Following McCartan [4], we define a space X to be T_1 -ordered if $i(x)$ and $d(x)$ are both closed for all $x \in X$, and T_2 -ordered if the partial order relation is a closed subset of $X \times X$. A T_1 -ordered space is T_4 -ordered (*normally ordered* in [5]) if, whenever A and B are closed disjoint subsets, the former decreasing and the latter increasing, there are disjoint open sets U and V , the former decreasing and the latter increasing, such that $A \subseteq U$ and $B \subseteq V$. The " T_3 -ordered" property is defined in [4], and " $T_{3,5}$ -ordered" can be taken to mean "completely regular ordered" as defined in [5], but it will not be necessary to repeat these latter definitions here.

Nachbin has constructed a Stone-Čech type order compactification β_0X of an arbitrary $T_{3,5}$ -ordered space X with the property that any continuous, increasing function from X into a T_2 -ordered, compact space can be lifted to β_0X . For details of the Nachbin compactification, see [3]. More recently, Choe and Park showed that X is T_4 -ordered whenever w_0X is T_2 -ordered, but were unable to prove the converse. Our main result establishes that w_0X is T_2 -ordered if and only if X is strongly T_4 -ordered

(this term is defined below), and consequently that w_0X and β_0X are equivalent compactifications of a strongly T_4 -ordered space X .

Let X be a topological ordered space. If $A \subseteq X$, let $I(A)$ (respectively, $D(A)$) be the smallest increasing (respectively, decreasing) closed set containing A , and let $A^\wedge = I(A) \cap D(A)$. Let $\mathcal{C}_X = \{A \subseteq X: A = A^\wedge\}$. Note that all members of \mathcal{C}_X are closed and convex; we shall call the members of \mathcal{C}_X *c-sets*. All monotone closed sets are *c-sets*, and thus \mathcal{C}_X is a closed subbase for τ . One can easily verify that every set of the form A^\wedge , for $A \subseteq X$, is a *c-set*, and also that \mathcal{C}_X is closed under finite intersections.

Let $F(X)$ be the set of all filters on X ; the fixed ultrafilter generated by $\{x\}$ will be denoted by \dot{x} for $x \in X$. If $\mathcal{F}, \mathcal{G} \in F(X)$, then $\mathcal{F} \vee \mathcal{G}$ will designate the filter generated by $\{F \cap G: F \in \mathcal{F}, G \in \mathcal{G}\}$ (assuming that the latter collection does not include \emptyset).

For $\mathcal{F} \in F(X)$, we denote by $i(\mathcal{F})$ the filter generated by $\{i(F): F \in \mathcal{F}\}$; the filters $d(\mathcal{F})$, $I(\mathcal{F})$, and $D(\mathcal{F})$ are defined analogously. A filter \mathcal{F} is a *c-filter* (respectively, a *convex filter*) if it has a filter base of *c-sets* (respectively, convex sets). Note that \mathcal{F} is a *c-filter* (respectively, a *convex filter*) iff $\mathcal{F} = I(\mathcal{F}) \vee D(\mathcal{F})$ (respectively, $\mathcal{F} = i(\mathcal{F}) \vee d(\mathcal{F})$). A *c-filter* which is not properly contained in any other *c-filter* will be called a *maximal c-filter*. A standard Zorn's Lemma argument establishes that every *c-filter* is contained in a maximal *c-filter*.

We can assume that X is a T_1 -ordered space and define $w_0(X)$ to be the set of all maximal *c-filters* on X . Note that the only convergent maximal *c-filters* are the fixed ultrafilters. It will be convenient to write $w_0X = \{\dot{x}: x \in X\} \cup X'$, where X' is the set of all non-convergent maximal *c-filters*. An order relation " \leq " for w_0X is defined as follows: $\mathcal{F} \leq \mathcal{G}$ iff $I(\mathcal{F}) \subseteq \mathcal{G}$ and $D(\mathcal{G}) \subseteq \mathcal{F}$. It is a simple matter to verify that (w_0X, \leq) is a poset and that the canonical map $\varphi: (X, \leq) \rightarrow (w_0X, \leq)$, defined by $\varphi(x) = \dot{x}$, is increasing.

We next introduce a topology on w_0X . For $A \subseteq X$, define $A^* = \{\mathcal{F} \in w_0X: A \in \mathcal{F}\}$. Then $\mathcal{C}^* = \{A^*: A \in \mathcal{C}_X\}$ is a closed subbase for a topology on w_0X which we shall denote by $w_0\tau$. Clearly, $(A \cap B)^* = A^* \cap B^*$ for all subsets A, B of X ; from this one easily deduces that w_0X is a topological ordered space. It is obvious that $A = \varphi^{-1}(A^*)$ for any $A \subseteq X$; therefore $\varphi: X \rightarrow w_0X$ is a topological embedding, and both φ and $\varphi^{-1}|_{\varphi(X)}$ are increasing functions.

Before proceeding further, it is desirable to compare our construction of w_0X with that of Choe and Park. They define a *bifilter* $(\mathcal{G}, \mathcal{H})$ on X to be a pair of filters such that \mathcal{G} has a base of decreasing closed sets, \mathcal{H} has a base of increasing closed sets, and $\mathcal{G} \vee \mathcal{H}$ exists; the set of all maximal

bifilters forms the underlying set for their compactification, which is also denoted by w_0X . It is easy to see that, for any bifilter $(\mathcal{G}, \mathcal{H})$ on X , the filter $\mathcal{F} = \mathcal{G} \vee \mathcal{H}$ is a c -filter, and that, for any c -filter \mathcal{F} , $(D(\mathcal{F}), I(\mathcal{F}))$ is a corresponding bifilter. If $(\mathcal{G}, \mathcal{H})$ is a maximal bifilter, then $\mathcal{F} = \mathcal{G} \vee \mathcal{H}$ is a maximal c -filter, and $(D(\mathcal{F}), I(\mathcal{F})) = (\mathcal{G}, \mathcal{H})$; thus a bijection exists between the set of maximal bifilters on X and the set of maximal c -filters on X . A comparison of the order relation and topology defined for w_0X in [2] with our definitions given above reveals the equivalence of these spaces both as posets and as topological spaces. Thus the results obtained concerning w_0X in [2] are applicable here, albeit with appropriate terminological alterations. The next two results are obtained in this way.

PROPOSITION 1.1. *For any T_1 -ordered space X , (w_0X, φ) is an order compactification of X , and w_0X is a T_1 topological space. If w_0X is T_2 -ordered, then X is T_4 -ordered.*

PROPOSITION 1.2. *Let X be a T_1 -ordered space, Y a T_2 -ordered compact space, and $f: X \rightarrow Y$ a continuous, increasing function. Then there is a unique, continuous, increasing function $\tilde{f}: w_0X \rightarrow Y$ such that $\tilde{f} \cdot \varphi = f$.*

We define a T_4 -ordered space X to be *strongly* T_4 -ordered if, whenever A and B are c -sets:

$$\begin{aligned}
 I(A) \cap B = \emptyset & \text{ implies } I(A) \cap D(B) = \emptyset \\
 D(A) \cap B = \emptyset & \text{ implies } D(A) \cap I(B) = \emptyset
 \end{aligned}$$

Note that a T_4 -ordered space X is strongly T_4 -ordered iff, for a c -set A and a decreasing open set U with $A \subseteq U$, $D(A) \subseteq U$ and dually.

Priestly [6] defines a C -space to be a topological ordered space X such that, for each closed subset A , $i(A)$ and $d(A)$ are also closed. The class of strongly T_4 -ordered spaces includes the T_4 C -spaces, among which are the T_2 -ordered compact spaces.

PROPOSITION 1.3. *A T_1 -ordered space X is strongly T_4 -ordered if and only w_0X is T_2 -ordered*

Proof. In Proposition 1, page 26, [5], Nachbin shows that a space is T_2 -ordered if, whenever $a \not\leq b$, there is an increasing neighborhood V of a and a decreasing W of b such that $V \cap W = \emptyset$.

Assume that \mathcal{F}, \mathcal{G} are elements of w_0X such that $\mathcal{F} \leq \mathcal{G}$ is false. Then either $I(\mathcal{F}) \subseteq \mathcal{F}$ or $D(\mathcal{G}) \subseteq \mathcal{F}$ is false. In the former case, since \mathcal{G} is a

maximal c -filter, there is $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $I(F) \cap G = \emptyset$. By the assumption that X is strongly T_4 -ordered, $I(F) \cap D(G) = \emptyset$, and so there are disjoint open neighborhoods U and V of $I(F)$ and $D(G)$, respectively, such that U is increasing and V decreasing. Then U^* and V^* are disjoint, open neighborhoods of \mathcal{F} and \mathcal{G} , respectively, in w_0X , the former increasing and the latter decreasing. This w_0X is T_2 -ordered.

Conversely, assume that w_0X is T_2 -ordered. Let A, B be c -sets and suppose $I(A) \cap B = \emptyset$. Then $I(A)^* \cap B^* = \emptyset$. $I(A)^*$ is a closed, increasing subset of w_0X and $B^* = D(B)^* \cap I(B)^*$ is a closed subset of w_0X . Let $d_w(B^*) = \{\mathcal{F} \in w_0X: \mathcal{F} \leq \mathcal{G} \text{ for some } \mathcal{G} \in B^*\}$. By Proposition 4, page 44, [5], $d_w(B^*)$ is a closed subset of w_0X , and it follows that $I(A)^* \cap d_w(B^*) = \emptyset$. Then $\varphi^{-1}(I(A)^* \cap d_w(B^*)) = \varphi^{-1}(I(A)^*) \cap \varphi^{-1}(d_w(B^*)) = \emptyset$. Since $\varphi^{-1}(I(A)^*) = I(A)$ and $D(B) \subseteq \varphi^{-1}(d_w(B^*))$, it follows that $I(A) \cap D(B) = \emptyset$. A similar argument shows that if $D(A) \cap B = \emptyset$, then $D(A) \cap I(B) = \emptyset$. This conclusion that X strongly T_4 -ordered now follows with the help of Proposition 1.1 □

COROLLARY 1.4. *A T_4 -ordered space X is strongly T_4 -ordered if and only if, for any c -set A , $d(A)$ and $i(A)$ are both closed.*

Proof. The condition is obviously sufficient. Suppose that X is strongly T_4 -ordered and $x \notin d(A)$. Then $i(x)^* \cap A^* = \emptyset$, and consequently $i(x)^* \cap d_w(A^*) = \emptyset$. It follows that $i(x) \cap \varphi^{-1}(d_w(A^*)) = \emptyset$. Since the closure of $d(A)$ in X is a subset of $\varphi^{-1}(d_w(A^*))$, x is not in the closure of $d(A)$. Thus $d(A)$ is closed. □

COROLLARY 1.5. *Let X be $T_{3.5}$ -ordered. Then the compactifications w_0X and β_0X are equivalent if and only if X is strongly T_4 -ordered.*

If the order relation of X is trivial, then the c -sets are simply the closed sets, and the compactification w_0X is identical with the ordinary Wallman compactification. In this case, Corollary 1.5 yields the well-known equivalence of the Wallman and Stone-Čech compactifications for T_4 topological space.

We conclude by considering the Wallman order compactification for a simple and familiar class of spaces. We define a *totally ordered space* to be a totally ordered set with its order topology. If X is a totally ordered space, then one can show that w_0X (and hence β_0X) is a totally ordered space and a complete lattice. If $X = R$ is the totally ordered space of real numbers, then w_0X can be identified with the extended real line $[-\infty, \infty]$.

If $X = Q$ is the space of rationals, then w_0X can also be regarded as the extended real line, but with each irrational “occurring twice”; by identifying these “irrational pairs”, one obtains w_0R as a quotient space of w_0Q .

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