NONISOTROPIC UNITARY SPACES AND MODULES WITH CAUCHY-SCHWARZ INEQUALITIES

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This subject is concerned with non-isotropic unitary spaces V over involutorial division rings D with characteristic not 2 and with non-trivial non-archimedean exponential valuations w, which are abelian. It will require a generalized Cauchy-Schwarz inequality relative to w. The dimension of V over D need not be finite. Treatments of the unitary module V_0 of finite vectors v in v (finite, in a technical sense), the ring L_0 of linear transformations of V that increase lengths, and the unitary group U yield information on the normal subgroup structure of this group and the factor group $U^{(r)}/U^{(r)} \cap Z$, where $U^{(r)}$ is the rth derived group of U and Z is the center of the ground division ring D.

Introduction. From a purely ring-theoretic viewpoint this subject arose from the treatment of primitive ring with involution L, in which, 2 is invertible and $1 - k$ is invertible for every skew-symmetric k in L. These invertibility assumptions ensure plenty of unitary elements $u = u^{*-1}$ in L, via the Cayley transform $u^{(k)} = (1 - k)/(1 + k)$ and one is interested in deciding whether or not the factor group $[U, U]/[U, U] \cap Z$ is simple, where $Z = \mathrm{center}(L)$ and U is the group of unitary elements in L. Another question which is of interest to me is the nature of the ring that is generated by U. From a more down to earth viewpoint, this subject specializes to the rings L of the form $L = L(V)$, the full ring of linear transformations of a certain left vector space V. It will be assumed throughout that V is a non-isotropic unitary space (in the sense of I. Kaplansky), where the involutorial ground division ring $(D; *)$ will be equipped with a non-trivial non-archimedean exponential valuation w, which is abelian. In fact, w will be a *-valuation (in the sense of S. S. Holland, Jr.). I will require, furthermore,

(1) $$2w(u \cdot v) \geq w(u \cdot u) + w(v \cdot v) + 2\varepsilon_0,$$

where $(\cdot)$ is the form of the unitary space $V, \cdots \leq \cdots$ is the ordering in G, the value group of w, and ε_0 is a constant (depending on V) in G. As shown by a theorem of Kaplansky, if L is as in the outset then L can be represented as a subring of $L(V)$, where the involution in L corresponds to the adjoint involution $\phi \to \phi^*$, provided L has a minimal left ideal. Of

course, given the valuation of w and/or assumption (1) are extra assumptions.

I shall call V an *elliptic space* if some inequality (1) holds true. As a special case of (1), there is, of course, the Cauchy-Schwarz inequality obtained by setting $\varepsilon_0 = 0$ in (1):

$$(2) \qquad\qquad 2w(u \cdot v) \geq w(u \cdot u) + w(v \cdot v).$$

Readers who are more familiar with (2), which has been dealt with recently by Holland [11], may stick to (2) throughout this work. My motivation in (1) will be explained as a concluding remark. I proceed to the material of this work.

This work is organized in four sections of which the first two were kept independent. Treatment of the unitary R-module V_0 of finite vectors cropping up in a given non-degenerate 0-elliptic space V (e.g. V satisfies (2)) occupies §1. Treatment of the ring that is generated by U has been omitted for reasons of space. Instead I will propose the ring L_0 of linear transformations ϕ that increase lengths

$$(3) \qquad\qquad w(u\phi \cdot u\phi) \geq w(u \cdot u) \qquad (u \in V).$$

The ideal structure of this ring will be investigated in an increasingly larger way in §§2 and 3. Treatment of the unitary group U of V and related groups will be done also by stages and exclusively in §4. Now to a synopsis of the results in the order these results will appear in the work.

Concerning Section 1. By *length* of $v \in V$, I mean the element $w(v \cdot v)$ in the group with infinity adjoined $G \cup \{\infty\}$. Those vectors v having non-negative lengths, or *finite* vectors, form an R-module, where R is the valuation ring in $(D; w)$. Questions such as when is this module V_0 a free module, when does V_0 contain some orthogonal basis and when orthogonal summands of the residue space $\overline{V}$ (in the sense of Holland) can be lifted to V_0 will be provided satisfactory solutions. In §1, Theorem 1.7 (1) will establish that if V has a denumerable basis then V_0 is free if and only if V_0 has an orthogonal basis, which relates to Kaplansky's [13, Theorems 37, 38, pp. 46–48]. In the cited theorem it will also be demonstrated that the freeness requirement has a neat number theoretic interpretation, namely, either the value group G is divisible by 2 or G has a first positive element $\varepsilon = 0^+$ such that one at least of these two elements 0, ε is represented by the form (e.g. occurs at length) in each one-dimensional subspace of V. As a corollary if 1 (= unity of D) is represented by the form in each one-dimensional subspace of V then V_0 has an orthogonal basis. This corollary is due to Holland (see [11, Theorem 5.4]).

Concerning Section 2. The treatment of the ring L_0 of all linear transformations ϕ that increase lengths corresponds to several needs. For one thing, for these spaces V for which the R-module V_0 is free L_0 will be shown to be isomorphic to the full endomorphism ring of V_0 (see §3, Theorem 3.25). For another, in a continuation of this work, I will show to what extent the ideal structure of L_0 is the same as that of the subring of L that is generated by U. A third need more pressing for this work is to use the ideal structure of L_0 for the normal subgroup structure of the group U. Last but not least one is interested in finding the analog for L of the valuation ring R in D. An attempt will be made to keep the treatment of L_0 coordinate-free. Instead of the usual vector valuation of the ring L viewed as a matrix ring over D, I will propose the notion of $*$-prevaluation $\cdots w \geq \cdots$. This is a binary relation between L and $G \cup \{\infty\}$ whose restriction to Z and $G \cup \{\infty\}$ is the same as the binary relation $\underline{w}$ defined by

(4) $$z\underline{w} \geq g \Leftrightarrow w(z) \geq g.$$

For general $\phi \in L$ and $g \in G \cup \{\infty\}$, $\phi w \geq g$ is defined by

(5) $$w(v\varphi \cdot v\phi) \geq w(v \cdot v) + 2g \qquad (v \in V).$$

Axioms of a $*$-prevaluation in general are listed and verifications of these axioms for the particular $*$-prevaluation $\cdots w \geq \cdots$ are stated without proofs (or almost). One can view the valuation ring R to be the set of elements $x \in D$ such that $xw \geq 0$. Likewise, L_0 is the set of linear transformations ϕ such that $\phi w \geq 0$. Evidently U is the unitary group of L_0 so that every 2-sided ideal I of L_0 which is $*$-closed (e.g. if $\phi \in I$ and if ϕ has adjoint ϕ^* then $\phi^* \in I$) gives rise to the congruence subgroup $U \cap (1 + I)$. Each $g \geq 0$ in G gives rise to the ideal L_g (resp. L_g^+) of all linear transformations ϕ such that $\phi w \geq g$ (resp. $\phi w > g$, that is, $w(v\phi \cdot v\phi) > w(v \cdot v) + 2g$, for every $0 \neq v \in V$). Among the non-formal theorems in §2, Theorem 2.7, asserts that L_0 is a prime ring which is an order in an overring of all linear transformations of finite rank, Theorem 2.12, asserts that L_0^+ in contained in the Jacobson radical of L_0 if V is finite-dimensional (or a more general result) and, Theorem 2.19, asserts that the congruence subgroup $U_0^+ = U \cap (1 + L_0^+)$ has all its non-trivial roots of unity (if any) outside $U_{\varepsilon_0}^+$, for a certain $\varepsilon_0 \in G$ depending on the characteristics of D and of the residue division ring $\overline{D}$ ($\varepsilon_0 = 0$ if charac$(D) \neq 0$; $\varepsilon_0 = w(p)$, if charac$(D) = 0$ and charac$(\overline{D}) = p \neq 0$; $\varepsilon_0 = 0$ if charac$(\overline{D}) = 0$).

Concerning Section 3. This section corresponds to several needs. (i) When the coordinate-free contribution of the $*$-prevaluation $\cdots w \geq \cdots$ as employed in the preceding section is a fact; still, on one or two

occasions in that section, the matrix delineation of this binary relation is as I believe, inevitable. (ii) In archimedean matrix representation of the n-dimensional orthogonal group $O(D; n(\cdot))$, where D is a field carrying an archimedean exponential valuation and $(\cdot)$ is an n-dimensional coordinate orthogonal form on D, each orthogonal transformation ϕ viewed as a matrix over D has all its entries from the valuation ring R in D. In the considered non-archimedean set up, if (f_i) is an orthogonal basis of V and if $\phi \in U$ what can be asserted about the matrix of ϕ relative to the basis (f_i)? (iii) While there is a parallel between R and L_0 from the formal point of view of $*$-prevaluations can one say something deeper; for instance, is $L_0^+ = \mathrm{Jac}(L_0)$ the unique maximal 2-sided ideal of the ring L_0? (iv) What is the nature of the factor group U/U_0^+? v) Of the mapping $g \rightarrow U_g = (1 + L_g) \cap U$? Taking the situation dealt with in (ii) as a fresh starting point, I will be dealing with those non-isotropic unitary spaces V (inequality (1) not required at the outset, $*$-valuation w required) equipped with some orthogonal bases (f_i) such that

$$(6) \qquad \phi \in U \Rightarrow w\big(\mathrm{entr}_{i,j}(\phi)\big) \geq \varepsilon_1 \quad (\text{all } i, j),$$

where ε_1 is a fixed element in G, $\varepsilon_1 \leq 0$; the case $\varepsilon_1 = 0$ being exactly analogous to the requirement in (ii). In Theorem 3.2, I will show that (6) implies V verifies some inequality (1); in the case $\varepsilon_1 = 0$ and $1/2$ exists in R, I will establish in Theorem 3.10, that V verifies the usual Cauchy-Schwarz inequality. For general ε_1, I will intertwine in Theorem 3.5, the $*$-prevaluation $\cdots w \geq \cdots$ with its matrix counterpart $\underline{w}$ defined by

$$(7) \qquad \phi \underline{w} \geq g \Leftrightarrow w\big(\mathrm{entr}_{i,j}(\phi)\big) \geq g \quad (\text{all } i, j).$$

This approximation theorem uses two parameters, namely an index of ellipticity ε_0 and a bound ε_2 for the basis (f_i), that is some $\varepsilon_2 \in G$ such that $|w(f_i \cdot f_i) - w(f_j \cdot f_j)| \leq \varepsilon_2$ for all pairs i, j ($|g| = g$ if $g \geq 0$ and $|g| = -g$ if $g < 0$). Both ε_0 and ε_2 depend fairly sharply on ε_1; in case $\varepsilon_1 = 0$ and $1/2$ exists in R, I will show in Theorem 3.10 that $\varepsilon_0 = 0$ and $\varepsilon_2 = 0$ or else ε_2 is the first positive element ε in G. For such a pair $(V; (f_i))$, I will provide satisfactory solutions to the questions in (iii), (iv) and (v) (see Theorems 3.13 through 3.23).

Concerning Section 4. I stated in the opening remarks that my initial motivation for this work can be traced back in the question asking if the group $[U, U]/[U, U] \cap Z$ is a simple group. For instance if V is an Euclidean space (in the sense of E. Artin) of finite dimension ≥ 3 then by Artin [1, Theorem 5.3, p. 17] the preceding group is a simple group. If,

contrary to the non-isotropic requirement, V is any non-degenerate unitary space with index ≥ 2, where the ground division ring D is deprived of the valuation w, but D contains at least 25 elements, then by J. Dieudonné [6, Theorems 1 and 4], the cited group is again a simple group. If V is any non-isotropic elliptic space which is orthogonal (e.g. $* =$ identity mapping of D) and of finite dimension $n \geq 3$ then by Artin [6, Theorem 5.8, p. 184] the group $[U, U]/[U, U] \cap Z$ is not simple. Treatment of the general projective group $U^{(r)}/U^{(r)} \cap Z$ $(r = 0, 1, 2, \ldots)$ for the considered space V arises as a conjunction of the cited results for, at least, $r \leq 1$. In §4, Theorem 4.4.7, I will establish that $U^{(r)}/U^{(r)} \cap Z$ does not verify the descending chain conditon for normal subgroups at the exception of the obvious cases where V is the 2-dimensional orthogonal space or V is 1-dimensional and $\dim_Z D \leq 4$. As an explanation of this negative fact I will propose a positive one, namely, the positive cone G^+ can be dually embedded in the lattice of normal subgroups of the group $U^{(r)}/U^{(r)} \cap Z$ for at least $r \leq 1$. This is done in Theorems 4.2.2 and Theorem 4.3.5, under a certain assumption familiar to §3, and a different assumption about the residue division ring $\overline{D}$, namely, the dimension of $\overline{D}$ over its own center exceeds 4. As an application of the foregoing theorems, I will derive that every torison normal subgroup of U is central; when D is finite-dimensional or $*$ is of the first kind then the same conclusion will hold for U replaced by $U^{(r)}/U^{(r)} \cap Z$. (See Theorems 4.4.7 and 4.4.8.)

Concerning assumption (1). In Artin's *Geometric Algebra*, it is stated that the old principle valid for Euclidean space V, namely, "small displacements on the unit sphere can be combined to give arbitrary displacements" (local cit. [1, Chapter V, Sect. 3, p. 179]) fails badly when the ground field D is equipped with a non-archimedean absolute value or, equivalently, a non-archimedean exponential valuation; it suffices to take V to be an orthogonal space of finite dimension $n \geq 3$ with assumption (1). The term used by Artin for such a space V is an *elliptic* space (see [1, Def. 5.2, p. 180])—I suggest no explanation of this term. My interest in assumption (1) arises more specifically from Artin's [1, Theorem 5.6, 5.7, pp. 18–183] which asserts that assumption (1) implies and is implied by the requirement some (and, hence, all) congruence subgroup $U_g = (1 + L_g) \cap U \neq 1$. Loosely speaking these theorems tell us that (1) has thus a ring-theoretic formulation and, by way of consequence, (1) serves the function of measuring the ring-theoretic incidence of (2). To close, let me add that as of the writing of this article no example of elliptic space V not

0-elliptic is known to me.[1] As a matter of fact, I will establish in a continuation of this work that if w is of rank 1 then (1) $\Leftrightarrow$ (2), which yields a full ring-theoretic formulation of (2) for such valuations w.

1. The module of finite vectors. I will begin with: a review of some of the undefined terms used in the Introduction; the main definitions and conventions; brief comments about non-isotropy, the valuation w, and assumption (1) in Introduction. The rest of this section specializes to the modulo V_0 (Definition 1.1 onward).

(a) *Involutorial division ring* D. Hereafter, D stands for a non-commutative skew-field or division ring with charac(D) $\neq$ 2. Let $x \rightarrow x^*$ be any fixed anti-automorphism of D of period 2 if D is not a field and period 1 or 2 otherwise. Let

$$Z = Z(D) \quad (= \{ z \in D | zx = xz \lor x \in D \}),$$

$$\mathscr{U}(D; *) = \{ w | x \in D | xx^* = 1 \},$$

be the center and unitary group of D respectively. If $x \in D$, denote by $x_{\mathscr{R}}$, the right translation of D induced by x and let

$$(0) \qquad\qquad D_{\mathscr{R}} = \{ x_{\mathscr{R}} | x \in D \},$$

be the division ring of right translations of D (*regular right representation of* D). Given the symmetric element $s = s^*$ in D, $s \neq 0$, let (s) stand for the new involution defined by

$$(1) \qquad\qquad x^{(s)} = sx^*s^{-1}.$$

(b) *Group with* ∞ *adjoined* $G^{\#}$. Hereafter, G stands for an additive group, $G \neq 0$, which is abelian and ordered. Denote the linear ordering on G by $\cdots \leq \cdots$. Extend the addition and the ordering to the set $G^{\#} = G \cup \{\infty\}$ ($\infty \notin G$) by the laws

$$(2) \qquad\qquad g < \infty, \quad \text{for every } g \in G, \quad \text{and}$$

$$(3) \qquad g + \infty = \infty + g = \infty = \infty + \infty, \quad \text{for every } g \in G.$$

(c) *$*$-Valuation* ([**11**, Sect. 2, p. 221]). Following Holland, by *$*$-valuation*, I mean a mapping $w: D \twoheadrightarrow G^{\#}$ verifying the following.

$$(4) \quad \text{If } D^{\times} = \{ x \in D | x \neq 0 \}, \text{ then } w(D^{\times}) \left(= \{ w(x) | x \in D^{\times} \}\right) = G.$$

$$(5) \qquad\qquad w(xy) = w(x) + w(y) \, (x, y \in D).$$

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(6) $$w(x + y) \geq \mathrm{Min}(w(x), w(y)).$$

(7) $$w(x^*) = w(x).$$

(d) *Elliptic space.* Let V be any unitary space over the involutorial division ring D. Hereafter, D carries a *-valuation w.

DEFINITION 1.0. I will call V an *elliptic space* (resp. *0-elliptic space*) if for some ε_0 in G (resp. $\varepsilon_0 = 0$)

(8) $$2w(u \cdot v) \geq w(u \cdot u) + w(v \cdot v) + 2\varepsilon_0,$$

for every pair u, v in V.

(e) *Concerning the valued division ring D.* By definition, w is a non-archimedean exponential valuation (in the sense of Jacobson) of D, which is abelian since G is abelian and w is non-trivial since $G \neq 0$. Let

(9) $$R = \{ x \in D | w(x) \geq 0 \};$$

(10) $$J = \{ x \in D | w(x) > 0 \}.$$

Then R is the *valuation ring* in D. Indeed, R is a subring of D, which is preserved under conjugation. Every one-sided ideal of R is 2-sided. Every finite set of ideals of R has a largest member. $J = \mathrm{Jac}(R)$ is the largest non-zero ideal $\neq R$ in R. For $g \in G$, let

(11) $$J_g = \{ x \in D | w(x) \geq g \}.$$

Then J_g is an additive subgroup of D; if, further, $g \geq 0$, then J_g is an ideal of R.

Although 2 is invertible in D it need not be so in R. Of course, $2 \in R$ since R has unity. The factor ring R/J, a division ring by the preceding, is denoted by $\overline{D}$ (*residue division ring*):

(12) $$\overline{D} = R/J.$$

(f) *Concerning the *-valuation w.* Inequality (6) carries over to algebraic sums:

(13) $$w\left(\sum_{i=1}^{n} x_i \right) \geq \mathrm{Min}_{i=1,\ldots,n} (w(x_i)) \quad (\textit{strong triangle inequality}).$$

In the special case $w(x_i) \neq w(x_j)$ for every pair $i \neq j$, one has the equality

(14) $$w\left(\sum_{i=1}^{n} x_i \right) = \mathrm{Min}_{i=1,\ldots,n} (w(x_i)) \quad (\textit{special triangle equality}),$$

Law (7) follows automatically from (5) plus D is finite-dimensional over Z with $*$ of the first kind. This is an immediate corollary to Dieudonné [6, Theorem 5].

(g) *Non-isotropic form.* Hereafter, V stands for any non-degenerate elliptic space. From the elliptic assumption alone follows that if $\mathrm{Rad}(V) = \{ v \in V \mid v \cdot V = 0 \}$, then

$$(15) \qquad\qquad \mathrm{Rad}\, V = \{ v \mid v \cdot v = 0 \}.$$

From non-degeneracy follows thus non-isotropy of the space V:

$$(16) \qquad\qquad v \cdot v = 0 \Rightarrow v = 0 \quad (v \in V).$$

(h) *Constant of ellipticity.* If $V \neq 0$ is elliptic then clearly the element ε_0 appearing in (8) is not positive. Hence $\varepsilon_0 \leq 0$. In the sequel, I refer to ε_0 as to *index of ellipticity* without insisting that ε_0 be the largest (as a negative element). Throughout this work V stands for a non-degenerate elliptic space and from Theorem 1.6 until the end of this section V is 0-elliptic.

DEFINITION 1.1. Call v a *finite* vector if for each $u \in V$

$$(17) \qquad\qquad 2w(u \cdot v) \geq (u \cdot u).$$

Denote by V_0, the set of all finite vectors.

Clearly the linear inequality (17) implies

$$(18) \qquad\qquad w(v \cdot v) \geq 0;$$

it suffices to set $u = v$ in (17). The quadratic inequality (18) implies back (17) under the 0-elliptic assumption. For the general assumption (1), the penalty is the constant ε_0:

$$(19) \qquad\qquad 2w(u \cdot v) \geq w(u \cdot u) + 2\varepsilon_0.$$

Call $v \in V_0$ *infinitesimal*, if

$$(20) \qquad\qquad 2w(u \cdot v) > w(u \cdot u) \quad (u \cdot u \neq 0).$$

Again, (20) implies

$$(21) \qquad\qquad w(v \cdot v) > 0,$$

which implies back (20) up to ε_0:

$$(22) \qquad\qquad 2w(u \cdot v) > (u \cdot u) + 2\varepsilon_0 \quad (u \cdot u \neq 0).$$

In the sequel, the subset of all infinitesimals v in V is denoted by V_0^+. In symbols:

$$(23) \quad v \in V_0^+ \Leftrightarrow (v \in V_0 \text{ and } u \cdot u \neq 0 \Rightarrow 2w(u \cdot v) > w(u \cdot u)).$$

THEOREM 1.2. *Let V be any unitary space over the $*$-valued division ring D—ellipticity of V not required. Then:*

(1) *If V_0 is the set of finite vectors on V then V_0 is a submodule of the R-module V, where R is the valuation ring in the valued division ring $(D; w)$.*

(2) *If V_0^+ is the subset of infinitesimal vectors then V_0^+ is a submodule of V_0.*

(3) *$V_0 \cdot V_0 \subset R$ and, hence, V_0 can be turned into a unitary module, relative to the form $(\cdot)$.*

(4) *V_0, V_0^+ and $J (= \mathrm{Jac}(R))$ are such that*

(i) *$JV_0 \subset V_0^+$,*

(ii) *$V_0 \cdot V_0^+ \subset J$, and*

(iii) *$V_0^+ \cdot V_0 \subset J$.*

Proof. (1) If v is a finite vecor and $\lambda \in R$, then for a given $u \in V$,

$$2w\big(u \cdot (\lambda v)\big) = 2w\big((u \cdot v)\lambda^*\big) = 2w(u \cdot v) + 2w(\lambda^*)$$
$$= 2w(u \cdot v) + 2w(\lambda) \geq 2w(u \cdot v) \geq w(u \cdot u);$$

as this holds for each $u \in V$, $\lambda v \in V_0$ follows. For u, v as before, if v' is another finite vector, then

$$2w\big(u \cdot (v + v')\big) = 2w(u \cdot v + u \cdot v') \geq 2\,\mathrm{Min}\big(w(u \cdot v), w(u \cdot v')\big)$$
$$= \mathrm{Min}\big(2w(u \cdot v), 2w(u \cdot v')\big) \geq w(u \cdot u)$$

implies $v + v' \in V_0$.

(2) is a trivial adaptation of (1).

(3) If v, $v' \in V_0$, then from $v \in V_0$, and $v' \in V_0 \subset V$, follows $2w(v \cdot v') \geq w(v' \cdot v') \geq 0$ so that $w(v \cdot v') \geq 0$ or, $v \cdot v' \in R$, for every pair v, $v' \in V$.

(4) (i) If $0 \neq j \in J$, and $0 \neq v \in V_0$, then for every $0 \neq u \in V$,

$$2w(u \cdot jv) = 2w(u \cdot v) + 2w(j) \geq w(u \cdot u) + 2w(j) > w(u \cdot u)$$

placing jv in V_0^+.

(ii) If $v \in V_0$ and $0 \neq u \in V_0^+$, then from $u \in V_0^+ \subset V_0$ follows $2w(u \cdot v) \geq w(u \cdot u)$. From $u \in V_0^+$ follows $w(u \cdot u) > 0$. Equivalently, $w(u \cdot v) > 0$ placing $u \cdot v$ in J.

(iii) $V_0^+ \cdot V_0 = (W_0 \cdot W_0^+)^* \subset J^* = J$. □

As a side remark, if $V \neq 0$ then $V_0 \neq 0$. This follows trivially from the elliptic axiom. Conversely, in §2 Theorem 2.18 it is shown that if $V_0 \neq 0$ then V is elliptic by parts (e.g. every finite dimensional subspace of V satisfies some inequality (1)). I turn to questions about the unitary R-module V_0. As a rule of thumb everything that will follow relies on the following extra assumption, which necessitates the 0-elliptic axiom (2):

$$(23) \qquad\qquad V_0 \setminus V_0^+ = \big\{ v \in V \,|\, w(v \cdot v) = 0 \big\}.$$

Equivalently, (21) asserts that if v is finite but not infinitesimal then and only then $w(v \cdot v) = 0$ (*medial* vector v).

DEFINITION 1.2. The family $(\lambda_i)_{i \in I}$, where $\lambda_i \in D$, is said to be a nullary row *vector* over D if all $\lambda_i = 0$ except for finitely many indices in I.

In the sequel, by basis of V (resp. V_0) over D (resp. R) I mean any family $(f_i)_{i \in I}$ in V (in V_0) such that given $v \in V$ (resp. $v \in V_0$) there is a unique nullary vector (λ_i) over D (resp. over R) such that

$$(24) \qquad\qquad v = \sum_{c \in I} \lambda_i f_i.$$

Can one say that V_0 is a free module (e.g. V_0 has some basis)? orthogonally free module (e.g. V_0 has some orthogonal basis)? The following examples will give some feeling for these questions.

EXAMPLE 1.4. (i) Up to isometry every unitary R-module V_0 where V is 1-dimensional is of the form

$$V_0 = \mathscr{I}_{g_0} = \{ x \in R | 2w(x) \geq g_0 \},$$

where $g_0 \in w(\mathscr{S}(D; *))$ and $g_0 \geq 0$.

(ii) Conversely, for any such g_0, the set $\mathscr{I}_{g_0}$ is a left (in fact, 2-sided) ideal of the valuation ring R, which can be realized as the R-module V_0 of some non-degenerate 0-elliptic space V.

(iii) To say that $\mathscr{I}_{g_0}$ viewed as a unitary R-module (in the sense of (ii)) is orthogonally free it is the same as saying that $\mathscr{I}_{g_0}$ is a principal left ideal of the ring R).

Proof. (1). Let V be any non-degenerate unitary space, which is 1-dimensional. Pick any $v \in V$ such that $v \cdot v \neq 0$. If $s_1 = v \cdot v$, then $s_1 = s_1^* \neq 0$. Replacing v by λv changes s_1 to $\lambda s_1 \lambda^*$. It follows that one can find some $0 \neq s = s^* \notin R$, which is represented by the form (e.g. $s = u \cdot u$, some $u \in V$). Then V is isometric to the space D relative to the coordinate form

$$x \cdot_s y = xsy^*.$$

Here, the valuation w is analytic, that is,

$$(25) \qquad\qquad w(v \cdot v) = w(\underline{v} \cdot_s \underline{v})$$

where $\underline{v} \in D$ is defined by the equation

$$(26) \qquad\qquad v = \underline{v}u \qquad (u \cdot u = s).$$

Since the space $\cdot_D D$ is evidently 0-elliptic relative to $(\cdot_s)$ it follows that V is 0-elliptic. Then V_0 corresponds to $V_0(_D D; (\cdot_s))$. Now,

$$x \in V_0(_D D; (\cdot_s)) \Leftrightarrow w(xsx^*) \geq 0 \Leftrightarrow 2w(x) \geq -w(s).$$

Thus if $g_0 = -w(s)$, then $g_0 \in w(\mathscr{S}(D; *))$, $g_0 \geq 0$, and

$$V_0 \cong V_0(_D D; (\cdot_s)) = \mathscr{I}_{g_0}.$$

(2) Pick any $s = s^*$ such that $w(s) = -g_0$. Equip $_D D$ with the form

(27) $$x \cdot_s y = xsy^*.$$

By construction, $V = _D D$ is a non-degenerate 0-elliptic space with $V_0 = \mathscr{I}_{g_0}$.

(3) Since $\mathscr{I}_{g_0}$ is an R-submodule of a 1-dimensional non-degenerate unitary space it is clear that $\mathscr{I}_{g_0}$ cannot contain any two non-zero orthogonal vectors. Hence, $\mathscr{I}_{g_0}$ is 1-dimensional. Equivalently, the left ideal $\mathscr{I}_{g_0}$ is a principal left ideal. □

EXAMPLE 1.5. (a) If (f_ι) is any orthogonal basis of the space V, then for every $v \in V$

(28) $$v = \sum_{\iota \in I} (v \cdot f_\iota) \frac{1}{f_\iota \cdot f_\iota} f_\iota.$$

(b) If V possesses some orthogonal basis and if every $g \in G$ that is represented by the form (e.g. there is $v \in V$ such that $v = w(v \cdot v)$) is divisible by 2 then the R-module V_0 is orthogonally free.

Proof. (a) For let (λ_ι) be the nullary vector over D such that $v = \sum \lambda_{\phi \iota}$. By construction, there is a unique finite subset I_1 of I such that $\lambda_\iota \neq 0$ for each $\iota \in I_1$ and $\cdot v = \sum_{\iota \in I_1} \lambda_\iota f_\iota$. If

$$v' = \sum_{\iota \in I_1} (v \cdot f_\iota) \frac{1}{f_\iota \cdot f_\iota} f_\iota$$

then $v - v'$ is orthogonal to each f_ι. Hence $v = v'$.

(b) Let (f_ι) be any orthogonal basis of V over D. Without disturbing the orthogonality relations each f can be replaced by g_ι with $w(g_\iota \cdot g_\iota) = 0$. The new basis (g) is in V_0. If $v \in V_0$ then by (a),

$$v = \sum_{\iota \in I} (v \cdot g_\iota) \frac{1}{g_\iota \cdot g_\iota} g_\iota$$

follows. If $v \cdot g_\iota \neq 0$ then from 0-ellipticity follows

$$2w(v \cdot g_\iota) \geq w(v \cdot v) + w(g_\iota \cdot g_\iota) = w(v \cdot v) \geq 0$$

giving $w(v \cdot g_\iota) \geq 0$ or, $v \cdot g_i \in R$. Hence, (g_ι) is a basis of V_0 over R. □

The treatment of bases of the R-module V_0 (if any) necessitates, of course, treatment of those $v \in V_0$ such that $\lambda v \in V_0$ implies $\lambda \in R$. Before I deal with such vectors v let me observe that if in the value group G there is some $g_0 > 0$ such that $g_0 < 2g$ for every $g > 0$ then G has a first positive element ε which is precisely g_0. Indeed, if $g < 0$ but $g < g_0$, then $g_0 - g > 0$ so that $g_0 < 2(g_0 - g)$, that is, $2g < g_0$, a contradiction.

THEOREM 1.6. *Let V be any nondegenerate 0-elliptic space, let $v \in V$ and let $g_0 = w(v \cdot v)$. The following requirements are equivalent.*
   (1) $g_0 < 2g$ *for every* $g \in G^+ (= \{h \in G | h > 0\})$.
   (2) $g_0 = 0$ *or* $g_0 = \varepsilon = $ *1st positive element of G.*
   (3) $g_0 \geq 0$ *and if $\lambda v \in V_0$, then $\lambda \in R$,*
   (4) $g_0 \geq 0$ *and if $u \in V_0$, then $w(u \cdot u) \geq g_0$.*
If one (and, hence all) requirements are verified then I will call v a pseudo-medial vector.

Proof. (1) $\Rightarrow$ (2). If v is not medial then $0 < g_0 < 2g$ for every $g \in G^+$. From this $g_0 = \varepsilon$.

(2) $\Rightarrow$ (3) From $\lambda v \in V_0$ follows that if $g = w(\lambda)$ then $2g + g_0 \geq 0$ or $\varepsilon = g_0 \geq -2g$; equivalently, $-g \leq 0$, that is, $g \geq 0$ or $\lambda \in R$.

(3) $\Rightarrow$ (4) Given $u \in V_0$ if u_1 is the projection of u on v:

$$u_1 = (u \cdot v)\frac{1}{v \cdot v}v,$$

then $w(u_1 \cdot u_1) = 2w(u \cdot v) - w(v \cdot v) \geq w(u \cdot u) \geq 0$ placing u_1 in v_0. If $\lambda = (u \cdot v)(v \cdot v)$ then since $\lambda v = u_1 \in V_0$, $\lambda \in R$ follows or $w(u \cdot v) \geq w(v \cdot v)$.

(4) $\Rightarrow$ (1) Let $g \in G$. If $g_0 \geq 2g$ then if $x \in D$ is such that $w(x) = g$ then $w(x^{-1}v \cdot x^{-1}v) = g_0 - 2g \geq 0$ placing $x^{-1}v$ in $V_0^+ \subset V_0$. Thus $w(x^{-1}v \cdot v) = g_0 - g \geq w(v \cdot v) = g_0$ or $g \leq 0$. The contrapositive of this is that if $g > 0$ then $g_0 < 2g$, as desired.   □

THEOREM 1.7. *Let V be any non-degenerate 0-elliptic space, which has some denumerable basis. The following requirements are equivalent.*
   (1) *The R-module V_0 is a free module (e.g. V_0 has some basis over R)*
   (2) *The R-module V_0 is orthogonally free (e.g. V_0 has some orthogonal basis over R).*

(3) *For every $g \in G$ which is represented by the form (e.g. $g = w(v \cdot v)$ for some $v \in V$) there is $g' \in G$ and $\varepsilon \in G$ such that $g = 2g' + \varepsilon$, where either $\varepsilon = 0$ or else ε is the first positive element in G.*

(4) *Given $0 \neq v \in V$ there is $\lambda \in D$ such that λv is a pseudo-medial vector (V contains enough pseudo-medial vectors).*

Proof. (1) $\Rightarrow$ (2) Evidently the R-module V is an essential extension of the R-module V_0 in that if $v \in V$ there is $0 \neq \lambda \in D$ such that $\lambda v \in V_0$. Equivalently, $V = DV_0$. Then every basis of V_0 is a basis of V over D. Hence V_0 contains a denumerable basis, say, $(e_i)_{i=1,2,\ldots}$. Now V is non-isotropic. Thus by induction on n one can define the following sequence $(f_i)_{i=1,2,\ldots}$:

(29) $$f_1 = e_1, \quad f_n = e_n - \sum_{i=1}^{n} e_n \cdot f_i \left(\frac{1}{f_i \cdot f_i} \right) f_i \quad (n > 1).$$

Put:

$$\alpha_{n,i} = (e_n \cdot f_i) \frac{1}{f_i \cdot f_i} \quad (i < n),$$

$v_{n,i} = \alpha_{n,i} f_i$. It is to be shown that $v_{n,i} \in V_0$. For since V is 0-elliptic it follows that $2w(e_n \cdot f_i) - w(f_i \cdot f_i) \geq w(e_n \cdot e_n) \geq 0$ ($e_n \in V_0$). Now, by a straight calculation $w(v_{n,i} \cdot v_{n,i}) = 2w(\alpha_{n,i}) + w(f_i \cdot f_i) = 2w(e_n \cdot f_i) - w(f_i \cdot f_i)$. The relations $v_{n,i} \in V_0$ will be used to show that, in turn, $\alpha_{n,i} \in R$ and $f_n \in V_0$ for every pair i, n with $i < n$. From the preceding $\alpha_{n,i} f_i = v_{n,i} \in V_0$. By a straight induction on n follows $f_n = e_n - \sum_{i=1}^{n-1} \alpha_{ni} f_i \in V_0$. For the relations $\alpha_{ni} \in R$, where $i < n$, use the $n - 1$ first equations of the f_j

$$f_1 = e_1,$$

$$f_2 = e_2 - \alpha_{2,1} f_1,$$

$$\cdots$$

$$f_{n-1} = e_{n-1} - \sum_{i=1}^{n-2} \alpha_{n-1,i} f_i,$$

to express the f_i, $1 \leq i \leq n - 1$, in terms of the e_j, $1 \leq j \leq n - 1$. Since the f_i with $1 \leq i \leq n - 2$ evidently do not use e_{n-1} it follows that

$$f_{n-1} = \beta_1 e_1 + \cdots + \beta_{n-2} e_{n-2} + 1 \cdot e_{n-1},$$

for some $\beta_i \leq D$. In the equation of f_n,

$$f_n = e_n - \alpha_{n,1} f_1 - \cdots - \alpha_{n,n-1} f_{n-1},$$

substituting for the f_i with $1 \leq i \leq f_{n-1}$ one obtains

$$f_n = \gamma_1 e_1 + \cdots + \gamma_{n-2} e_{n-2} - \alpha_{n,n-1} e_{n-1} + e_n,$$

where the $\gamma_i \in D$, $1 \leq i \leq n - 2$. Since $e, \ldots, e_n, \ldots$ is a basis of the R-module V_0 it follows that all coefficients in the preceding equation are in R, particularly $-\alpha_{n,n-1} \in R$. Hence,

$$f_n + \alpha_{n,n-1} f_{n-1} = f_n' = e_n - \alpha_{n1} f_1 - \cdots - \alpha_{n,n-2} f_{n-2}$$

where $f_n' \in V_0$. Repeating the preceding argument for f_n replaced by f_n' shows that $\alpha_{n,n-2} \in R$. Step by step $\alpha_{n,n-3}, \ldots, \alpha_{n,1} \in R$ follow.

Since the α_{ni} are all in R it follows that the e_j are R-expressible in terms of the $f_i \in V_0$. Since the e_j span V_0 so will be the f_i and since the f_i are orthogonal they form hence an orthogonal basis of V_0 over R.

$(2) \Rightarrow (3)$. I will first establish the following identity, which was suggested to me by the referee:

$$(29)' \qquad w\left(\sum_{i=1}^{r} v_i \cdot v_i\right) = \operatorname*{Min}_{j=1,\ldots,r} \left(w(v_j \cdot v_j)\right)$$

$$(0 \neq v_1, \ldots, v_r \text{ orthogonal vectors in } V).$$

One-half of (22) follows automatically from the generalized triangle inequality. Conversely, for any fixed i,

$$2w(v_i \cdot v_i) = 2w\left(v_i \cdot \sum_{j=1}^{r} v_j\right) \geq w(v_i \cdot v_i) + w\left(\sum_{j=1}^{r} v_j \cdot v_j\right)$$

giving after cancellation by $w(v_i \cdot v_i)$, $w(v_i \cdot v_i) \geq w(\sum_{j=1}^{r} v_j \cdot v_j)$. As this holds for every i it follows that $\operatorname{Min}_{i=1,\ldots,r}(w(v_i \cdot v_i)) \geq w(\sum_{j=1}^{r} v_j \cdot v_j)$. Next, I observe that if (f_i) is any orthogonal basis of the R-module V_0 then by a straight adaptation of the preceding if (λ_i) is any nullary row vector over D, then

$$(30) \qquad w\left(\sum_{i \in I} \lambda_i f_i \cdot \lambda_i f_i\right) = \operatorname*{Min}_{i \in I} w(\lambda_i f_i \cdot \lambda_i f_i),$$

$((f_i)$ any orthogonal family in $V)$. Now since (f_i) is a basis of V_0 over R it follows that for each fixed $i \in I$, $f_i \in V_0$ and if $\lambda f_i \in V_0$ then $\lambda \in R$. In view of Theorem 1.6, f_i is then a pseudo-medial vector of V for every $i \in I$. If then $\infty \neq g$ is represented by the form, say, $g = w(v \cdot v)$ then for some $0 \neq \lambda' \in D$, $\lambda' v \in V_0$. If $v' = \lambda' v$ then $v' = \sum \lambda_i f_i$ for a certain nullary row vector (λ_i) over R. From (30), follows

$$w(v' \cdot v') = w\left(\lambda_0 (f_{i_0} \cdot f_{i_0}) \lambda_0^*\right)$$

for some $i_0 \in I$ and some $\lambda_0 \in R$. Since $v \neq 0$ and $\lambda' \neq 0$ it follows that $v' \neq 0$ so that $w(v' \cdot v') \neq \infty$ and, hence, $\lambda_0 \neq 0$. Then

$$g = w(v' \cdot v') - 2w(\lambda') = w\big(\lambda_0(f_{i_0} \cdot f_{i_0})\lambda_0^*\big) - 2w(\lambda')$$

$$= 2w\big(\lambda_0 \lambda'^{-1}\big) + \varepsilon$$

where $\varepsilon = f_{i_0} \cdot f_{i_0} = 0$ or else ε is the first positive element in G (Theorem 1.6).

(3) $\Rightarrow$ (4) This is a corollary to Theorem 1.6.

(4) $\Rightarrow$ (1) That V possesses an orthogonal basis as soon as V possesses a denumerable basis was shown for the R-submodule V_0 itself so, V has some orthogonal basis. Given any orthogonal basis $(g_i)_{i \in I}$ for the space V over D, scale this basis to an orthogonal basis of V over D, where the new g_i are pseudo-medial vectors. I assert that these g_i form a basis of V_0 over R. For to begin with the $g_i \in V_0$. They are linearly independent over D and, hence, over R. To see that the g_i span V_0 over R proceed as follows. Given $0 \neq v \in V_0 \subset V$, since $(g_i)_{i \in I}$ is an orthogonal basis in V, it follows (Example 1.5) that

$$v = \sum_{i \in I} (v \cdot g_i)\left(\frac{1}{g_i \cdot g \cdot_i}\right)g_i,$$

where it is understood that all the $(v \cdot f_i)(f_i \cdot f_i) = 0$ but for a finite number of indices. Since each g_i is pseudo-medial $w(v \cdot g_i) \geq w(g_i \cdot g_i)$ follows (Theorem 1.7, point 4.) or, $(v \cdot g_i)(g_i \cdot g_i) \in R$. Hence the preceding equation shows that v is R-expressible in terms of the g_i, as desired. $\square$

As stated earlier in Introduction, the equivalence 1 $\Leftrightarrow$ 2 holds for any unitary R-module V_0, where R is any valuation ring containing $1/2$ and V_0 is finite-dimensional (cf. [13; Theorems 37, 38]). Note, however, that the argument as given in [13] makes essential use of these two extra assumptions (extra assumptions for Theorem 1.7).

Using arguments similar to the argument in Theorem 1.7 and standard arguments one can show the following corollaries.

COROLLARY 1.8. *Let V be any non-degenerate 0-elliptic space. Then*:

(1) *If V has some orthogonal basis then V_0 is orthogonally free if and only if V contains enough pseudo-medial vectors.*

(2) *If the subgroup $2G$ is of index ≤ 2 in G (in particular if G is isomorphic to the ordered additive group of integers) then given $g \in G$ there*

is $g' \in G$ and ε such that $g = 2g' + \varepsilon$, where either $\varepsilon = 0$ or ε is the first positive element in G so that V contains enough pseudo-medial vectors.

(3) If V is spanned by pseudo-medial vectors it does not generally follow that V contains enough pseudo-medial vectors.

COROLLARY 1.9. Let V be as in Corollary 1.8. Then:
(1) If V_0^+ is an orthogonal summand in V_0 then, in fact, $V_0 = V_0^+$.
(2) If V_0^+ has denumerable basis as an R-module then so must be V_0.

To find orthogonal summands of the R-module V_0 less forbidding than V_0^+, I will first recall a construction due to Holland of the residue space $\overline{V}$ ([11, Theorem 5.4]). Let $\overline{V} = V_0/V_0^+$. Then annihilator in R of $\overline{V}$ can be turned into a left space over $\overline{D} = R/J$. By Theorem 1.2, again,

$$V_0^+ \cdot V_0 \subset J$$

so that $\overline{V}$ can be equipped with the form

(31) $(v + V_0^+) \cdot (u + V_0^+) = vu + J (v, u \in V_0).$

Then $(\overline{V}; (\cdot))$ is a unitary space over $\overline{D}$, relative to the induced involution of $\overline{D}$. By construction, if $\bar{v} \cdot \bar{v} = 0$ then for $\bar{v} = v + V_0^+$, $v \cdot v \in J$ follows so that $v \in V_0^+$ or $\bar{v} = 0$. Thus $\bar{v}$ is non-isotropic. This is the

REMARK 1.10 (Holland [11, Theorem 5.4]). Let V be any unitary space over the $*$-valued division ring D. Then the factor R-module V_0/V_0^+ can be turned into a non-isotropic unitary space over the residue division ring $\overline{D}$ (residue space $\overline{V} = (V_0/V_0^+; (\cdot))$), relative to the form

$$(v + V_0^+) \cdot (u + V_0^+) = v \cdot u + J.$$

THEOREM 1.11. Let V be any non-degenerate 0-elliptic space having some orthogonal basis. Let $\overline{V}^{(1)}$ be any non-zero finite dimensional subspace of the residue space V. Then $\overline{V}^{(1)}$ can be lifted to an R-submodule $V_0^{(1)}$ of the R-module V_0 such that:
(1) $V_0^{(1)}/V_0^+ = \overline{V}^{(1)}$.
(2) $V_0^{(1)}$ possesses an orthogonal basis formed by medial vectors with cardinality equal to $\dim_{\overline{D}} \overline{V}^{(1)}$.
(3) $V_0^{(1)}$ is an orthogonal summand of V_0 with orthogonal complement $V_0^{(2)}$ mapping onto the orthogonal of $\overline{V}^{(1)}$ in $\overline{V}$.

Proof 1. *and* 2. Let $\bar{f}_1, \ldots, \bar{f}_n$ be a basis of $\overline{V}^{(1)}$ over $\overline{D}$ and let $f_1, \ldots, f_n \in V_0$ map onto $\bar{f}_1, \ldots, \bar{f}_n$ respectively. Denote by $V_0^{(1)}$ the R-submodule of V_0 that is spanned by the f_i. By construction, $V_0^{(1)}/V_0^+ = \overline{V}^{(1)}$.

Also, the f_i form a basis of the R-module $V_0^{(1)}$. For let $\lambda_1, \ldots, \lambda_n \in R$ be such that $\sum \lambda_i f_i = 0$. Then $\sum \bar{\lambda}_i \bar{f}_i = 0$. Thus $\lambda_i \in J$ for every i. If some $\lambda_i \neq 0$ pick one of least value, say, λ_1. Then $f_1 + \sum_{i>1} \lambda_1^{-1} \lambda_i f_i = 0$. By construction, $\lambda_1^{-1} \lambda_i \in R$. The preceding argument shows that $\bar{f}_1 + \sum_{i>1} \overline{\lambda_1^{-1} \lambda_i} \bar{f}_i = 0$, which is nonsense. This shows that all $\lambda_i = 0$. Therefore (f_i) is a basis $V_0^{(1)}$ over R. I proceed to show that the basis (f_i) can be transformed to an orthogonal basis (g_i) of $V_0^{(1)}$, where each g_i is a medial vector. Put:

$$g_1 = f_1,$$

$$g_2 = f_2 - (f_2 \cdot g_1) \frac{1}{g_1 \cdot g_1} g_1,$$

$$\cdots$$

$$g_n = f_n - \sum_{i<n} (f_n \cdot g_i) \frac{1}{g_i \cdot g_i} g_i.$$

I claim that (g_i) is the desired basis. The agrument goes by induction on n. For $n = 1$, $g_1 = f_1 \in V_0^{(1)} \subset V_0$ and $g_1 \notin V_0^+$. Thus g_1 is pseudo-medial. Assume that $g_1, \ldots, g_{n-1}$ are medial vectors. Since for each $i < n$, g_i is medial and since $f_n \in V_0^{(1)} \subset V_0$, it follows by Theorem 1.7, that $(f_n \cdot g_i)(g_i \cdot g_i) \in R$. Hence, $\sum_{i<n}(f_n \cdot g_i)(1/(g_i \cdot g_i)g_i \in V_0^{(1)})$. Thus $g_n \in V_0^{(1)} \subset V_0$. Assume that $g_n \in V_{0+}$. In the equation

$$g_n = f_n - \sum_{i<n} (f_n \cdot g_i) \frac{1}{g_i \cdot g_i} g_i$$

substituting for all the g_i with $i < n$ in terms of the f_j will not disturb the term f_n since each g_i is R-expressible in terms of f_j with $j \leq i \leq n$. Going down to the residue space $\overline{V}$, one has

$$0 = \bar{f}_n + \sum_{i<n} \bar{\gamma}_i \bar{f}_i \qquad \left(\gamma_i \in \overline{D} \right),$$

contrary to the fact that the $(\bar{f}_j)_{j \in I}$ form a basis over $\overline{D}$. This shows that $g_n \notin V_{0+}$ and, hence, $g_n \in V_0^{(1)} \subset V_0$ is a medial vector. By construction, the sequence $g_1, \ldots, g_n$ is an orthogonal sequence of medial vectors in $V_0^{(1)}$. Since the f_i are R-expressible in terms of the g_j and since the f_i span $V_0^{(1)}$ it follows that $g_1, \ldots, g_n$ is a basis of $V_0^{(1)}$ over R.

(3) It is to be shown that if $V^{(2)} = V_0^{(1)\perp}$ is the orthogonal of $V_0^{(1)}$ in V then $V_0^{(2)} = V_0^{(1)\perp} \cap V_0$ is an orthogonal complement of $V_0^{(1)}$ in V_0.

Since $V_0^{(1)\perp} = \{ g_1, \ldots, g_n \}^\perp$ it follows that $V^{(2)} \oplus DV_0^{(1)} = V$. Since $V_0^{(1)}$ has an orthogonal basis $g_1, \ldots, g_n$ as an R-module it follows that the projection of any given $v \in V_0 \subset V$ on $DV_0^{(1)}$ is

$$v_1 = \sum_{i=1}^{n} (v \cdot g_i)(1/(g_i \cdot g_i)g_i).$$

Thus

$$v = v_1 \oplus v_2, \qquad \left(v_1 \in DV_0^{(1)}; \, v_2 \in V^{(2)}\right).$$

Now the g are medial vectors and $v \in V_0$. By Theorem 1.7, again, follows $(v \cdot g_i)(g_i \cdot g_i) \in R$. Hence

$$v_1 = \sum_i (v \cdot g_i) \frac{1}{g_i \cdot g_i} g_i \in V_0^{(1)}.$$

Thus $v_2 = v - v_1 \in V_0$ so that $v_2 \in V_0^{(2)}$. Therefore $V_0 = V_0^{(1)} \oplus V_0^{(2)}$.

(4) $V_0^{(2)}$ maps onto the orthogonal $\overline{V}^{(2)}$ of $V^{(1)}$ in $\overline{V}$.

Clearly, $\overline{V}^{(1)} \oplus \overline{V}^{(2)} = \overline{V}$. Since $V_0^{(1)} \oplus V_0^{(2)} = V_0$ it follows that $\overline{V}_0^{(1)} \oplus \overline{V}_0^{(2)} = \overline{V}_0 = \overline{V}$. By construction, $\overline{V}_0^{(1)} = \overline{V}^{(1)}$. Hence, $\overline{V}^{(1)} \oplus \overline{V}_0^{(2)} = \overline{V}$. Then $\overline{V}_0^{(2)} \subset \overline{V}^{(1)\perp} = \overline{V}^{(2)}$ forces $\overline{V}_0^{(2)} = V^{(2)}$. $\qquad \square$

THEOREM 1.12. *Let V be any non-degenerate 0-elliptic space, which is finite-dimensional. Then*:

(1) *Any two maximal orthogonal families of medial vectors have equal cardinalities n_1.*

(2) *Any two maximal orthogonal families of truly pseudo-medial vectors have equal cardinalities n_2 provided V contains enough pseudo-medial vectors.*

(3) $n_1 + n_2 = \dim_D V.$

Proof 1. Let $f_1, \ldots, f_{n_1}$ be any maximal orthogonal family of medial vectors. Denote by $\overline{V}^{(1)}$ the span of the $\overline{f}_i$ in $\overline{V}$. Clearly $\overline{f}_1, \ldots, \overline{f}_{n_1}$ is a basis of $\overline{V}^{(1)}$ over $\overline{D}$. If $V_0^{(1)}$ is the span of the f_i over R then by Theorem 1.11, $V_0^{(1)}$ is an orthogonal summand of V_0. Assume that $n_1 \neq \dim_{\overline{D}} \overline{V}$. by the cited theorem, the orthogonal of $V_0^{(1)}$ in V_0 maps onto $\overline{V}^{(1)\perp}$. Since $n_1 \leq \dim_{\overline{D}} \overline{V}$ and since $\overline{V}$ is non-isotropic and, hence, non-degenerate, it follows that $\overline{V}^{(1)\perp} \neq 0$ so that $\overline{V}^{(1)\perp}$ contains some non-zero vector $\overline{f}$. If $f \in (V_0^{(1)})^{\perp}$ maps onto $\overline{f}$ then f is a medial vector. Now $f_1, \ldots, f_{n_1}, f$ is an orthogonal family consisting solely of medial vectors, a contradiction. This shows that $n_1 = \dim_{\overline{D}}(\overline{V})$.

(2) Let $g_1, \ldots, g_{n_2}$ be any maximal orthogonal family of truly pseudo-medial vectors. Let $V_0^{(2)}$ be the span of the g_i over the ring R. I claim that $V_0^{(2)}$ is an orthogonal summand of the R-module V_0. For let $v \in V_0$. As a vector in V, $v = v_1 \oplus v_2$, where v_2 is the projection of v onto the subspace $DV_0^{(2)}$, which has the orthogonal basis $g_{n_1}, \ldots, g_{n_2}$. Now,

$$v_2 = \sum_{i=1}^{n_2} (v_2 \cdot g_i)\left(\frac{1}{g_i \cdot g_i}\right) g_i.$$

Since g_i is pseudo-medial it follows by Theorem 1.7 that $(v_2 \cdot g_i)(g_i \cdot g_i)$ $\in R$. Hence, $v_2 \in V_0^{(2)}$. From this $v_1 = v - v_2 \in V_0$. Hence $V_0^{(2)}$ is an orthogonal summand in V_0. Let $V_0^{(1)}$ be its orthogonal complement. Every $0 \neq v \in V_0^{(1)}$ can be scaled to a pseudo-medial vector $\lambda v \in V_0$. By construction, $\lambda v \in V_0^{(1)}$. By maximality of the family $g_1, \ldots, g_{n_2}$ follows λv is a medial vector. Therefore, $V_0^{(1)}$ possesses an orthogonal basis of medial vectors $f_1, \ldots, f_{n_1^*}$. Going down to the residue space $\overline{V}$ one finds that $n_1^* = \dim_{\overline{D}} \overline{V} = n_1$. Therefore $n_2 = n - n_1$. $\qquad \square$

2. Ring of linear transformations increasing lengths.

The goal of this section is two-fold. Firstly, a parallel between the valuation ring R in D and a certain subring L_0 of the full ring L of linear transformations of the space V is drawn. Here V is any non-degenerate elliptic space and the parallel is obtained by means of a certain binary relation $\cdots w \geq \cdots$ (hereafter called *-prevaluation*) between L and the value group G. Secondly, the ideal structure of the ring L_0 is dealt with with a view to apply the results to the normal subgroup structure of the group U. At the end of the section certain torsion-free congruence subgroup of U is pointed out. I will begin with recalling one or two facts about L.

(a) *Notation.* Members of L generally written ϕ, σ, $\tau, \ldots$ always operate on the right of their domain, which is the space V.

(b) *Ideal of linear transformations of finite rank.* If $\phi \in L$ is such that $V\phi$ is finite-dimensional then ϕ is a linear transformation of *finite rank*. Let

$$(1) \qquad \mathscr{F}L = \left\{ \phi \in L \mid \dim V\phi < \infty \right\}.$$

Then $\mathscr{F}L$ is a 2-sided ideal of the ring L. By a result of Jacobson, every subring A of L containing L is a primitive ring A.

(c) *Ring of linear transformations having adjoints.* For V any non-degenerate unitary space, the *adjoint* ϕ^* (if any) of $\phi \in L$ is defined to be the linear transformation ϕ^* of V such that

$$(2) \qquad u \cdot v\varphi = u\phi \cdot v \qquad (u, v \in V).$$

As is well-known, every linear transformation in $\mathscr{F}L$ has adjoint in $\mathscr{F}L$. Generally, if ϕ_1, $\phi_2 \in L$ have adjoints in L, then so must be ϕ_1^*, $\phi_1 + \phi_2$ and $\phi_1\phi_2$. Then:

$$(3) \qquad \phi_1^{**} = \phi_1$$

$$(4) \qquad (\phi_1 + \phi_2)^* = \phi_1^* + \phi_2^*$$

$$(5) \qquad (\phi_1\phi_2)^* = \phi_2^*\phi_1^*.$$

Hence, the set of linear transformations $\phi \in L$ having adjoints is a primitive subring of L. This ring is the domain of the partial operator $\phi \to \phi^*$. I refer to the partial mapping $\phi \to \phi^*$ as to a *partial involution* and to $(L; *)$ as to *partial involutorial ring*.

(d) *Unitary group* $U - U(V)$. The partial involutorial ring $(L; *)$ has a unitary group (general notation: $\mathscr{U}(L; *)$), which is the group of elements $\phi \in L$ such that (i) ϕ is invertible (ii) ϕ^* is defined and (iii) $\phi^* = \phi^{-1}$. This group I will denote apart by $U = U(V)$.

(e) *Center of* L. Recall that Z = center of D. Then L can be turned into an algebra over Z, where the scalar multiplication $z \in Z$, $\phi \in L \to z\phi \in L$ is:

$$(6) \qquad\qquad z\phi = v \to z(v\phi).$$

The algebra L is *central* in that $Z(L) = Z1$, where 1 = unity of D. Incidentally, if $z \in Z$, then taking $*$ in D or $*$ in L agree in that

$$(7) \qquad\qquad (z \cdot 1)^* = z^* \cdot 1.$$

DEFINITION 2.1. Let A be any central algebra over the valued field $(Z; w)$. Assume that A carries a partial involution $x \to x^*$ (partial anti-automorphism whose domain is a subring of A such that $x^{**} = x$ for every $x \in A$ and $(z \cdot 1)^* = z^* \cdot 1$). The binary relation $\mathscr{P}$ between A and the group with infinity adjoined $G^{\#}$ is said to be a ($*$-*prevaluation*) (resp. *strict* $*$-*prevaluation*) if:

    (i) For each pair $0 \neq z \in Z$ and $g \in G^{\#}$, $z \cdot 1 \mathscr{P} g \Leftrightarrow w(z) \geq g$ (resp. $w(z) > g$).

    (ii) For each triple $x \in A$, $g, g' \in G^{\#}$, if $x \mathscr{P} g$ and $g \geq g'$ then $x \mathscr{P} g'$.

    (iii) For each quadruple $x, x' \in A$, $g, g' \in G^{\#}$, if $x \mathscr{P} g$ and $x' \mathscr{P} g'$ then $xx' \mathscr{P}(g + g')$.

    (iv) For each quadruple as in (iii), $x + x' \mathscr{P} \text{Min}(g, g')$.

    (v) For each pair $x, g \in A$, $G^{\#}$ with x^* defined if $x \mathscr{P} g$ then $x^* \mathscr{P} g$.

    (vi) $x \mathscr{P} g$ for every $g \in G$ if, and only if, $x \mathscr{P} \infty$ if, and only if, $x = 0$.

The following examples will motivate Definition 2.1.

EXAMPLE 2.2. Let A be the Z-algebra D, relative to the ground involution $*$. Denote by $\cdots \underline{w} \geq \cdots$ (resp. $\cdots \underline{w} > \cdots$) the binary relation between D and $G^{\#}$ that is defined by

$$(8) \qquad\qquad x\underline{w} \geq g \Leftrightarrow w(x) \geq g$$

$$(9) \qquad\qquad (\text{resp. } xw > g \Leftrightarrow w(x) > g).$$

Then $\cdots \underline{w} \geq \cdots$ (resp. $\cdots \underline{w} > \cdots$) is a $*$-prevaluation (resp. a strict $*$-prevaluation).

EXAMPLE 2.3. Let $D_{\mathscr{R}}$ be the division ring of right translations of D viewed as an algebra over Z, relative to the involution

$$x_{\mathscr{R}}^* = x_{\mathscr{R}}^{(s)},$$

where $0 \neq s = s^*$ is fixed and $x^{(s)} = sx^*s^{-1}$. Denote by $\cdots w \geq \cdots$ (resp. $\cdots w > \cdots$) the binary relation between $D_{\mathscr{R}}$ and $G^\#$ that is defined by

$$(10) \qquad x_{\mathscr{R}}w \geq g \Leftrightarrow 2w(u \cdot vx_{\mathscr{R}}) \geq w(u \cdot u) + w(v \cdot v) + 2g,$$

for every pair u, v in D, where $u \cdot v = usv^*$.

$$(11) \qquad \left(\text{resp. } x_{\mathscr{R}}w > g \Leftrightarrow 2w(u \cdot uv_{\mathscr{R}}) > w(u \cdot u) + w(v \cdot v) + 2g\right),$$

for every pair u, v in D such that $u \cdot u \neq 0$ and $v \cdot v \neq 0$). Then $\cdots w \geq \cdots$ (resp. $\cdots w > \cdots$) is a *-prevaluation (resp. a strict *-prevaluation).

Example 2.1 is formal. Example 2.2 reduces to the preceding example since, in effect, $xw \geq g \Leftrightarrow x_{\mathscr{R}}\underline{w} \geq g$ and $x_{\mathscr{R}}w > g \Leftrightarrow x\underline{w} > g$. Every 1-dimensional unitary space V, which is non-degenerate, can be identified to $_DD$, relative to the coordinate form $u \cdot v = usv^*$, where $0 \neq s$ is any element that is represented by the form in the sense $s = v \cdot v$, some $v \in V$. Then $L(V) = D_{\mathscr{R}}$ and the adjoint involution in L is precisely the one in Example 2.2. By analogy with that example one is led to the

THEOREM 2.4. *Let V be any non-degenerate elliptic space and let L be the ring of linear transformations of V considered as an algebra over the center Z of the ground division ring. Equip L with the partial adjoint involution $\phi \to \phi^*$ and Z with the ground valuation w. Then:*

(1) *The binary relation $\cdots w \geq \cdots$ between L and $G^\#$ that is defined by*

$$(12) \qquad \phi w \geq g \Leftrightarrow 2w(u \cdot v\phi) \geq w(u \cdot u) + w(v \cdot v) + 2g,$$

*for every pair $u, v \in V$, is a *-prevaluation.*

(2) *The binary relation $\cdots w > \cdots$ between L and $G^\#$ that is defined by*

$$(13) \qquad \phi w > g \Leftrightarrow 2w(u \cdot v\phi) > w(u \cdot u) + w(v \cdot v) + 2g;$$

*for every pair $u, v \in V$ with $u \cdot u \neq 0$, $v \cdot v \neq 0$, is a strict *-prevaluation.*

Theorem 2.4 is essentially evident. It is appropriate to add that the linear inequality in (12) and its strict version in (13) contain as particular cases the quadratic inequalities

$$(14) \qquad w(u\phi \cdot u\phi) \geq w(u \cdot u) + 2g,$$

(15) $w(u\phi \cdot u\phi) > w(u \cdot u) + 2g(u \cdot u \neq 0),$

respectively. Under the assumption V is 0-ellitpic, (14) (resp. (15)) implies
back

(16) $2w(u \cdot v\phi) \geq w(u \cdot u) + w(v \cdot v) + 2(g + \varepsilon_0)$

(17) $(\text{resp. } 2w(u \cdot v\phi) > w(u \cdot u) + w(v \cdot v) + 2(g + \varepsilon_0),$

$$u \cdot u \neq 0, v \cdot v \neq 0).$$

For future reference notice that (12) (resp. (13)) implies equally

(18) $w(u \cdot u\phi) \geq w(u \cdot u) + g$

(19) $(\text{resp. } w(u \cdot u\phi) > w(u \cdot u) + g, u \cdot u \neq 0).$

DEFINITION 2.5. Let:

(20) $L_g = L_g(V; w \geq) = \{ \phi \in L | \phi w \geq g \} \quad (g \in G^{\#});$

(21) $L_g^+ = L_g^+(V; w \geq) = \{ \phi \in L | \phi w > g \} \quad (g \in G);$

(22) $L_{-\infty} = \bigcup_{g \in G} L_g = \{ \phi \in L \exists g \in G^{\#} | \phi w \geq g \}.$

Refer to L_g (resp. L_g^+) as to a *congruence ideal* if, further, $g \geq 0$.
The following results are, again, formal results.

(23) $L_g \left(\text{resp. } L_g^+\right)$ is an additive subgroup of L,

which is closed under $*$.

(24) $g \geq g' \Rightarrow L_g \subset L_{g'}, \quad (\text{resp. } L_g^+ \subset L_{g'}^+).$

(25) $\bigcap_{g \in G} L_g = \bigcap_{g \in G} L_g^+ = L_\infty = 0.$

(26) $L_{-\infty}$ is a $*$-closed subring of L containing Z and

evidently L_g for every $g \in G$.

(27) $L_g \supset L_g^+ \quad (g \in G).$

(28) $zL_g = L_{g+w(z)} \quad (z \in Z).$

(29) $zL_g^+ = L_{g+w(z)}^+ \quad (g \in G, 0 \neq z \in Z).$

(30) $L_g + L_{g'} \subset L_{\text{Min}(g,g')}.$

(31) $L_g L_{g'} \subset L_{g+g'}.$

(32) $L_g + L_{g'}^+ \subset L_{\text{Min}(g,g')}^+ \quad (g \in G^{\#}, g' \in G).$

(33) $\qquad L_g L_{g'}^+ \subset L_{g+g'}^+ \qquad (g \in G^{\#}, g' \in G).$

(34) $\qquad L_g^+ L_{g'} \subset L_{g+g'}^+ \qquad (g \in G, g' \in G^{\#}).$

For the rest of this section I will specialize to the congruence ideals L_g or L_g^+, where $g \geq 0$. They are indeed 2-sided ideals of the ring L_0. By construction, the members $\phi \in L_0$ are the linear transformations ϕ such that

(35) $\qquad 2w(u \cdot v\phi) \geq w(u \cdot u) + w(v \cdot v).$

I refer to such a ϕ as to a linear transformation increasing lengths. Indeed as a special case of (15) one has the quadratic inequality

(36) $\qquad w(u\varphi \cdot u\phi) \geq w(u \cdot u) \qquad (u \in V).$

In L_0 sits the 2-sided ideal L_0^+ of linear transformations ϕ strictly increasing lengths since by construction $\phi \in L_0^+$ if, and only, if,

(37) $\qquad 2w(u \cdot v\phi) > w(u \cdot u) + w(v \cdot v)$

for every pair u, v with $u \cdot u \neq 0$, $v \cdot v \neq 0$. Relationship between $L_0^+ \subset L_0 \subset L_{-\infty}$ will follow. Let me first recapitulate some of the facts:

THEOREM 2.5. *Let V be any non-degenerate elliptic space. Then*:
(1) *The set L_0 of linear transformations ϕ that* increase lengths

$$2w(u \cdot v\phi) \geq w(u \cdot u) + w(v \cdot v) \qquad (u, v \in V),$$

is a subrng of L, which is $*$-closed.

(2) *In L_0 sits the 2-sided ideal of linear transformations ϕ that* strictly increase lengths:

$$2w(u \cdot v\phi) > w(u \cdot u) + w(v \cdot v) \qquad (u \cdot u \neq 0, v \cdot v \neq 0)$$

(3) *For each $g \geq 0$, L_g (resp. L_g^+) is a 2-sided ideal of the rng L_0, which is contained in L_0^+, if $g > 0$.*

(4) *If $L_{-\infty} = \bigcup_{g \in G} L_g$, then $L_{-\infty}$ is a $*$-closed subring of L containing Z and evidently L_0.*

A key result for the considered treatment of the tower $L_0^+ \subset L_0 \subset L_{-\infty}$ will be the

THEOREM 2.6. *Let $(V_\iota)_{\iota \in I}$ be any orthogonal decomposition of V and suppose that (g_ι) is a family in the value group such that $g_i \geq g$ for every $\iota \in I$. If, further, V is a non-degenerate ε_0-elliptic space then*

$$L_{g+2\varepsilon_0}(V) \supset \bigoplus_{\iota \in I} L_{g_\iota}(V_\iota).$$

Proof. Let $\phi_\iota \cap L_{g_\iota}(V_\iota)$, $\iota \in I$. Let $\phi = \bigoplus_{\iota \in I} \phi_\iota$. It is to be shown that $\phi \in L_{g+2\varepsilon_0}(V)$. If $v \in V$ there is (v_ι) such that $v = \sum v_i$, where all $v_\iota = 0$ but for finitely many indices. From $\phi_\iota \in L_{g_\iota}(V_\iota)$ follows $w(v_\iota\phi_\iota \cdot v_\iota\phi_\iota) \geq w(v_\iota \cdot v_\iota) + 2g_\iota$. Then

$$w(v\phi \cdot v\phi) = 2\left(\sum v_\iota\phi_\iota \cdot v_\iota\phi_\iota\right) \geq \operatorname*{Min}_{\iota \in I}(v_\iota\phi_\iota \cdot v_\iota\phi_\iota)$$

$$\geq \operatorname*{Min}_{\iota \in I}(w(v_\iota \cdot v_\iota)) + 2g_\iota \geq \operatorname*{Min}_{\iota \in I}(w(v_\iota \cdot v_\iota)) - 2g.$$

Since (v_ι) is an orthogonal family in V and since V is ε_0-elliptic it follows by a trivial adaptation of §1, (29) (Theorem 1.7, implication 2. $\Rightarrow$ 3.) that

$$\operatorname*{Min}_{gi \in I}(w(v_\iota \cdot v_\iota)) \geq 2\left(\sum_{\iota \in I} v_\iota \cdot v_\iota\right) + 2\varepsilon_0$$

so that

$$w(v\phi \cdot v\phi) \geq w(v \cdot v) + 2\varepsilon_0 + 2g,$$

for every v in V. From ε_0-ellipticity of V follows $\phi \in L_{g+2\varepsilon_0}(V)$, as desired. $\square$

THEOREM 2.7. *If the rng L_0 has some one-sided unity then L_0 is a subring of L. Equivalently, V is 0-elliptic.*

Proof. For let ϕ_0 be any right unity of the rng L_0. If ϕ is any projection (e.g. $\phi = \phi^* = \phi^2 \in L$) with V_ϕ is finite-dimensional I assert that $\phi\phi_0 = \phi$. For to begin with if $V^{(1)} = V\phi$ and $V^{(2)} = V(1 - \phi)$ then $V = V^{(1)} \oplus V^{(2)}$. From Theorem 2.6 follows

$$L_0 \supset L_{-2\varepsilon_0}(V^{(1)}) \oplus L_{-2\varepsilon_0}(V^{(2)}).$$

Now $V^{(1)}$ has some orthogonal basis f_i, $i = 1,\ldots,n$. Using this basis one can turn $V^{(1)}$ into a bi-space over D, relative to the right scalar multiplication

$$(38) \qquad v = \sum_{\iota=1}^{n} x_i f_i, \qquad \lambda \in D \to v\lambda = \sum_{\iota=1}^{n} x_i\lambda f_\iota.$$

Then $L(V^{(1)})$ is a right vector space over D, relative to the scalar multiplication

$$(39) \qquad \phi \in L(V^{(1)}) \qquad \lambda \in D \to \phi \cdot \lambda = v \to (v\phi)\lambda.$$

If $\tilde{\lambda} = 1_{V_1} \cdot \lambda$, then $\tilde{\lambda} \in L(V^{(1)})$. I assert that in the ring $L(V^{(1)})$

$$\tilde{\lambda}w \geq (2\varepsilon_0 + w(\lambda))$$

follows. For if $v \in V^{(1)}$, then

$$v\tilde{\lambda} \cdot v\tilde{\lambda} = \sum_{j=1}^{n} \lambda_j \lambda s_j (\lambda_j \lambda)^* \qquad (s_j = f_j \cdot f_j).$$

Thus

$$w(v\tilde{\lambda} \cdot v\tilde{\lambda}) \geq 2w(\lambda) + \mathrm{Min}\!\left(w\!\left(\lambda_j s_j \lambda_j^4\right)\right) \geq 2w(\lambda) + 2\varepsilon_0 + w(v \cdot v)$$

giving

$$2w(u \cdot v\tilde{\lambda}) \geq w(u \cdot u) + w(v \cdot v) + 2(2\varepsilon_0 + w(\lambda)),$$

as desired.

Define $\lambda\phi$ to be the linear transformation on V

$$v = v_1 \oplus v_2 \rightarrow v(\lambda\phi) = v_1\tilde{\lambda} \oplus 0.$$

For $\lambda \neq 0$ with $w(\lambda)$ large enough it is clear that $\lambda\phi \in L_{-2\varepsilon_0}(V_1) \oplus L_{-2\varepsilon_0}(V_2)$ hence, $\lambda\phi \in L_0(V)$. Thus $(\lambda\phi)\phi_0 = \lambda\phi$. Hence if $v = v_1 \oplus v_2$, then

$$v(\lambda\phi)\phi_0 = (v_1\tilde{\lambda})\phi_0 = v_1\tilde{\lambda}.$$

Since λ is evidently an onto transformation of $V^{(1)}$ it follows that $v_1\phi_0 = v_1$ for every $v_1 \in V^{(1)}$ or, $\phi\phi_1 = \phi$, for every projection ϕ of the space V with ϕ of finite rank. Now, for every non-zero vector u in V if $\phi_0: V \rightarrow V$ is defined by

$$(40) \qquad\qquad v\phi_u = (v \cdot u)\frac{1}{u \cdot u}u,$$

then as is well-known and easy, ϕ_u is a projection of rank 1. Hence, $\phi_u\phi_0 = \phi_u$ or $u\phi_0 = u$. As this holds for every u, $\phi_0 = 1$ follows; since $1 \in L_0$, V is evidently 0-elliptic. $\qquad\square$

THEOREM 2.7. *Let V be any non-degenerate elliptic space. Then*:

(1) *$L_{-\infty}$ is a primitive ring since, in fact, $L_{-\infty}$ contains all linear transformations of finite rank.*

(2) *If V possesses some orthogonal basis then L_0 is an order in the ring $L_{-\infty}$ so that L_0 is a prime rng.*

Proof. (1) (§3, Theorem 3.5, required). I will assume in what will follow that if V is finite-dimensional then $L_{-\infty}(V) = L$. For general V proceed as follows. Pick $\phi \in \mathcal{F}L$. Put: $\phi_1 = \phi - \phi^*$, $\phi_2 = \phi - \phi^*$, $V_\iota = V_{\phi_\iota}$, $\iota = 1, 2$. Here, $V_\iota^\perp = \mathrm{Ker}(\phi_\iota)$ and since V_ι is finite-dimensional it follows that $V = V_\iota \oplus V_\iota^\perp$. If $\phi_{\iota 1} = \phi_\iota / V_\iota$ and $\phi_{\iota 2} = 0_{V_\iota^\perp}$ then evidently

$\phi_\iota = \phi_{\iota 1} \oplus \phi_{\iota 0}$. Since V_ι is finite-dimensional it follows that $\phi_{\iota_1} w \geq g$ for some $g \in G$ ($i = 1, 2$). By Theorem 2.5 follows $\phi_\iota \in L_{-\infty}$ so that $2\phi = \phi_1 + \phi_2 \in L_{-\infty}$. Since evidently $1/2 \in L_{-\infty}$, $\phi \in L_{-\infty}$ follows.

(2) Define a right scalar multiplication via the basis (f_ι) of V by

$$(41) \qquad v = \sum x_\iota f_\iota, \qquad \lambda \in D \to v\lambda = \sum_{\iota \in I} x_\iota \lambda f_\iota.$$

Then define the right scalar multiplication

$$(42) \qquad \phi \in L \qquad \lambda \in D \to \phi\lambda v \to (v\phi)\lambda.$$

By a trivial adaptation of the calculation in Theorem 2.7 point 2, follows that if $\tilde{\lambda} = v \to v\lambda$ then $\tilde{\lambda} w \geq (2\varepsilon_0 + w(\lambda))$. Thus

$$(42)' \qquad \tilde{\lambda} w \geq g \Rightarrow \phi\lambda w \geq (g + w(\lambda) + 2\varepsilon_0).$$

Given $\phi \in L_{-\infty}$ there is $g \in G$ such that $\phi w \geq g$. Then for g' large enough $g + g' + 2\varepsilon_0 \geq 0$. If $\lambda \in d$ is such that $w(\lambda) = g' \neq \infty$ then $\lambda \neq 0$ and by the preceding $\phi\lambda \in L_0$. Then for $w(\lambda)$ large enough follows

$$\phi = (\phi\lambda)\lambda^{-1}$$

where both $\phi\lambda, \lambda \in L_0$. Hence L_0 is an order in $L_{-\infty}$ so that L_0 is a prime rng.   □

DEFINITION 2.8. Call $\phi \in L$ medial or say ϕ *preserve lengths*, if

$$w(v\phi \cdot v\phi) = w(v \cdot v),$$

for every $v \in V$.

EXAMPLE 2.9. Let V be the left space ${}_D D$, relative to the coordinate form

$$x \cdot y = xsy^*.$$

To say that $\phi \in L$ is medial is to say that $\phi = x_{\mathscr{R}}$, where x is a *unit* in the valuation ring R.

THEOREM 2.10. *The set $\overline{M} = \overline{M}(V)$ of all medial transformations has the following features.*

(1) *For each $0 \neq z \in Z$, $z \cdot 1 \in \overline{M}$ if and only if $w(z) = 0$.*

(2) *$\overline{M}$ is a multiplicative monoid consisting only of right invertible transformations ϕ.*

(3) *$\overline{M}$ contains inverses (e.g. if $\phi \in \overline{M}$ and if ϕ is invertible in L then $\phi^{-1} \in M$)*

(4) *For each $\sigma \in L_0^+$, if $\phi = 1 + \sigma$, then $\phi \in \overline{M}$.*

(5) *For each $g \in G$, $ML_g \subset L_g$ (resp. $ML_g^+ \subset L_g^+$).*
(6) *If ϕ^* exists in $\overline{M}$ then $L_g\phi \subset L_g$ (resp. $L_g^+ \subset L_g^+$).*
(7) *If V is finite-dimensional then $\overline{M}$ is a group.*
(8) *If V is 0-elliptic then the group of invertible elements in $\overline{M}$ is also the group of units of the ring L_0.*

Proof. (1) To say that $z \cdot 1$ is medial is to say that $w(zv \cdot zv) = w(v \cdot v)$ for each $v \in V$; if, and only if, $2w(z) = 0$ or $w(z) = 0$ ($if\ v \neq 0$).

(2) This is evident.

(3) This is, again, evident.

(4) From $\sigma \in L_0^+$ follows for each $0 \neq v \in V$:

$$w(v\sigma \cdot v\sigma)\pounds(v \cdot v); \quad w(v\sigma \cdot v) = w(v \cdot v\sigma) > w(v \cdot v).$$

By the strong triangle inequality follows

$$w(v \cdot v\sigma) + v\sigma \cdot v + (v\sigma \cdot v\sigma) > w(v \cdot v),$$

and by the special triangle equality follows

$$w(v \cdot v + (v \cdot v\sigma + v\sigma \cdot v + v\sigma \cdot v\sigma)) = w(v \cdot v)$$

or,

$$w(v(1 + \sigma) \cdot v(1 + \sigma)) = w(v\sigma \cdot v\sigma) = w(v \cdot v),$$

as desired.

(5) This is the same as showing that

$$\tau w \geq g \Rightarrow \phi\tau w \geq g\ (\text{resp. } \tau w > g \Rightarrow \phi\tau w > g)$$

for every triple τ, ϕ, $g \in L$, M, G. For instance, if $\tau w > g$ then for every pair $u, v \neq 0$,

$$2w(u \cdot v(\phi\tau)) = 2w(u \cdot (v\phi)\tau).$$

If $v\phi = 0$ then $v\phi\tau = 0$ so that

$$2w(u \cdot v(\phi\tau)) \geq w(u \cdot u) + w(v \cdot v) + 2g.$$

If $v\phi \neq 0$ then,

$$2w(u \cdot v(\phi\tau)) = 2w(u \cdot (v\phi)\tau) \geq 2(u \cdot u) + w(v\phi \cdot v\phi) + 2g$$
$$= w(u \cdot u) + w(v \cdot v) + 2g$$

placing $\phi\tau$ in L_g.

(6) Again, if $\tau w > g$ and $\phi^* \in M$ then for $u\phi^* \neq 0$,

$$2w(u \cdot v\tau\phi) = 2w(u\phi^* \cdot v\tau) > w(u\phi^* \cdot u\phi^*) + w(v \cdot v) + 2g$$
$$= w(u \cdot u) + w(v \cdot v) + 2g;$$

while $u\phi^* = 0$ implies $2w(u \cdot v\tau\phi) = 2w(u\phi^* \cdot v\tau) = \infty > w(u \cdot u) + w(v \cdot v) + 2g$.

(7) Let ϕ be any invertible medial transformation. Then for every $v \in V$,

$$w(v\phi \cdot v\phi) = w(v \cdot v) \geq w(v \cdot v).$$

Since V is 0-elliptic it follows that

$$2w(u \cdot v\phi) \geq w(u \cdot u) + w(v\phi \cdot v\phi) = w(u \cdot u) + w(v \cdot v)$$
$$\geq w(u \cdot u) + w(v \cdot v),$$

placing ϕ in L_0. Since ϕ^{-1} is medial then, again, $\phi^{-1} \in L_0$. Hence ϕ is a unit of L_0. Conversely, let ϕ be a unit of the ring L_0. For each $v \in V$,

$$w(v \cdot v) = w\big((v\phi^{-1}j)\phi \cdot (v\phi^{-1})\phi\big) \geq w(v\phi^{-1} \cdot v\phi^{-1}) \qquad (\phi \in L_0)$$
$$\geq w(v \cdot v) \qquad \big(\phi^{-1} \in L_0\big).$$

Hence $w(v \cdot v) = w(v\phi^{-1} \cdot v\phi^{-1})$. By symmetry, $w(v \cdot v) = w(v\phi \cdot v\phi)$. Hence ϕ and ϕ^{-1} are medial transformations, as desired. $\qquad\square$

The preceding theorem suggests that the medial transformations behave as potential units of the ring L_0 for, at least, the finite dimensional 0-elliptic spaces V.

THEOREM 2.11. *Let V be any finite-dimensional non-isotropic unitary space over the involutorial division ring D with characteristic $\neq 2$. Then*:
(1) *If $\phi \in L = L(V)$ has adjoint ϕ^* and if $\phi\phi^* = 0$ then $\phi = 0$.*
(2) *If $\phi = \phi^*$ is one-to-one then ϕ is invertible in L.*
(3) *If $\phi = -\phi^*$ then $1 - \phi$ is invertible in L.*

Proof. (1) and (2) are well-known and easy.
(3) Let ϕ be as in the statement. If $v \in V$ is such that $v\phi = v$ then $v = v\phi^2$ follows so that

$$v \cdot v = v\phi \cdot v\phi = v \cdot (v\phi)\phi^* = v \cdot \big(-v\phi^2\big) = -v \cdot v$$

giving $v \cdot v = 0$ or $v = 0$. This shows that $1 - \phi$ is one-to-one. Repeating for ϕ replaced by $-\phi$ one gets that $1 + \phi$ is one-to-one. Hence, $1 - \phi^2 = (1 - \phi)(1 + \phi)$ is one-to-one. Since $1 - \phi^2$ is symmetric it follows by 2. that $1 - \phi^2$ is invertible and, hence, $1 - \phi$ is invertible. $\qquad\square$

THEOREM 2.12. *Let V be any finite-dimensional non-degenerate elliptic space. Then*:
(1) *The subset M of medial transformations ϕ having medial adjoints ϕ^* is a group (medial group) such that $L_g\phi = \phi L_g = L_g$ (resp. $L_g^+\phi = \phi L_g^+ = L_g^+$) for every $\phi \in M$ and $g \in G$.*

(2) *If $\mathscr{I}$ is any right ideal of L_0 which is contained in L_0^+ then* $\mathscr{I} \subset \mathrm{Jac}(L_0)$.

Proof. (1) Since both ϕ, ϕ^* are medial and, hence, one-to-one, it follows that $\phi^*\phi$ is one-to-one. From Theorem 2.11 follows that $\phi^*\phi$ is invertible. Then ϕ is invertible. Similarly, ϕ^* is invertible. Also ϕ^{-1} is medial and has medial adjoint. The rest of the assertion follows readily from Theorem 2.10.

(2) Let $\sigma \in \mathscr{I}$. Since $\sigma \in \mathscr{I} \subset L_0^+$ it follows that $1 + \sigma$ is medial. Since σ^* is defined, and, hence, $\sigma^* \in L_0^+$ it follows that $(1 + \sigma)^* = 1 + \sigma^*$ is medial. Thus $1 + \sigma \in M$ has inverse in M. If $1 + \sigma'$ is the inverse of $1 + \sigma$ then

$$-\sigma' = (1 + \sigma')\sigma \in (1 + \sigma')L_0^+ \subset L_0^+ l.$$

Hence $\sigma' \in L_0^+$ is a quasi-inverse of σ. As this holds for every $\sigma \in \mathscr{I}$ it follows that $\mathscr{I} \subset \mathrm{Jac}(L_0)$. $\qquad\square$

DEFINITIONS 2.13. (a) By *congruence subgroup* of the unitary group I mean any subgroup N of U of the form

$$N = U \cap (1 + \mathscr{I}),$$

for some 2-sided ideal $\mathscr{I}$ of the rng L_0.

(b) The congruence subgroup

$$(44) \qquad\qquad N' = U \cap (1 + L_g) \qquad (g \geq 0)$$

is denoted by U_g.

(c) The congruence subgroup

$$(45) \qquad\qquad N'' = U \cap (1 + L_g^+) \qquad (g \geq 0)$$

is denoted by U_g^+.

Two important congruence subgroups:

$$(46) \qquad\qquad U_0 = \{ \phi \in U | (\phi - 1)w \geq 0 \};$$

$$(47) \qquad\qquad U_0^+ = \{ \phi \in U | (\phi - 1)w > 0 \}.$$

Evidently, $U_0 = U$ if, and only if, v is 0-elliptic. One can, of course, extend formulas (45), (46) for $g \leq 0$; but, there is only little gain to do so for what will follow here or elsewhere.

THEOREM 2.14. *Let V be any non-degenerate elliptic space. The congruence subgroups U_1 (resp. U_g^+), $g \geq 0$, form a chain of normal subgroups of the unitary group U such that:*

(i) $g \geq g' \Rightarrow U_g \subset U_{g'}$, (resp. $U_g^+ \subset U_{g'}^+$);

(ii) $\bigcap U_g = \bigcap U_g^+ = (1)$;

(iii) $[U_{g'}, U_{g'}] \subset U_{g+g'}$ (*resp.* $[U_{g'}, U_{g'}^+] \subset U_{g+g'}^+$).

Proof. (0) U_g (resp. U_g^+) is a normal subgroup of U. For let $\phi \in U_g$. Then $\phi - 1 \in L_g \subset L_0$. Since ϕ has adjoint it follows that $\phi^* - 1 \in L_g$ and since $\phi^* = \phi^{-1}$, $\phi^{-1} \in U_g$ follows. Let $\phi, \sigma \in U_g$. Then

$$1 - \phi\sigma = \phi(1 - \sigma) + (1 - \phi) \in \phi L_g + L_g.$$

Since $\phi \in U \subset M$, it follows that $L_g \subset L$ (Theorem 2.12, point (1)) so that $1 - \phi\sigma \in L_g$. Thus $\phi\sigma \in U_g$. Let $\tau \in U$. Then $\tau L_g \tau^{-1} = L_g$ (Theorem 2.12, point 1.). Hence $\tau U_g \tau^{-1} = U_g$ and $U_g \lhd U$. For the assertion $U_g^+ \lhd U$ use similar argument.

(i), (ii) This is evident.

(iii) Let me show, say, that $\phi \in U_{g*}$, $\tau \in U_g^+$, together imply $[\phi, \tau]$ $(= \phi^{-1}\tau^{-1}\phi\tau) \in U_{g+g*}^+$. Indeed,

$$[\phi, \tau] = \phi^{-1}\tau^{-1}(\phi\tau - \tau\phi)$$

$$= \phi^{-1}\tau^{-1}((\phi - 1)(\tau - 1) - (\tau - 1)(\phi - 1))$$

$$\in \phi^{-1}\tau^{-1}\left(L_g L_{g*}^+ L_g\right)$$

$$\subset \phi^{-1}\tau^{-1}L_{g+g*}^+ \subset L_{g+g*}^+. \qquad \square$$

THEOREM 2.15. (i) *Each congruence subgroup U_g^+ with $g \geq w(2)$, consists only of unitary transformations ϕ that can be Cayley parametrised.*

(ii) *Hence no $\phi \in U_g^+$ can reverse $0 \neq v \in V$ and, consequently, U_g^+ excludes any involution $\sigma \neq 1$.*

Proof. (i) It suffices to establish the theorem for $g = w(2)$. Given $\phi \in U_{w(2)}^+$ and $v \in V$ if $v(1 + \phi) = 0$ but, $v \neq 0$ then since $\phi - 1 \in L_{w(2)}^+$,

$$w(v(\phi - 1) \cdot v(\phi - 1)) > w(v \cdot v) + 2w(2),$$

follows; but, the left member of this inequality is precisely $w(2v \cdot 2v) = w(v \cdot v) + 2w(2)$. This shows that $1 + \phi$ is one-to-one. Again $\phi^* \in U_{w(2)}^+$ forces $1 + \varphi^*$ is one-to-one. By Theorem 2.12, $1 + \phi$ is then invertible. Let $\tau = (1 - \phi)/(1 + \phi)$. By construction, τ^* is defined and $\tau^* = -\tau$. Here, $1 + \tau$ is invertible as this follows from Theorem 2.12. Moreover, $\phi = (1 - \tau)/(1 + \tau)$, the usual Cayley parametrisation of the unitary ϕ.

(ii) This is partly shown in (i) and partly evident. $\qquad \square$

The following theorem is technical and will be used here and elsewhere.

THEOREM 2.16. *Let V be any non-degenerate elliptic space. Let $\mathscr{C}$ be the partial operator of the ring L which is defined at $\phi \in L$ if and only if $1 + \phi$ is invertible and then*

$$(48) \qquad\qquad \mathscr{C}(\phi) = 2\frac{1 - \phi}{1 + \phi}.$$

If $\mathscr{C}^{-1}$ is the partial operator of L which is defined at $\tau \in L$ if and only if $1 + \tau/2$ is invertible and then

$$(49) \qquad\qquad \mathscr{C}^{-1}(\tau) = \frac{1 - \tau/2}{1 + \tau/2}$$

then:

(1) $\mathscr{C}(\mathscr{C}^{-1}|\tau) = \tau$ *and* $\mathscr{C}^{-1}(\mathscr{C}(\phi)) = \phi$ *for every pair τ, ϕ with $\mathscr{C}^{-1}$ and $\mathscr{C}(\tau)$ defined.*

(2) $\mathscr{C}$ *(resp. $\mathscr{C}^{-1}$) is one-to-one.*

(3) $\mathscr{C}$ *is entirely defined at U_g (resp. U_g^+) for every $q > w(2)$ (resp. $g \geq w(2)$) and*

$$\mathscr{C}(U_g) = \ell_g(V) \left(resp. \left(U_g^+\right) = \ell_g^+\right)$$

where $\ell_g(V) = \{\tau = -\tau^* \in L | \tau w \geq g\}$ provided V is finite dimensional.

(4) $\mathscr{C}^{-1}$ *is entirely defined at ℓ_g (resp. k_g^+) and*

$$\mathscr{C}^{-1}\left(\ell_g\right) = U_g \left(resp. \mathscr{C}^{-1}\left(\ell_g^+\right) = U_g^+\right),$$

where $g \geq w(2)$ (resp. $g > w(2)$) provided V is finite dimensional.

Proof. (1) and (2) are formal.

(3) and (4) If $g \geq w(2)$ then for each $\phi \in U_g^+$ it is clear that $\phi \in U_{w(2)}^+$. From Theorem 2.15 follows that $1 + \phi$ is invertible. Thus $\mathscr{C}(\phi)$ is defined. If

$$\tau = \mathscr{C}(\phi) = 2\frac{1 - \phi}{1 + \phi}$$

then by construction $\tau^* = -\tau$. Also,

$$\tau = 2\frac{1 - \phi}{2 + (\phi - 1)} = \frac{1 - \phi}{1 + ((\phi - 1)/2)}.$$

From $\phi \in U_g^+$ follows $(\phi - 1) \in L_g^+$ so that $(\phi - 1)/2 \in L_{q-w(2)}^+ \subset L_0^+$. Thus $1 + (\phi - 1)/2 \in M$ so that

$$\tau \in ML_g^+ = L_g^+$$

and, hence, $\tau \in \ell_g^+$.

Similarly if $g > w(2)$ then for each $\phi \in U_g$, $\tau = \mathscr{C}(\phi)$ is defined and $\tau \in \ell_g$.

Conversely, if $\tau \in \ell_g^+$ where $g \geq w(2)$ then evidently $\mathscr{C}^{-1}$ is defined at $\tau/2$. (In fact, $1 + \tau$ is invertible for every $\tau = -\tau^*$). If $\phi = \mathscr{C}^{-1}(\tau)$ then by construction ϕ is unitary and

$$\phi - 1 = \frac{1 - \tau/2}{1 + \tau/2} - 1 = \frac{-\tau}{1 + \tau/2} \in M\ell_g^+ \subset ML_g^+ = L_g^+.$$

Thus $\phi \in U_g$. Similarly if $\tau \in \ell_g$ where $g > w(2)$ then $\tau \cdot \phi = \mathscr{C}^{-1}(\tau) \in U_g$. $\qquad \square$

THEOREM 2.17. *Let V be any non-degenerate elliptic space, which is not the 1-dimensional orthogonal space. For every $g \in G$, the congruence subgroup U_g (resp. U_g^+) is not the identity subgroup.*

Proof. By Theorem 2.16 it suffices to show that $\ell_g(V) \neq 0$ (resp. $\ell_g^+(V) \neq 0$). As a second reduction, since $\ell_g^+ \subset \ell_g$ it suffices to show that $\ell_g^+ \neq 0$. As a third reduction, still, since from a trivial adaptation of Theorem 2.6, follows that if $V^{(1)}$ is any orthogonal summand of V then

$$\ell_g^+(V) \supset \ell_{g-2\varepsilon_0}(V^{(1)}) \oplus 0,$$

one may assume, further, that V is finite dimensional. By the same token as soon as a given subspace $V^{(2)}$ of V verifies the theorem so will be all of V. If then V is not orthogonal, then picking any 1-dimensional subspace $V^{(3)}$ of V and observing that $\ell_g^+(V^{(3)}) \neq 0$ the theorem follows. If, on the other hand, V is an orthogonal space one can quote Artin's [1, Theorem 5.7]. To motivate the reader, let me give a direct proof. From finite-dimensionality of V follows that $L_{-\infty}(V) = L$. Pick any $0 \neq \tau = -\tau^*$ (possible since $\dim_D V \geq 2$). Then $\tau w \geq g$ for some g_0. Choose any $0 \neq z \in Z = D$ with $w(z)$ large enough so as $w(z) + g_0 > g$. Then $z\tau \in L_{g_0 w(z)}^+ \subset L_g^+$. By construction, $0 \neq z\tau = -(z\tau)^*$. $\qquad \square$

THEOREM 2.18. *Let V be any non-isotropic unitary space over the $*$-valued division ring $(D; *; w)$, which is finite dimensional but other than the 1-dimensional orthogonal space. The following requirements are equivalent.*

(1) *There is $0 \neq v \in V$ and $g \in G$ such that if $u \in V$ then $2w(v \cdot v) \geq w(u \cdot u) + 2g$.*

(2) *There is some $0 \neq \phi \in L$ and $g \in G$ such that $\phi w \geq g$.*

(3) *$U_g \neq 1$ for some $g \in G$.*

(4) *V is elliptic.*

Proof. By Theorem 2.17, (4) $\Rightarrow$ (3). Evidently, (3) $\Rightarrow$ (2) $\Rightarrow$ (1). Assume (1). Quoting [2, Theorem 1] or directly one can show that U acts

irreducibly on V. Hence there is a basis of V of the form $v_i = v\phi_i$, where $\phi_1 = 1$, $\phi_2, \ldots, \phi_n \in U$. Given $u \in V$ there is a unique row vector $\underline{u} = (x_1, \ldots, x_n)$ over D such that $u = \sum x_i v_i$. I proceed to establish that, if $u_1, u_2 \in V$ then

$$2w(u_1 \cdot u_2) \geq w(u_1 \cdot u_1) + 2\underline{w}(\underline{u}_2) + 2g,$$

where $\underline{w}(x_1, \ldots, x_n) = \operatorname{Min}_{i=1,\ldots,n}(w(x_i))$. For let $\underline{u}_2 = (x_1, \ldots, x_n)$. Then

$$2w(u_1 \cdot x_i v_i) = 2w(u_1 \cdot x_i v\phi_i) = 2w(u_1\phi_i \cdot x_i v)$$

$$= 2w(u_1\phi_i \cdot v) + 2w(x_i) \geq w(u_1\phi_i \cdot u_1\phi_i) + 2w(x_i) + 2g$$

$$= w(u_1 \cdot u_1) + 2w(x_i) + 2g \geq w(u_1 \cdot u_1) + 2(\underline{w}(\underline{u}_2) + g);$$

$$2w(u_1 \cdot u_2) = 2w\Big(u_1 \cdot \sum x_i v_i\Big) \geq 2\operatorname{Min}_i\big(w(u_1 \cdot x_i v_i)\big)$$

$$\geq w(u_1 \cdot u_1) + 2(\underline{w}(\underline{u}_2) + g),$$

as desired. It remains then to show that for some $g' \in G$, $2\underline{w}(\underline{u}) \geq w(u \cdot u) + 2g'$, for every $u \in V$. This can be readily seen from the observations:

(50) $$\underline{u} = (x_1, \ldots, x_n) = (u \cdot v_1, \ldots, u \cdot v_n)T,$$

where $T = (S^{\mathrm{tr.}*})^{-1}$;

(51) $$\underline{w}(\underline{u}) \geq \underline{w}(u \cdot v_1, \ldots, u \cdot v_n) + \underline{w}(T),$$

where $\underline{w}(T)$ is the minimum value of the entries in T. $\square$

As stated in Introduction, Theorem 2.18 is due to Artin in the orthogonal case and the proof is inspired by Artin's. The last item for this section will be the treatment of roots of unity in the congruence subgroup L_0^+.

THEOREM 2.19. *Let V be any non-degenerate elliptic space and let P be the prime subfield of the ground division ring D. Then:*

(1) *If the valuation w induces the trivial valuation on P, then U_0^+ is torsion-free.*

(2) *If w induces a non-trivial valuation on P then if $p = \operatorname{charac}(\overline{D})$ then $U_{w(p)}^+$ is torsion-free.*

Proof. (1) It is to be shown first that if $\phi \in L_0^+$ is algebraic over P then, in fact, ϕ is nilpotent. For then there is a polynomial $p(t)$ over P without constant term such that $\phi^n = p(\phi)\phi^{2n}$ for some n. If $\tau = p(\phi)\phi^n$ then τ is an idempotent transformation. Since $w(P^\times) = w\{r | 0 \neq r \in P\}$ is evidently contained in the medial group M it follows that $\tau \in L_0^+$ and,

hence, τ strictly decreases lengths. If now $v\tau \neq 0$ for some v it would follow

$$w(v\tau \cdot v\tau) = w(v\tau^2 \cdot v\tau^2) > w(v\tau \cdot v\tau),$$

a contradiction. This shows that $V\tau = 0$ so that $\tau = 0$ and, hence, $\phi^n = 0$.

Let then $\sigma \in U_0^+$ be any root of unity. If $\phi = \sigma - 1$ then by construction $\phi \in L_0^+$ is algebraic over P. By the preceding ϕ is nilpotent. Now evidently $w(2) = 0$ so that $L_0^+ = L_{w(2)}^+$ and, hence, σ is Cayley parametrisable. If $\sigma = (1 - \tau)/(1 + \tau)$ with $\tau = -\tau^*$ then $\tau = (1 - \sigma)/(1 + \sigma)$ is nilpotent. In view of Theorem 2.11, $\tau = 0$ or $\sigma = 1$, as desired.

(2) To say that w/p is not trivial is to say that $\mathrm{charac}(\overline{D}) = p \neq 0$ (where $\overline{D} = R/J$) but, $\mathrm{charac}(D) = 0$ so that $w(p)$ is a well-defined element in G.

Claim 1. Let $1 \neq \phi \in U_0^+$ be any root of unity. The multiplicative order of ϕ is a power of p.

For let m be the multiplicative order of ϕ. Let p^r be the highest power of p dividing m. Then $m = p^r l$, where $p \nmid l$. Suppose that $l \neq 1$. If $\sigma = \phi^{p^r}$, then $\sigma \neq 1$ and σ has order l. Moreover, $\sigma \in U_0^+$ for σ is a power of $\phi \in U_0^+$. Let $\tau = \sigma - 1$. From the relation $\sigma^l = 1$ follows $((\sigma - 1) + 1)^l = (\tau + 1)^l = 1$. Thus there are integers $r_2, r_3, \ldots, r_l$ such that

$$l\tau + \sum_{i=2}^{l} r_i \tau^i = 0.$$

Dividing through the preceding equation by l this gives after factorisation by τ

$$\tau\left(1 + \sum_{i=2}^{l} r_i' \tau^{i-1}\right) = 0 \qquad (r_i' = r_i/l).$$

Now, from $\sigma \in U_0^+$ follows $\tau = \sigma - 1 \in L_0^+$. Hence $\tau^{i-1} \in L_0^+$ ($i \geq 2$). Also,

$$r_i' \tau^{i-1} \in r_i' L_0^+ \in \frac{1}{l} r_i L_0^+ \subset \frac{1}{l} L_0^+ = L_{w(1/l)}^+ = L_0^+.$$

Hence, $\sum_{i \geq 2}^{l} r_i \tau^{i-1} \in L_0^+$. From this, if $\tau_0 = \sum_{i \geq 2}^{l} r_i \tau^{i-1}$, then $1 + \tau_0$ is medial. By the basic equation, $\tau(1 + \tau_0) = 0$. Now $1 + \tau$ is one-to-one and τ commutes with $1 + \tau_0$. Hence $\tau = 0$ or, $\sigma = 1$, a contradiction. This shows that the order m of ϕ is a power of p, as desired.

Claim 2. If w/P is the 2-adic valuation then $U_{w(2)}^+$ is torsion-free.

Deny this claim. Pick $1 \neq \phi \in U_{w(2)}^+$ and let $m \neq 1$ be the multiplicative order of ϕ. By the preceding claim, $m = 2^r$ for some integer $r \neq 0$. If $\phi_1 = \phi^{2^{r-1}}$ then $\phi_1 \neq 1$. Hence, ϕ_1 is an involution belonging evidently to $U_{w(2)}^+$, contradicting Theorem 2.15.

Claim 3. If w/p is the p-adic valuation with p odd then $U_{w(p)}$ is torsion-free so that $U_{w(p)}^+$ is torsion-free.

Deny this claim. There is $1 \neq \phi \in U_{w(p)}$ with multiplicative order p. If $\tau = \phi - 1$ and if the $\binom{p}{i}$ are the usual binomial coefficients then from $\phi^p = 1$ follows

$$p\tau + \binom{p}{2}\tau^2 + \cdots + \binom{p}{p}\tau^p = 0.$$

Since p is prime, $p | \binom{p}{i}$ follows for $i = 2, \ldots, p - 1$. Dividing through by p and isolating the last term this gives

$$\frac{1}{p}\tau^p = \tau(1 + \tau_0),$$

where $\tau_0 = -(r_2\tau + \cdots + r_{p-1}\tau^{p-2})$ and $r_2, \ldots, r_{p-1}$ are integers. Now $w(p) > 0$ for $w(p) \geq 0$ and w/P is p-adic. Hence, $U_{w(p)} \subset U_0^+$. Thus, $\tau \in L_0^+$. From this $\tau_0 \in L_0^+$ so that $1 + \tau_0$ is medial. In terms of lengths the factor $1 + \tau_0$ can be thus neglected, that is,

$$w(v(\tau(1 + \tau_0)) \cdot v(\tau(1 + \tau_0))) = w(v\tau \cdot v\partial t).$$

Since $(1/p)\tau^p = \tau(1 + \tau_0)$ it follows that

$$w\left(v\frac{1}{p}\tau^p \cdot v\frac{1}{p}\tau^p\right) = w(v\tau \cdot v\tau);$$

or,

$$w(v\tau^p \cdot v\tau^p) = w(v\tau \cdot v\tau) + 2w(p).$$

Restricting the preceding equality to $V^{(1)} = V\tau$ this gives

$$w(v\tau^{p-1} \cdot v\tau^{p-1}) = w(v \cdot v) + 2w(p),$$

for every $v \in V^{(1)}$. Now, the restriction $\tau^{(1)}$ of τ to $V^{(1)}$ is evidently a linear transformation of $V^{(1)}$. From $\tau \in L_{w(p)}$ follows $\tau^{(1)} \in L_{w(p)}(V^{(1)})$. Hence $(\tau^{(1)})^{p-1} \in L_{(p-1)w(p)}(V^{(1)})$ (Equation (33)). Thus

$$w(v\tau^{p-1} \cdot v\tau^{p-1}) \geq w(v \cdot v) + 2(p - 1)w(p)$$

for every $v \in V^{(1)}$. Hence,

$$w(v \cdot v) + 2w(p) \geq w(v \cdot v) + 2(p-1)w(p).$$

If now $v \neq 0$, $v \in V^{(1)}$, this gives

$$2w(p) \geq 2(p-1)w(p); \; 2(p) \geq (p-1)w(p); \; (p-2)w(p) \leq 0.$$

However p is an odd prime. Thus $p - 2$ is a non-zero natural number. In view of the relation $w(p) > 0$, $(p-2)w(p) > 0$ follows, a contradiction. This shows that $V^{(1)} = 0$ or $V\tau = 0$, that is, $\tau = 0$ or, $\phi = \tau + 1 = 1$, a contradiction.   □

Question (open). Must U_0^+ be always torsion-free?

To conclude this section, let me make two side remarks. The information about the normal subgroup structure of the unitary group as obtained in Theorem 2.14 can be carried over (almost verbatim) to the normal subgroup structure of the medial group M (that is, the group of units of the medial semi-group $\overline{M}$ which have adjoints). The only difference arises in the fact that while $\sigma - 1$ nilpotent with $\sigma \in U_{w(2)}^+$ implies $\sigma = 1$, one cannot decide as neatly in the case $\sigma \in M_{w(2)}^+$. As a result, in the preceding theorem, roots of unity in M_0^+ can occur when charac$(D) = p \neq 0$; these are, of course, p roots of unity. If charac$(\overline{D}) = p$ and charac$(D) = 0$ then $M_{w(p)}^+$ is torsion-free. The second remark is of different nature. I stated in the Introduction that my motivation in the ring L_0 arises equally from the ring A that is spanned by U. In a continuation of this work I will show to what extent A coincides with L_0 and/or L_0 and A have essentially the same ideal structure.

3.   **Bounded orthogonal bases.**   In this section, the following situation will be analysed: Relative to a given orthogonal basis (f_i) the non-isotropic space V is such that for each unitary transformation ϕ the matrix of ϕ is with entries from a fixed additive subgroup J_g of the valued division ring D. Special attention is given to the case $g = 0$. In that case and assuming that $1/2 \in R$ the results will be quite complete. I begin with notations to be used throughout this section.

(a) *Row finite matrix* ϕ. All matrices Φ are row-finite matrices. However, the number of rows of Φ is any cardinal number. Thus $\Phi = [\phi_{ij}]_{i,j \in I}$ is such that:

(0) $\phi_{ij} = \text{entr}_{i,j}(\Phi) \in D$;

(1)   For each $i \in I$, $(\phi_j)_{j \in I}$ is a nullary row vector over D.

(b) *Matrix of inner products S.* Let $(f_i)_{i \in I}$ be a fixed orthogonal basis of V. I will consistently denote the inner product $f_i \cdot f_i$ by s_i. Thus the matrix of the inner products $f_i \cdot f_j$ is the diagonal matrix

(2) $$S = \mathrm{Diag}\{s_i\}_{i \in I} \qquad (s_i = f_i \cdot f_i)$$

(c) *Matrix involutions.* The standard matrix involution tr.* is the partial involution of the ring $D_{I \times I}$ of all $I \times I$ row-finite matrices $\Phi = [\phi_{ij}]_{i,j \in I}$ over D, which is defined at Φ if, and only if, $\Phi^{\mathrm{tr.}} \in D_{I \times I}$. Then

$$\Phi^{\mathrm{tr.*}} = \left[\phi_{ji}^* \right]_{i,j \in I}.$$

The matrix of inner products $S = \mathrm{Diag}\{s_i\}_{i \in I}$ induces a new partial involution (S) of $D_{I \times I}$ which is defined exactly when tr.* is defined and then

(3) $$\Phi^{(S)} = S \Phi^{\mathrm{tr.*}} S^{-1}.$$

Explicitly,

(4) $$\mathrm{entr}_{i,j}(\Phi^{(S)}) = s_i \, \mathrm{entr}_{j,i}(\Phi)^* s_j^{-1} \qquad (i, j \in I).$$

(d) *Matrix prevaluation.* Let $\cdots \underline{w} \geq \cdots$ be the binary relation between the Z-algebra $D_{I \times I}$ and the group with infinity adjoined $G^{\#}$, which is defined by

(5) $$\Phi \underline{w} \geq g \Leftrightarrow \mathrm{entr}_{i,j}(\Phi) \underline{w} \geq g \text{ (all } i, j \in I)$$

$$\left(\text{e.g. } w\big(\mathrm{entr}_{i,j}(\Phi)\big) \geq g, \text{ all } i, j \in I \right).$$

Then $\cdots \underline{w} \geq \cdots$ is a *-prevaluation relative to the involution tr.*; but, $\cdots \underline{w} \geq \cdots$ is generally *not* a *-prevaluation relative to the matrix involution (S). Hence it is better to refer to this case as *matrix prevaluation.*

(e) *Bounded basis* (f). I shall say that the orthogonal basis (f) is *bounded* if there is $g_0 \in G$, which I call a bound for (f_i), such that

(6) $$w(s_i) - w(s_j) \leq g_0$$

for every pair $i, j \in I$—Recall that $s = f \cdot f$. Evidently every finite-dimensional unitary space V over the *-valued division ring D has all its orthogonal bases bounded, the case ε is the *least* bound for the given basis (f) is noteworthy but not required in what will follow (unless otherwise specified).

(f) *Matrix of a linear transformation.* Throughout the rest of this section V stands for any non-isotropic unitary space over the *-valued division ring. I will assume, further, that V is equipped with the fixed

orthogonal basis (f_i). To the vector $v \in V$ associate the nullary row vector

(7) $$\underline{v} = (x_i) \Leftrightarrow v = \sum_{i \in I} x_i f_i.$$

To $\phi \in L$ associate the matrix of ϕ, Mtrx(ϕ) (or $\underline{\phi}$) $\in D_{I \times I}$ determined by the equations

(8) $$\mathrm{Row}_i(\mathrm{Mtrx}(\phi)) = (\phi_{ij})_{j \in I} \Leftrightarrow f_i \phi = (\phi_{ij})_{j \in I} \qquad (\iota \in I).$$

DEFINITION 3.1. I will call the basis (f_i) ε_0'-elliptic, where $\varepsilon_0' \in G$ is fixed, if

(9)   For each $\phi \in U = U(V)$, Mtrx$(\phi)\underline{w} \geq \varepsilon_0'$.

THEOREM 3.2. *Let V be any non-isotropic unitary space and let (f_i) be any orthogonal basis of V. The following requirements are equivalent.*
(1) (f_i) is an elliptic basis (e.g. $(f_i)_{i \in I}$ is ε_0'-elliptic for some $\varepsilon_0' \in G$)
(2) V is an elliptic space and (f_i) is bounded.

Proof $(1) \Rightarrow (2)$.

Claim 1. There is $\varepsilon_0 \in G$ such that if $(x_i)_{i \in I}$ is any nullary row vector over D, then

(10) $$\min_{i \in I} w(x_i s_i x_i^*) \geq w\left(\sum_{j \in I} x_j s_j x_j^*\right) + \varepsilon_0.$$

If $(x_i) = 0$ there is nothing more to show. If $(x_i) \neq 0$ then if $v = \sum x_i f_i$ then $v \neq 0$ so that $v \cdot v \neq 0$. Hence, the linear transformation $\sigma = \sigma_v$ defined by

$$u\sigma = u \cdot v \frac{1}{v \cdot v} v$$

is a projection of V (that is, $\sigma^2 = \sigma = \sigma^*$). Thus $\phi = 1 - 2\sigma$ is a unitary transformation of V. Let then ε_0' be such that (f_i) is ε_0'-elliptic. By definition, if Φ is the matrix ϕ then $\Phi\underline{w} \geq \varepsilon_0'$. Now, $\Phi = 1 - 2\,\mathrm{Mtrx}(\sigma)$. From this follows

$$\mathrm{entr}_{i,j}(\underline{\sigma}) \geq \varepsilon_0' - w(2) \qquad (i, j \in I),$$

where $\underline{\sigma} = \mathrm{Mtrx}(\sigma)$.
    A simple calculation shows that

$$\mathrm{entr}_{i,j}(\underline{\sigma}) = (f_i \cdot x_i f_i) \frac{1}{v \cdot v} x_j.$$

If $\varepsilon_0 = \varepsilon_0' - w(2)$ then by the preceding,

$$w(s_i) + w(x_i) + w(x_j) \geq w(v \cdot v) + \varepsilon_0;$$

for $i = j$, this gives

$$w(x_i s_i x_i^*) \geq w(v \cdot v) + \varepsilon_0.$$

As this holds for every $i \in I$, it follows that

$$\operatorname*{Min}_{i \in I} w(x_i s_i x_i^*) \geq w(v \cdot v) + \varepsilon_0 = w\left(\sum_{i \in I} x_i s_i x_i^*\right) + \varepsilon_0.$$

Claim 2. Let ε_0 be as in Claim 1. Then V is ε_0-elliptic.
Let (x_i), (y_i) be any two nullary vectors over D. For $i \in I$ fixed:

$$2w(x_i s_i y_i^*) = w(x_i s_i x_i^*) + w(y_i s_i y_i^*)$$

$$\geq \operatorname*{Min}_{j \in I} w(x_j s_j x_j^*) + \operatorname*{Min}_{j \in I} w(y_j s_j y_j^*).$$

The preceding inequality holding true for every $i \in I$, it follows that

$$2 \operatorname*{Min}_{i \in I} w(x_i s_i y_i^*) = \operatorname*{Min}_{i \in I} \left(2w(x_i s_i y_i^*)\right)$$

$$\geq \operatorname*{Min}_{j \in I} w(x_j s_j x_j^*) + \operatorname*{Min}_{j \in I} w(y_j s_j y_j^*)$$

$$\geq w\left(\sum x_j s_j x_j^*\right) + w\left(\sum y_j s_j y_j^*\right) + 2\varepsilon_0.$$

Given $u, v \in V$ it is clear that there are (x_i), (y_i) such that $\underline{u} = (x_i)$, $\underline{v} = (y_i)$ (e.g. $u = \sum y_i f_i$). Then

$$2w(u \cdot v) = 2w\left(\sum x_i s_i y_i^*\right) \geq 2\operatorname{Min}(x_i s_i y_i^4)$$

$$\geq w\left(\sum x_j s_j x_j^*\right) + w\left(\sum y_j s_j y_j^*\right) + 2\varepsilon_0$$

$$= w(u \cdot u) + w(v \cdot v) + 2\varepsilon_0,$$

and V is ε_0-elliptic.

Claim 3. $(f_i)_{i \in I}$ is bounded.

For suppose that (f_i) is ε_0'-elliptic. Taking up the calculation in Claim 1, in the case $v = f_i + xf_j$, where x is to be fixed later and $i, j \in I$ are such that $w(s) \leq w(s_j)$, then

$$w(s_i) + w(x) \geq w(x_i + xs_j x^*) + \varepsilon_0 \qquad \left(\varepsilon_0 = \varepsilon_0' - w(2)\right).$$

Pick $\varepsilon_1 > |\varepsilon_0|$, where as usual $|\varepsilon_0| = \varepsilon_0$ if $\varepsilon_0 \geq 0$ and $|\varepsilon_0| = -\varepsilon_0$ if $\varepsilon_0 < 0$. Put $\varepsilon_2 = 2\varepsilon_1$. I claim that $\infty \neq g \geq \varepsilon_1$ implies $w(s_j) - w(s_i) \leq 2g + \varepsilon_1$. For, otherwise, for such a $g \in G$, there is $0 \neq x \in D$ such that $w(x) = -g$. Then

$$w\left(xs_jx^*\right) = w(s_j) - 2g > \left(w(s_i) + 2g + \varepsilon_1\right) - 2g$$

$$= w(s_i) + \varepsilon_1 > w(s_i).$$

By the special triangle equality follows

$$w\left(s_i + xs_jx^*\right) = w(s_i).$$

Then

$$w(s_j) + 2(x) \geq w(s_i) + \varepsilon_0;$$

$-g = w(x) \geq \varepsilon_0$. Equivalently, $q \leq -\varepsilon_0 < \varepsilon_1$, contrary to the relation $g \geq \varepsilon_1$. Hence the asserted implication holds. If $g = \varepsilon_1$, this gives $w(s_j) - w(s_i) \leq s\varepsilon_1 + \varepsilon_1 = 3\varepsilon_1 = \varepsilon_2$, as desired. Hence $w(s_j) \geq w(s_i)$ implies $w(s_j) - w(s_i) \leq \varepsilon_2$. Now, if $w(s_j) < w(s_i)$ it is clear that since $\varepsilon_2 \geq 0$, $w(s_j) - w(s_i) \leq \varepsilon_2$. Therefore (f_i) is bounded with bound ε_2.

$(2) \Rightarrow (1)$. I will establish something more, namely,

(11) $\phi w \geq g \Rightarrow \underline{\phi}\underline{w} \geq g + \varepsilon_0 - \varepsilon'$,

where: ε' is a bound for (f_i), ε_0 is an index of ellipticity for V; $\underline{\phi} = $ Mtrx(ϕ)—Recall that $\underline{\phi}w \geq g'$ means $w(\text{entr}_{i,j}(\underline{\phi})) \geq g'$, for every pair $i, j \in I$.

Claim 1. If V is ε_0-elliptic then for each nullary vector (x_i) over D,

$$2\varepsilon_0 + w\left(\sum_{\iota \in I} x_\iota s_\iota x_\iota^*\right) \geq \text{Min}_{\iota \in I}\left(x_\iota s_\iota x_\iota^*\right).$$

It suffices to apply inequality (22), Theorem 2.6, §2, for $v_i = x_i f_i$ $(i \in I)$.

Claim 2. $w \geq g \Rightarrow \underline{\phi}\underline{w} \geq g + \varepsilon_0 - \varepsilon'$.
Put $u_i = f_i\phi$. From $\phi w \geq g$ follows

$$w(u_i \cdot u_i) = w(f_i\phi \cdot f_i\phi) \geq w(f_i \cdot f_i) + 2g = w(s_i) + 2g.$$

Now,

$$w(u_\iota \cdot u_\iota) = w\left(\sum_{j \in I} \phi_{\iota j} s_j \phi'_{\iota j}\right).$$

From Claim 1, follows

$$w\left(\phi_{ij'}s_{j'}\phi_{ij'}^*\right) \geq \operatorname*{Min}_{j\in I} w\left(\phi_{ij}s_j\phi_{ij}^*\right) \geq w\left(\sum \phi_{ij}s_j\phi_{ij}^*\right) + 2\varepsilon_0$$

$$= w(u_i \cdot u_i) + 2\varepsilon_0 \geq w(s_i) + 2g + 2\varepsilon_0,$$

$$2w\left(\phi_{ij'}\right) \geq 2(s_i) - w(s_j) + 2g + 2\varepsilon_0$$

$$\geq -\varepsilon' + 2g + 2\varepsilon_0 \geq -2\varepsilon' + 2g + 2\varepsilon_0;$$

$$w\left(\phi_{ij'}\right) \geq g + \varepsilon_0 - \varepsilon',$$

as desired.

Claim 3. If $\phi \in U$ then $\underline{\phi w} \geq 2\varepsilon_0 - \varepsilon'$.
Since V is ε_0-elliptic, it follows that

$$2w(u \cdot v\phi) \geq w(u \cdot u) + w(v\phi \cdot v\phi) + 2\varepsilon_0,$$

and since ϕ is unitary,

$$2w(u \cdot v\phi) \geq w(u \cdot u) + 2(v\phi \cdot v\phi) + 2\varepsilon_0$$

$$= w(u \cdot u) + w(v \cdot v) + 2\varepsilon_0,$$

so that,

$$\phi w \geq \varepsilon_0.$$

By Claim 2,

$$\underline{\phi w} \geq \varepsilon_0 + \left(\varepsilon_0 - \varepsilon'\right) = 2\varepsilon_0 - \varepsilon'$$

follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\Box$

The proof of the preceding theorem revealed few facts that are appropriate to separate for future reference. This is the content of

REMARKS 3.3. Let (f_i) be any orthogonal basis of the non-isotropic unitary space V. Then:
(a) If V is ε_0-elliptic and if ε_1 is a bound for (f_i), then

(12) $$\phi w \geq g \Rightarrow \underline{\phi w} \geq (g + \varepsilon_0 - \varepsilon_1).$$

(b) If V is ε_0-elliptic then for every nullary vector (x_i) over D,

(13) $$w\left(\sum_{i\in I} x_i s_i x_i^*\right) - 2\varepsilon_0 \leq \operatorname*{Min}_{i\in I}\left(w\left(x_i s_i x_i^*\right)\right).$$

(c) Conversely, if the preceding inequality holds then V automatically verifies a generalized Cauchy-Schwarz inequality. In effect, V is (ε_0)-elliptic.

I proceed to a converse of inequality (12) in Remarks 3.3.

THEOREM 3.4. *Let V be any non-degenerate ε_0-elliptic space with orthogonal basis (f_i) bounded by ε_1. If $\phi \in L$ is such that $\underline{\phi}w \geq g$, then $\phi w \geq g + 2\varepsilon_0 - \varepsilon_1$ follows.*

Proof. As a first reduction, one can replace the asserted relation by the quadratic inequality

$$w(v\phi \cdot v\phi) \geq w(v \cdot v) + 2(g + \varepsilon_0 - \varepsilon_1).$$

This readily follows from ε_0-ellipticity of V. As a second reduction, one may assume V is finite-dimensional. The argument to that effect runs as follows. If $\underline{v}$ is the row vector of $v \in V$ and $\underline{v\phi}$ that of $v\phi$, then for $\underline{u} = \underline{v\phi}$ it is clear that

$$\underline{v}\underline{u} = \underline{u}.$$

By construction, $\underline{u}$ is a nullary vector over D. If $\underline{F}^{(i,j)}$ is the usual (i, j) standard matrix in $D_{I \times I}$ and $\phi_{ij} = \text{entr}_{i,j}(\phi)$ then evidently there is a finite subset I_1 of I such that if $\Phi' = \sum_{i,j \in I_1} \underline{E}^{(i,j)}\phi_{ij}$ then

$$\underline{u} = \underline{v}\phi'.$$

If $\phi' \in L$ is such that $\Phi' = \text{Mtrx}(\phi')$, then $v\phi' = v\phi$ follows. By construction, $\Phi' = \underline{\phi}'\underline{w} \geq g$. If the assertion can be shown for ϕ replaced by ϕ' then evidently

$$w(v\phi \cdot v\phi) = w(v\psi' \cdot v\phi') \geq w(v \cdot v) + 2(g + \varepsilon_0 - \varepsilon_1).$$

Hence one can replace ϕ by ϕ'. If $V^{(1)}$ is the subspace that is spanned by the f_i as i ranges over I_1, then without loss of generality $v \in V^{(1)}$, by construction $V^{(1)}\phi' \subset V^{(1)}$, and $(f_i)_{i \in I_1}$ is, again, orthogonal with bound ε_1. Since $V^{(1)}$ is finite-dimensional the asserted reduction follows. Hence in what follows I can be taken to be a finite set, say, $I = \{1, 2, \ldots, n\}$. Let $S = \text{Diag}\{s_1, \ldots, s_n\}$. Then

$$v\phi \cdot v\phi = \underline{v\phi}S\left(\underline{v\phi}^{\text{tr},*}\right),$$

so that

$$w(v\phi \cdot v\phi) = w\left(\underline{v\phi}S\left(\underline{v\phi}\right)^{\text{tr},*}\right).$$

Observing that if $\underline{w}(\Phi)$ is the minimum value of the entries of Φ then $\underline{w}$ is sub-multiplicative one gets

$$w\left(v\phi S(v\phi)^{\text{tr.}*}\right) = \underline{w}\left(v\phi S(v\phi)^{\text{tr.}*}\right) \geq 2\underline{w}(v\phi) + \underline{w}(S).$$

Hence,

$$w(v\phi \cdot v\phi) \geq \underline{ww}(v\phi) + w(S) \geq 2\underline{w}(v) + 2\underline{w}(\phi) + \underline{w}(S).$$

If $v = \sum x_i f_i$, then by Remarks 3.3b) follows

$$\text{Min } w\left(x_i s_i x_i^*\right) \geq w\left(\sum x_i s_i x_i^*\right) + 2\varepsilon_0 = w(v \cdot v) + 2\varepsilon_0.$$

If $\overline{w}(S)$ is the maximum value of the diagonal entries in S (all $\neq 0$), then

$$2\underline{w}(v) + \overline{w}(S) \geq \text{Min } w\left(x_i s_i x_i^*\right) \geq w(v \cdot v) + 2\varepsilon_0.$$

Hence,

$$
\begin{aligned}
w(v\phi \cdot v\phi) &\geq \left(w(v \cdot v) + 2\varepsilon_0 - \overline{w}(S)\right) + 2\underline{w}(\phi) + \underline{w}(S) \\
&\geq w(v \cdot v) + 2\varepsilon_0 - \overline{2}(S) + \underline{w}(S) + 2g \\
&= w(v \cdot v) + 2\varepsilon_0 + \left(\underline{w}(S) - \overline{w}(S)\right) + 2g \\
&\geq w(v \cdot v) + 2\varepsilon_0 - \varepsilon_1 + 2g \\
&\geq w(v \cdot v) + 2\varepsilon_0 - 2\varepsilon_1 + 2g \\
&= w(v \cdot v) + 2(g + \varepsilon_0 - \varepsilon_1),
\end{aligned}
$$

as desired. $\square$

Combining (12) with Theorem 3.4:

THEOREM 3.5. *Let V be any non-degenerate ε_0-elliptic space and let (f_i) be any orthogonal basis, which is bounded by ε_1. Then:*
1. $\phi w \geq \Rightarrow \underline{\phi w} \geq g \, (= g + 2\varepsilon_0 - \varepsilon_1)$;
2. $\underline{\phi w} \geq g \Rightarrow \phi w \geq g$; *where ϕ is the matrix of ϕ and $\underline{\phi w} \geq g$ means $w(\text{entr}_{i,j}(\underline{\phi})) \geq g$ for every pair $1, j \in I$.*

COROLLARY 3.6. *Under the assumptions in Theorem 3.5, if $g_1, g_2 \in G$ are such that $|g_1 - g_2| > 2(\varepsilon_1 - 2\varepsilon_0)$ then $L_{g_1} \neq L_{g_2}$.*

Proof. Assume, say, $g_2 > g_1 + 2(\varepsilon_1 - 2\varepsilon_0)$. If now $L_{g_1} = L_{g_2}$ define $\bar{g}_1 = g_1 - 2\varepsilon_0 + \varepsilon_1$. If $(J_{\bar{g}_1})_{I \times I}$ is the additive subgroup of $I \times I$ matrices $\Phi \in D_{I \times I}$ such that each entry of Φ belongs to $J_{\bar{g}}$ then by Theorem 3.5, point 2. follows $\text{Mtrx}^{-1}(J_{\bar{g}_1})_{I \times I} \subset L_{g_1}$. Also, by Theorem 3.5, point 1.

follows $\mathrm{Mtrx}^{-1}(J_{g_2}) \supset L_{g_2}$. Thus

$$\mathrm{Mtrx}^{-1}(J_{g_2})_{I \times I} \mathrm{Mtrx}^{-1}(J_{\bar{g}_1})_{I \times I}.$$

Hence,

$$(J_{g_2})_{I \times I} \supset (J_{\bar{g}_1})_{I \times I}.$$

Thus,

$$J_{g_2} \supset J_{\bar{g}_1}.$$

From this $\underline{g}_2 \leq \bar{g}_1$. Recalling that $\underline{g}_2 = g_2 + 2\varepsilon_0 - \varepsilon_1$ and $\bar{g}_1 = g_1 - 2\varepsilon_0 + \varepsilon_1$, then

$$g_2 - g_1 \leq (-2\varepsilon_0 + \varepsilon_1),$$

which is a contradiction. □

In the course of the proof of the preceding corollary, certain matrix approximation of the additive subgroup L_g was offered; namely,

(14) $\mathrm{Mtrx}(L_g) \supset (J_{\bar{g}})_{I \times I}$ $(\bar{g} = g - 2\varepsilon_0 + \varepsilon_1).$

One can offer related bound for the congruence subgroup U_g of the unitary group U. For observe that the passage from $\phi \in (L; *)$ to $\mathrm{Mtrx}(\phi) \in (D_{I \times I}; (S))$ is an isomorphism of partial involutorial rings. Hence

(15) $\mathrm{Mtrx}(U) = \mathscr{U}(D_{I \times I}; (S))$ $(=$ Unitary group of $(D_{I \times I}; (S))).$

Related to the congruence subgroup U_g (resp. U_g^+) is the additive subgroup $\mathscr{K}_g(V)$ (resp. $\mathscr{K}_g^+(V)$) which, I recall, is the set of skew-symmetric transformations $\psi = -\psi^*$ such that $\psi \in L_g(V)$ (resp. $\psi \in L_g^+(V)$).

REMARK 3.7. In the partial involutorial matrix ring $D_{I \times I}$ relative to (S), the matrix Φ is skew-symmetric if, and only if, Φ is of the form

(17) $\Phi = S\Sigma,$

where Σ is a row and column finite matrix such that

(18) $\Sigma^{\mathrm{tr}.*} = -\Sigma.$

Proof. From (18) follows

$$\Phi^{(S)} = (S\Sigma)^{(S)} = S(\Sigma^{\mathrm{tr}.*}S)S^{-1} = -S\Sigma.$$

Conversely, if $\Phi^{(S)}$ is defined then Φ is both row and column finite and then

(19) $\Phi^{(S)} = S\Phi^{\mathrm{tr}.*}S^{-1} = -\Phi.$

If $\Sigma = S^{-1}\Phi$, then by construction $\Phi = S\Sigma$ and,

$$\Sigma^{\text{tr.*}} = \Phi^{\text{tr.*}}S^{-1} = -S^{-1}\Phi \quad (\text{from (19)}) = -\Sigma. \qquad \square$$

THEOREM 3.9. *Let V be as in Theorem 3.5. Assume, further, that V is not the 1-dimensional orthogonal space. If g_1, $g_2 \in G$ are such that $|g_1 - g_2| > 3\varepsilon_1 - 4\varepsilon_0$ and if g_1, $g_2 > w(2)$ then $U_{g_1} \neq U_{g_2}$.*

Proof. Let $\mathscr{K}_g(D_{I\times I}; (S))$ be the additive subgroup of skew-symmetric matrices Φ relative to (S) such that $\Phi\underline{w} \geq g$. From Theorem 3.5, point 2., follows

(20) $$\text{Mtrx}(\mathscr{K}_g(V)) \supset_{\bar{g}}(D_{I\times I}; (S)),$$

where $\bar{g} = g - 2\varepsilon_0 + \varepsilon_1$. From the cited theorem point 1. follows

(21) $$\text{Mtrx}(\mathscr{K}_g(V)) \subset \mathscr{K}_{\underline{g}}(D_{I\times I}; (S)),$$

where $\underline{g} = g + 2\varepsilon_0 - \varepsilon_1$. Assume that $\mathscr{K}_{g_1}(V) = \mathscr{K}_{g_2}(V)$. Applying (20) to $g = g_1$ and (21) to $g = g_2$ this gives

$$\mathscr{K}_{\bar{g}_1}(D_{I\times I}; (S)) \subset \text{Mtrx}(\mathscr{K}_{g_1}(V)) = \text{Mtrx}(\mathscr{K}_{g_2}(V)) \subset \mathscr{K}_{\underline{g}_2}(D_{I\times I}; (S)).$$

From Remark 3.7 follows

$$\mathscr{K}_g(D_{I\times I}; (S)) = \{ S\Sigma | \Sigma^{\text{tr.*}} = -\Sigma(S\Sigma)\underline{w} \geq g\}.$$

When the ground involution $*$ in D is the identity mapping then by hypothesis $\dim_D V \geq 2$. In that case choose Σ of the form

$$\Sigma = \begin{vmatrix} 0 & x \\ -x & 0 \end{vmatrix} \oplus 0;$$

where $w(s_1 x)$ and $w(s_2 x) \geq \bar{g}_1$. By construction, $S\Sigma \in \mathscr{K}_{\bar{g}_1}(D_{I\times I}; (S))$. Hence, $S\Sigma \in \mathscr{K}_{g_2}(D_{I\times I}; (S))$ and so, $w(s_1 x)$ and $w(s_2 x) \geq \underline{g}_2$. What I have shown is that

(22) $$\begin{cases} w(s_1 x) \geq \bar{g}_1 \\ w(s_2 w) \geq \bar{g}_1 \end{cases} \Rightarrow \begin{cases} w(s_1 x) \geq \underline{g}_2 \\ w(s_2 x) \geq \underline{g}_2 \end{cases} \quad (x \in D).$$

Now $|w(s_1) - w(s_2)| \leq \varepsilon_1$. If then x is such that $w(x) = \bar{g}_1 - w(s_1) + \varepsilon_1$, then

$$w(s_1 x) = \bar{g}_1 + \varepsilon_1 > \bar{g}_1$$

and

$$w(s_2 x) = w(x) + w(s_2) = \bar{g}_1 + \varepsilon_1 + (w(s_2) - w(s_1))$$
$$\geq \bar{g}_1 + \varepsilon_1 - \varepsilon_1 = \bar{g}_1.$$

From (22) follows $w(s_1 x) = \bar{g}_1 + \varepsilon_1 \geq g_2$. Hence,

$$\bar{g}_1 + \varepsilon_1 = g_1 - 2\varepsilon_0 + 2\varepsilon_1 \geq g_2 + 2\varepsilon_0 - \varepsilon_1.$$

or,

(23) $$g_1 - g_2 \geq 4\varepsilon_0 - 3\varepsilon_1; \ g_2 - g_1 \geq 3\varepsilon_1 - 4\varepsilon_0.$$

By symmetry,

$$g_1 - g_2 \leq 3\varepsilon_1 - 4\varepsilon_0,$$

contradicting the hypothesis $|g_1 - g_2| > 3\varepsilon_1 - 4\varepsilon_0$.

When the ground involution $*$ in D is not the identity mapping one can choose in that case Σ to be of the form

$$\Sigma = [x] \oplus 0,$$

where $x^* = -x$. By a trivial adaptation of the preceding argument follows $|g_1 - g_2| \leq 2\varepsilon_1 - 4\varepsilon_0$, which contradicts again the hypothesis. This shows that $\mathscr{H}_{g_1}(V) \neq \mathscr{H}_{g_2}(V)$. Since $g_1, \iota_2 > w(2)$ it follows from Theorem 2.16, §2, that $U_{g_1} \neq U_{g_2}$. $\square$

The proof of the preceding theorem evidenced the fact that the smaller the bounds ε_0, ε the better the matrix counterparts of L_g, U_g will be. Hence, it seems desirable to work out still further the special case where the basis (f_i) is 0-elliptic which means, I recall, every unitary transformation ϕ of V has matrix over the valuation ring R. From Theorem 3.2, the space V automatically verifies a generalized Cauchy-Schwarz inequality where the index is not less than $-w(2)$. When $1/2 \in R$, then V becomes 0-elliptic and the bound ε for (f_i) should be quite small since by the cited theorem $|w(s_i) - (s_j)| \leq 2g$ for each positive $g \in G$. In the theorem to follow it is shown that indeed ε can be taken to be either 0 or else, ε is the first positive element in G. This is the

THEOREM 3.10. *Let V be any non-isotropic unitary space with orthogonal basis (f_i). If $1/2 \in R$ and if the basis is 0-elliptic then*:
(1) *V is 0-elliptic.*
(2) *The basis (f_i) is bounded by ε, where either $\varepsilon = 0$ or else ε is the first positive element in the value group G.*

Proof (of (2)). If the basis (f_i) is uniform in the technical sense all the f_i have same lengths there is nothing more to prove. If to the contrary there is a pair $i, j \in I$ such that f_i and f_j have distinct lengths then $|w(s_i) - w(s_j)| \neq 0$. Since $|w(s_i) - w(s_j)| \leq 2g$ for every $g > 0$ it follows

that if $\varepsilon_1 = |w(s_i) - w(s_j)|$ then ε_1 is the first positive element of G if $\varepsilon_1 \notin 2G$; otherwise, $1/2\varepsilon_1$ is the first positive element of G. In both cases, G has definitely a first positive element ε and what is left to show is that the case $\varepsilon_1 \in 2G$ can be dismissed. For assume, say, $w(s_j) = w(s_i) + 2\varepsilon$. Evidently V has dimension at least 2. Pass to the 2-dimensional subspace $V^{(1)}$ spanned by f_j and f_i. Then $V^{(1)}$ has orthogonal complement in V. by standard argument, (f_i, f_j) is then a 0-elliptic basis for the space $V^{(1)}$. Put $v = x_1 f_i + x_2 f_j$, where $x_1, x_2 \in D$ are to be fixed later. Writing out the matrix of the projection τ induced by v one gets for matrix of τ the following matrix:

$$(24) \qquad \mathrm{Mtrx}(\tau) = \begin{vmatrix} s_i x_1^* \dfrac{1}{v \cdot v} x_1 & s_i x_1^* \dfrac{1}{v \cdot v} x_2 \\ s_j x_2^* \dfrac{1}{v \cdot v} x_1 & s_j x_2^* \dfrac{1}{v \cdot v} x_2 \end{vmatrix}$$

Since by hypothesis $w(2) = 0$ it follows as observed earlier that $\mathrm{Mtrx}(\tau)\underline{w} \geq 0$ and, hence, $w(\mathrm{entr}_{1,2}(\mathrm{Mtrx}(\tau))) \geq 0$. Since V is a 0-elliptic space one knows in that case that (Theorem 1.7, §1, equality $(29)'$)

$$w(v \cdot v) = w\left(x_1 s_i x_1^* + x_2 s_j x_2^*\right) = \mathrm{Min}\left(w\left(x_1 s_i x_1^*\right), w\left(x_2 s_j x_2^*\right)\right).$$

If x_1, x_2 are such that $w(x_1) = 0$, $w(x_2) = -\varepsilon$ then one should get for such x_i, $\mathrm{entr}_{1,2}(\underline{\tau}^{(1)}) \in R$. Equivalently,

$$w(s_i) + w(x_1) + w(x_2) \geq w(v \cdot v).$$

Now,

$$w\left(x_2 s_j x_2^*\right) = 2w(x_2) + 2(s_j) = -2\varepsilon + w(s_j)$$

$$= (-2\varepsilon) + (2(s_i) + 2\varepsilon) = w(s_i)$$

$$= w(s_i) + 2w(x_1) = w\left(x_1 s_i x_1^*\right).$$

Hence,

$$\mathrm{Min}\left(w\left(x_1 s_i x_1^*\right), w\left(x_2 s_j x_2^*\right)\right) = w(s_i).$$

Thus,

$$w(v \cdot v) = w(s_i).$$

Substituting for $w(v \cdot v)$ in the preceding inequality this gives

$$w(x_1) + w(x_2) \geq 0 \quad \text{or} \quad 0 - \varepsilon \geq 0,$$

which is nonsense. □

THEOREM 3.11. *Conversely, let V be any non-degenerate 0-elliptic space —$w(2)$ need not be 0. If $(f_i)_{i \in I}$ is any orthogonal basis with bound ε, where $\varepsilon = 0$ or ε is the first positive element in G, then $(f_i)_{i \in I}$ is a 0-elliptic basis for V.*

Proof. It is to be shown that if $\phi \in U$ then $\mathrm{Mtrx}(\phi) \in R_{I \times I}$. I will show something more, namely, if ϕ is integral (e.g. $\phi \in L_0$) then $\mathrm{Mtrx}(\phi) \in R_{I \times I}$. For let $\phi \in L_0$. Then ϕ increases lengths. Hence, for each $i \in I$,

$$w(f_i \phi \cdot f_i \phi) \geq w(s_i).$$

If ϕ_{ij} is the (i, j) entry in the matrix $\underline{\phi}$ of ϕ then as in the proof of Theorem 3.2, point 1, Claim 2, one has

$$w\left(\phi_{ij} s_j \phi_{ij}^*\right) \geq \operatorname*{Min}_{j_0 \in I} w\left(\phi_{ij_0} s_{ij_0} \phi_{ij_0}^*\right)$$

$$= w\left(\sum_{j_0 \in I} \phi_{ij_0} s_{j_0} \phi_{ij_0}^*\right) = w(f_i \phi \cdot f_i \phi) \geq w(s_i).$$

Hence,

$$2w(\phi_{ij}) \geq w(s_i) - w(s_j) \geq -\varepsilon.$$

If $\varepsilon = 0$ from $2w(\phi_{ij}) \geq 0$ follows $w(\phi_{ij}) \geq 0$. If ε is the first positive element of G, then by the usual arithmetic argument from $2w(\phi_{ij}) \geq -\varepsilon$ follows $w(\phi_{ij}) \geq 0$, as desired.

If now ϕ is any unitary matrix then from 0-ellipticity of V follows $\phi \in L_0$ so that $\mathrm{Mtrx}(\phi) \in R_{I \times I}$. $\qquad\square$

To recapitulate the foregoing theorems in the special case under consideration:

THEOREM 3.12. *Let V be any non-isotropic unitary space. Suppose that $1/2 \in R$ and let (f_i) be any orthogonal basis of V. The following requirements are equivalent.*

(1) *V is 0-elliptic and (f_i) has bound ε with ε either 0 or else ε is the first positive element in G.*

(2) *If ϕ is unitary transformation then the matrix of ϕ is over the valuation ring R in D.*

(3) *If ϕ increases lengths (e.g. $w(v\phi \cdot v\phi) \geq w(v \cdot v)$ for every $v \in V$) then $\mathrm{Mtrx}(\phi)$ is over R.*

(4) *$(\phi w \geq \iota \Rightarrow \mathrm{Mtrx}(\phi)\underline{w} \geq g)$ and $(\mathrm{mtrx}(\phi)\underline{w} \geq g \Rightarrow \phi w \geq g - \varepsilon)$.*

Although Theorem 3.12 point 4. asserts that the prevaluations $\cdots w \geq \cdots$ and $\cdots \underline{w} \geq \cdots$ can be identified up to ε, still, in the inequality

(23) $\mathrm{Mtrx}(\phi)\underline{w} \geq g \Rightarrow \phi w \geq g - \varepsilon,$

it would be wrong to drop the term $-\varepsilon$. For if one neglects it, then w and $\underline{w}$ would be isomorphic $*$-prevaluations. Hence $\underline{w}$ would be an (S)-prevaluation; by inspection, this happens exactly when S has all its diagonal

entries with same values. Equivalently, the basis (f) is uniform (i.e. all f_i have same lengths). Because of the residual term $-\varepsilon$ one has less control on the strict prevaluation $\cdots w > \cdots$ and it is again wrong to infer that $\phi w > g$ implies $\underline{\phi w} > g$. The kind of information which is lacking here is how one can decide whether or not $\phi \in L_0$ looking at the matrix of ϕ. This problem will be solved in what will follow; but, first some additional notations and definitions.

(g) *Uniform and mixed bases.*
Without loss of generality the index set I can be turned into a linearly ordered set such that

$$(24) \qquad\qquad i \geq j \Rightarrow w(s_i) \geq w(s_j).$$

By construction, if $E = \{w(s_i)\}_{i \in I}$, then either

$$(25) \qquad\qquad \#E = 1 \quad \text{(uniform basis)},$$

or

$$(26) \qquad\qquad \#E = 2 \quad \text{(mixed basis)}$$

In the case the basis is uniform, put

$$(27) \qquad\qquad I_0 = I, \qquad I_\varepsilon = \varnothing .$$

In the case the basis is mixed, put

$$(28) \qquad I_0 = \{ i \in I | w(s_i) = \text{Min}(E)\}, \quad I_\varepsilon = \{ i \in I | w(s_i) = \text{Min}(E)\}.$$

By construction, I is the ordinal product of the ordered subsets I_0 and I_ε. As a memonic device for the considered matrix constructions to follow let me make the

(h) *Convention.* Given $g = -\varepsilon$, 0, or ε let

$$J^g = \begin{cases} R, & \text{if } g = -\varepsilon \text{ or } 0, \\ J, & \text{if } g = \varepsilon. \end{cases}$$

For instance, if G is the ordered additive group of integers then:

$$J^0 = R; \qquad J^1 = J;$$

and by convention,

$$J^{-1} = R.$$

There shall not be confusion in what will follow with inverses of fractional ideals.

(i) *w-Matrix.* By *w-matrix*, I mean any $I \times I$ matrix W over the value group G such that if $w_{ij} = \mathrm{entr}_{i,j}(W)$, then:

  (i) If G has no first positive element then $w_{ij} = 0$ for every pair $i, j \in I$. ($0 =$ zero of G);

  (ii) If G has a first positive element ε, then $w_{ij} = -\varepsilon, 0$ or ε, and;

  (iii) $i \geq j \Rightarrow w_{ij} \geq 0$;

  (iv) $w_{ji} = -w_{ij}$ $(i, j \in I)$.

(j) *The ring R_W.* Given the *w*-matrix W over G denote by R_W the subset of matrices $\Phi = [\phi_{ij}]_{i,j \in I}$ such that

$$(29) \qquad \begin{cases} \phi_{ij} \in R^{ij} & (\text{all } i, j \in I), \\ \Phi \in D_{I \times I}. \end{cases}$$

By inspection, R_W is a subring of $D_{I \times I}$, which is closed under the partial matrix involution (S).

(k) *The ideal J_W.* Given the *w*-matrix W, denote by W^+ the $I \times I$ matrix over G such that if $w_{ij}^+ = \mathrm{entr}_{i,j}(W)$ then

$$(30) \qquad w_{ij}^+ = \begin{cases} w_{ij}, & \text{if } w_{ij} \neq 0, \\ w_{ij}^+, & \text{if } w_{ij} = 0. \end{cases}$$

Define J_W to be the subset of matrices $\Phi = [\phi_{ij}]_{i,j \in I}$ such that

$$(31) \qquad \begin{cases} \phi_{ij} \in R^{w_{ij}^+} & (\text{all } i, j \in I), \\ \Phi \in D_{I \times I} \end{cases}$$

By inspection, J_W is a (S)-closed 2-sided ideal of the ring R_W.

(l) *Illustration.* Every orthogonal basis (f_i) of the non-isotropic space V such that (f_i) is bounded by the considered ε gives rise to the *w*-matrix $w = w((f_i))$, where

$$(32) \qquad w_{ij} = w(s_i) - w(s_j) \qquad (s_k = f_k \cdot f_k, k \in I).$$

Hence, if I is the ordered segment $I = \{1, 2, 3, 4\}$, $I_0 = \{1, 2\}$ and $I_\varepsilon = \{3, 4\}$, then:

$$W = \begin{vmatrix} 0 & 0 & -\varepsilon & -\varepsilon \\ 0 & 0 & -\varepsilon & -\varepsilon \\ \varepsilon & \varepsilon & 0 & 0 \\ \varepsilon & \varepsilon & 0 & 0 \end{vmatrix}; \qquad W^+ = \begin{vmatrix} \varepsilon & \varepsilon & -\varepsilon & -\varepsilon \\ \varepsilon & \varepsilon & -\varepsilon & -\varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{vmatrix};$$

$$R_W = \begin{vmatrix} R & R & R & R \\ R & R & R & R \\ J & J & R & R \\ J & J & R & R \end{vmatrix}; \qquad J_W = \begin{vmatrix} J & J & R & R \\ J & J & R & R \\ J & J & J & J \\ J & J & J & J \end{vmatrix}.$$

That, again, R_W is an (S)-closed subring of $D_{I \times I}$ and J_W is an (S)-closed 2-sided ideal of R_W this is also a consequence of the

THEOREM 3.13. *Let V be any non-degenerate 0-elliptic space with orthogonal basis $(f_i)_{i \in I}$ and suppose that either (f_i) is uniform or (f_i) is mixed (e.g. (f_i) is bounded by ε where either $\varepsilon = 0$ or $\varepsilon = 1$st positive element of the value group G). If W is the corresponding w-matrix $(w_{ij} = \text{entr}_{i,j}(W) = w(s_i) - w(s_j), s_k = f_k \cdot f_k)$ then:*
(1) *The ring R_W of all row-finite $I \times I$ matrices $\Phi = [\phi_{ij}]$ with $\phi_{ij} \in J^{w_{ij}}$ is isomorphic to the ring L_0 of linear transformations ϕ increasing lengths.*
(2) *The ideal J_W of all row-finite $I \times I$ matrices $\Phi = [\phi_{ij}]$ with $\phi_{ij} \in J^{w_{ij}^+}$ is isomorphic to the ideal L_0^+ of linear transformations ϕ strictly increasing lengths.*

Proof. Claim 1. $\text{Mtrx}(L_0) \subset R_W$.
From Theorem 3.2 implication $(2) \Rightarrow (1)$, claim 2, follows that if $\phi \in L_0$ then if $\text{Mtrx}(\phi) = [\phi_{ij}]$ then $2w(\phi_{ij}) \geq w(s_i) - w(s_j) = w_{ij}$.
If $w_{ij} \leq 0$ then $J^{w_{ij}} = R$ and, hence, $\phi_{ij} \in J^{w_{ij}}$. If $w_{ij} > 0$ then $w_{ij} = \varepsilon$ so that $2w(\phi_{ij}) \geq \varepsilon$. From this $w(\phi_{ij}) \geq \varepsilon$ and, consequently, $\phi_{ij} \in J = R^{w_{ij}}$.

Claim 2. Turn V into a right vector space over the division ring D using the basis (f_i) as follows:

$$v = \sum_{\iota \in I} x_\iota f_\iota, \qquad \lambda \in D \to v\lambda = \sum_{\iota \in I} x_\iota \lambda f_\iota.$$

Given $0 \neq \lambda \in D$, let $\tilde{\lambda}: V \to V$ be the linear transformation sending v to $v\lambda$. Then $w(\lambda) \geq g \Rightarrow \tilde{\lambda}w \geq g$.
This was established in §2, Theorem 2.7, point 2. inequality $(27)'$.

Claim 3. For $0 \neq \lambda \in D$, $i, j \in I$ fixed let $\phi_\lambda^{(i,j)}: V \to V$ be the linear transformation sending $v = \sum x_\iota f_\iota$ onto $v\phi_\lambda^{(i,j)} = x_i \lambda f_j$. Then $\text{Mtrx}(\phi_\lambda^{(i,j)}) = E^{(i,j)}\lambda$, and $\phi_\lambda^{(i,j)} \in L_0$ (resp. $\phi_\lambda^{(i,j)} \in L_0^+$) as soon as $\lambda \in J^{w_{ij}}$ (resp. $\lambda \in J^{w_{ij}^+}$).
That the matrix of $\phi_\lambda^{(i,j)}$ is the standard (i, j) matrix $E^{(i,j)}$ times λ this is formal. Choose $\lambda \in J^{w_{ij}}$ (resp. $\lambda \in J^{w_{ij}^+}$). The assertion is equivalent to $\phi_\lambda^{(i,j)}$ increases (strictly increases) lengths. Let then $0 \neq v \in V$. If $v = \sum_{\iota \in I} x_\iota f_\iota$ then

$$w(v \cdot v) = w\left(\sum_{k \in I} x_k s_k x_k^* \right) = \underset{k \in I}{\text{Min}}\, w\left(x_k s_k x_k^* \right) \leq w\left(x_i s_i x_i^* \right);$$

$$w\big(v^{(i,j)} \cdot v^{(i,j)}\big) = w\big(x_i\lambda f_j \cdot x_i\lambda f_j\big)$$

$$= w\big(x_i\lambda\big(f_j \cdot f_j\big)(x_i\lambda)^*\big) = 2w(x_i) + 2w(\lambda) + w(s_j)$$

$$= w\big(x_i s_i x_i^*\big) + 2w(\lambda) + w(s_j) - w(s_i)$$

$$= w\big(x_i s_i x_i^*\big) + 2w(\lambda) + w_{ji} \ge w(v \cdot v) + \big(2w(\lambda) - w_{ij}\big).$$

From the preceding inequality all that is left to show is that $2w(\lambda) - w_{ij} \ge 0$ (resp. $2w(\lambda) - w_{ij} > 0$). Now, in the case $\lambda \in J^{w_{ij}}$, $w_{ij} \le 0$, then

$$2w(\lambda) - w_{ij} \ge 2w(\lambda) \ge 0;$$

in the case $\lambda \in J^{w_{ij}}$, $w_{ij} > 0$, then $w(\lambda) \ge \varepsilon$ so that

$$2w(\lambda) - w_{ij} \ge 2\varepsilon - \varepsilon = \varepsilon > 0.$$

This shows that $\phi^{(i,j)}$ increases lengths. In the case $\lambda \in J^{w_{ij}^+}$ when $w_{ij} = 0$, $2w(\lambda) - w_{ij} = 2w(\lambda)$ follows and then $w_{ij}^+ = \varepsilon$ so that $w(\lambda) \ge \varepsilon$, whence $2w(\lambda) - w_{ij} > 0$; when $w_{ij} = -\varepsilon$ then $2w(\lambda) - w_{ij} = 2w(\lambda) + \varepsilon \ge \varepsilon > 0$; when $w_{ij} = \varepsilon$, then $2w(\lambda) - w_{ij} \ge \varepsilon > 0$. This shows that $\phi^{(i,j)}$ strictly increases lengths, as desired.

Claim 4. $R_W \subset \mathrm{Mtrx}(L_0)$ so that $R_W = \mathrm{Mtrx}(L_0)$.

Let $\phi \in R_W$. There is $\phi \in L$ with $\Phi = \mathrm{Mtrx}(\phi)$. It is to be shown that $\phi \in L_0$ or, it is the same, ϕ increases lengths. Put $v = \Sigma_i x_i f_i$. The equation $v\phi = v'$, $v' \in V$ translated into matrix equation is now $\underline{v} = \underline{v}'$. Since $\underline{v}'$ is nullary vector over D there is a finite set I_1 of I such that if $\Phi' = \Sigma_{i,j \in I} E^{(i,j)}\phi_{ij}$ then $\underline{v}\Phi' = \underline{v}'$. By construction, $\phi_{ij} \in J^{w_{ij}}$, $i, j \in I_1$. If $\phi^{(i,j)} = \mathrm{Mtrx}^{-1}(E_{\phi_{ij}}^{(i,j)})$ then $\phi^{(i,j)} = \phi_{\lambda_{ij}}^{(i,j)}$, where $\lambda_{ij} = \phi_{ij} \in J^{w_{ij}}$. By the preceding claim, $\phi^{(i,j)} \in L_0$. If $\phi' = \mathrm{Mtrx}^{-1}(\Phi')$ then $\phi' = \Sigma_{i,j \in I_1} \phi_{\lambda_{ij}}^{(i,j)} \in L_0$. Since $v = v\phi'$ it follows that $w(v\phi \cdot v\phi) = w(v\phi' \cdot v\phi') \ge w(v \cdot v)$, for every $v \in V$, as desired.

Claim 5. $J_W = \mathrm{Mtrx}(L_0^+)$.

For the inclusion $J_W \subset \mathrm{Mtrx}(L_0^+)$ use a similar argument as in the preceding claim based on claim 3. For the inclusion $\mathrm{Mtrx}(L_0^+) \subset J_W$ proceed as follows. From $\phi \in L_0^+$ follows

$$2w(\phi_{ij}) > w(s_i) - w(s_j) = w_{ij}.$$

When $w_{ij} = 0$ then $w(\phi_{ij}) > 0$ follows giving $w(\phi_{ij}) \ge \varepsilon$ so that $\phi_{ij} \in J = J^{w_{ij}^+}$. When $w_{ij} < 0$ then $\phi_{ij} \in R = J^{w_{ij}} = J^{w_{ij}^+}$. When $w_{ij} < 0$ then $2w(\phi_{ij}) < w_{ij}$ implies $2w(\phi_{ij}) \ge 2\varepsilon$ so that $w(\phi_{ij}) \ge \varepsilon$ whence $\phi_{ij} \in J = J^{w_{ij}} = J^{w_{ij}^+}$. $\qquad\square$

DEFINITION 3.14. (*Residue matrix module* W.) Let W be any w-matrix over G. Given $\Phi \in R_{I \times I}$ denote by $\overline{\Phi}_W$ the $I \times I$ matrix over the residue division ring $\overline{D}$ such that

(32) $\quad \begin{cases} \mathrm{entr}_{i,j}(\overline{\Phi}_W) = \overline{0}, & \text{if } w_{ij} \neq 0. \\ \mathrm{entr}_{i,j}(\overline{\Phi}_W) = \mathrm{entr}_{i,j}(\Phi) + J, & \text{if } w_{ij} = 0. \end{cases}$

I refer to $\overline{\Phi}_W$ as to *residue matrix modulo* W.

For instance, if W is as in Illustration (l), then if $\Phi = [\phi_{ij}]_{1 \le i, j \le 4}$, then

$$\overline{\Phi}_W = \begin{vmatrix} \overline{\phi}_{11} & \overline{\phi}_{12} & & 0 \\ \overline{\phi}_{21} & \overline{\phi}_{22} & & \\ \hline & 0 & & \overline{\phi}_{33} & \overline{\phi}_{er} \\ & & & \overline{\phi}_{43} & \overline{\phi}_{44} \end{vmatrix},$$

where $\overline{\phi}_{ij} = \phi_{ij} + J$. Hence,

(33) $\quad \left(\overline{R_{I \times I}}\right)_W = \begin{vmatrix} \overline{D} & \overline{D} & & 0 \\ \overline{D} & \overline{D} & & \\ \hline & 0 & & \overline{D} & \overline{D} \\ & & & \overline{D} & \overline{D} \end{vmatrix} = \overline{D}_{2 \times 2} \oplus \overline{D}_{2 \times 2}$

where $\oplus$ is the block diagonal sum and $\mathrm{Diag}_2(\overline{D})$ is the ring of 2×2 diagonal matrices over $\overline{D}$ ($= R/J$).

THEOREM 3.15. *Let V be as in Theorem 3.13. Denote by $\overline{D}_W$, the block diagonal sum of the ring $\overline{D}_{I_0 \times I_0}$ of row-finite $I_0 \times I_0$ matrices over the residue division ring $\overline{D}$, and the ring $\overline{D}_{I_\varepsilon \times I_\varepsilon}$ of row-finite $I_\varepsilon \times I_\varepsilon$ diagonal matrices over $\overline{D}$ (if $I_\varepsilon = \varnothing$ then discard the right summand $\mathrm{Diag}_{I_\varepsilon}(\overline{D})$). Then:*
(1) $R_W/J_W \cong (R/J)_W = \overline{D}_W$.
(2) $L_0/L_0^+ \cong \overline{D}_W$.

Proof. Given $\phi \in L_0$ define $\overline{\phi}_W$ to be the residue matrix modulo W of the matrix of ϕ. As ϕ ranges over R_W the matrix of ϕ ranges over R_W (Theorem 3.13). Hence $\overline{\phi}_W$ ranges over $\overline{D}_W$. Now, $\overline{\phi}_W = 0 \Leftrightarrow \mathrm{Mtrx}(\phi) \in J_W \Leftrightarrow (\phi) \in L_0^+$ (Theorem 3.13). Since $\phi \to \overline{\phi}_W$ is a ring homomorphism

$$L_0/L_0^+ \cong \overline{D}_W$$

follows, which also shows that $R_W/J_W \cong \overline{D}_W$, as this can be checked directly. $\quad \square$

From the preceding theorem one can deduce several new facts about the pair $L_0 \supset L_0^+$, some of which are of global meaning. Here are some.

COROLLARY 3.16. (i) L_0^+ *is an intersection of maximal ideals in L_0 and, consequently, $L_0^+ \supset \text{Jac}(L_0)$. (ii) Hence, $L_0^+ \cap \mathscr{F}\mathscr{L} = \text{Jac}(L_0) \cap \mathscr{F} L$.*

Proof. (i) by Theorem 3.15,

$$L_0/L_0^+ = R_W/J_W = (R/J)_W = \overline{D}_W = \overline{D}_{I_0 \times I_0} \oplus \overline{D}_{I_\varepsilon \times I_\varepsilon}$$

$$= \overline{D}_{I_0 \times I_0} \times \overline{D}_{I_\varepsilon \times I_\varepsilon}.$$

Now $\overline{D}_{I_0 \times I_0}$ is primitive and so is $\overline{D}_{I_\varepsilon \times I_\varepsilon}$. Thus $\overline{D}_W$ is semi-primitive. Hence L_0/L_0^+ is semi-primitive. Equivalently, L_0^+ is an intersection of maximal ideals. (ii) follows from the preceding and §2, Theorem 2.12. □

COROLLARY 3.17. L_0^+ *is a prime ideal if, and only if L_0^+ is primitive. Equivalently, some (and, hence, all) orthogonal basis of V which is bounded by ε is uniform.*

Proof. If L_0^+ is a prime ideal then L_0/L_0^+ is a prime ring. Hence, $\overline{D}_W \cong \overline{D}_{I_0 \times I_0} \times \overline{D}_{I_\varepsilon \times I_\varepsilon}$ is prime; if, and only if, $I_\varepsilon = \varnothing$. Equivalently, (f_i) is uniform. □

COROLLARY 3.18. L_0^+ *is a maximal ideal if and only if V is finite-dimensional and every orthogonal basis of V which is bounded by ε is uniform.*

Proof. If L_0^+ is maximal then $\overline{D}_W \cong \overline{D}_{I_0 \times I_0} = \overline{D}_{I \times I}$ is a simple ring. Hence, I is finite so that V is finite-dimensional. □

COROLLARY 3.19. (i) *If V is finite-dimensional then $L_0^+ = \text{Jac}(L_0)$. (ii) Conversely, if $L_0^+ = \text{Jac}(L_0)$ and if V possesses a uniform basis necessarily V is finite-dimensional.*

Proof. (i) follows from Corollary 3.16. (ii) From the equality $L_0^+ = \text{Jac}(L_0)$ and the isomorphisms $L_0 \cong R_{I_0 \times I_0}$, $L_0^+ \cong J_{I_0 \times I_0} = \text{Jac}(R)_{I_0 \times I_0}$ follows the equality $\text{Jac}(R_{I_0 \times I_0}) = \text{Jac}(R)_{I_0 \times I_0}$. Since evidently R contains no nilpotent elements it follows by Patterson's [14, Theorem 1] that I_0 is a finite set. Equivalently, V is finite-dimensional. □

Question (Open). If $L_0^+ = \text{Jac}(L_0)$ must V be finite-dimensional?

COROLLARY 3.20. *Let V be any non-degenerate 0-elliptic space with some orthogonal basis (f_i) which is bounded by ε, where as usual $\varepsilon = 0$ or else ε is the first positive element in the value group G. Then the prime ring L_0 contains in its lattice of 2-sided ideals an isomorphic copy of the positive cone of G.*

Proof. When the basis (f_i) is uniform the result is evident. Assume to the contrary that (f_i) is mixed so that ε is the first positive element in G. Extend the convention about J^ε to J_g^ε, specifically, put:

(34) $$J_g^0 = J_g^{-\varepsilon} = J_g, \quad J_g^\varepsilon = J_{g+\varepsilon} \qquad (g \geq 0).$$

Define $(J_g)_W$ (and $(J_g)_W^+$) similarly to R_W (and J_W respectively). Thus $\Phi = [\phi_{ij}] \in (J_g)_W$ if, and only if,

(35) $$\begin{cases} \Phi \in R_{I \times I}, \\ \text{entr}_{\iota, j}(\Phi) \cap J_g^{w_{ij}} \qquad (\iota, j \in I). \end{cases}$$

Exactly as for the case $g = 0$ one can show that $\text{Mtrx}(L_g) = (J_g)_W(\text{Mtrx}(L_g^+) = (J_g)_W^+)$. Since $(J_g)_W = (J_{g'})_W \Leftrightarrow g = g'$ this makes it clear that $g \in G^+ \to L_g$ is a dual isomorphism of ordered sets from G^+ into $\text{Lat.}(L_0)$. $\qquad \square$

Since this section began with an assumption about the unitary group $U = U(V)$ it seems appropriate to close it with information that can be derived about this group. In what will follow I will deal with a certain matrix representation of U over the residue division ring $\overline{D} = R/J$, where I will assume that, again, V has some orthogonal basis (f_i), which is bounded by ε with either $\varepsilon = 0$ or $\varepsilon = 1$st positive element of G. The unitary version of Corollary 3.22 will be left to upcoming §4.

THEOREM 3.21. *Let V be any non-degenerate 0-elliptic space possessing some orthogonal basis (f_i), which is bounded by ε, where either $\varepsilon = 0$ or ε is the first positive element in the value group G. If W is the w-matrix associated to (f_i) and if $\mathcal{M}^\times \colon U \to \overline{D}_W$ is the mapping sending $\phi \in U$ onto $\text{Mtrx}(\phi)_W$, the residue matrix modulo W of the matrix of ϕ relative to the basis, then $\mathcal{M}^\times$ is a representation of U over $\overline{D}$ such that:*

(1) $\text{Ker}(\mathcal{M}^\times) = U_0^+$;

(2) *$\text{Im}(\mathcal{M}^\times)$ is a subgroup of unitary matrices of the unitary group of $\overline{D}_W$, relative to the partial matrix involution $(\overline{S})$ of $\overline{D}_W$ induced by (S).*

Proof (1) Recall that $\sigma \to \sigma_W = \text{Mtrx}(\sigma)_W$ is a ring homomorphism from L_0 onto $\overline{D}_W$ with kernel precisely L_0^+ (Theorem 3.15). Since V is

0-elliptic, it follows that $U \subset L_0$ so that $\mathcal{M}^\times$ is the restriction of $\sigma \to \sigma_W$ to U and, hence, $\mathcal{M}^\times$ is a group homomorphism from U into the group of invertible matrices in $\overline{D}_W$. Now

$$\mathrm{Ker}(\mathcal{M}^\times) = \left(1 + \mathrm{Ker}(\sigma \to \sigma_W)\right) \cap U = \left(1 + L_0^+\right) \cap U = U_0^+.$$

(2) The ring L_0 carries a partial involution $*$ which can be identified to the partial matrix involution (S) of the ring R_W. Since J_W is (S)-closed it follows that $\overline{D}_W \cong R_W/J_W$ carries a partial involution (S). By construction $\mathcal{M}^\times$ takes U onto $\mathscr{U}(R_W; S))$ and $\mathscr{U}(R_W; (S))$ is mapped onto a subgroup of $\mathscr{U}(\overline{D}_W; (\overline{S}))$ under the residue homomorphism $\Phi \to \Phi_W$. $\quad\square$

The following theorem is fairly formal.

THEOREM 3.22. *The unitary group of the ring $\overline{D}_W$ relative to the induced involution $(\overline{S})$ is of the form*

$$\mathscr{U}\left(\overline{D}_W; (\overline{S})\right) = \mathscr{U}\left(\overline{D}_{I_0 \times I_0}; \left(\overline{S}_0\right)\right) \times \mathscr{U}\left(\overline{D}_{I_\varepsilon \times I_\varepsilon}; \left(\overline{S}_\varepsilon\right)\right),$$

where $(\overline{S}_0)$ takes $\overline{\Phi}_{(0)} = [\phi_{ij} + J]_{i,j \in I_0}$ onto $\overline{\Phi}_{(0)}^{(S_0)} = [s_i \phi_{j,i}^* s_j^{-1} + J]_{i,j \in I_0}$ (when $\overline{\Phi}_0$ is both row and column finite) and, similarly, $(\overline{S}_\varepsilon)$ takes $\Phi_{(1)} = [\phi_{i,j}^* + J]_{i,j \in I_\varepsilon}$ onto $\overline{\Phi}_{(1)}^{(S_\varepsilon)} = [s_i \phi_{j,i}^* s_j^{-1} + J]_{i,j \in I_\varepsilon}.$

THEOREM 3.23. *Let V be any non-degenerate 0-elliptic space and suppose that the R-module V_0 has an orthogonal basis (f_i) all of whose members are medial vectors. If, further, $1/2 \in R$, then the normal subgroup $\mathscr{CF}(U)$ of unitary transformations ϕ such that $\phi - 1$ is of finite rank maps onto the corresponding subgroup $\mathscr{CF}(\overline{U})$ of $\overline{D}$.*

Proof. It is to be shown first that the group homomorphism $\phi \in U \to \overline{\phi} \in \overline{U}$ extends to an onto homomorphism $\sigma \in L_0 \twoheadrightarrow \overline{\sigma} \in \overline{L} = L(\overline{V})$. For if $\sigma \in L_0$ it is clear that $V_0 \sigma \subset V_0$ and $V_0^+ \sigma \subset V_0^+$. Hence if

$$(43) \qquad\qquad \overline{\sigma} = v + V_0^+ \to v\sigma + V_0^+$$

then $\sigma \to \overline{\sigma}$ is a ring homomorphism from L_0 into $\overline{L}$ with correspondence of the adjoint involution in L_0 and $\overline{L}$ respectively. To show that $\sigma \to \overline{\sigma}$ is onto proceed as follows. If $\overline{\sigma} \in \overline{L}$ then using the basis $(\overline{f_i}) = (f_i + V_0^+)$ of $\overline{V}$ lift $\overline{\sigma}$ to a linear transformation σ_0 of the R-module V_0 sending f_i onto $\overline{f_i}\overline{\sigma}$. By construction σ_0 maps V_0^+ into V_0^+. Extend σ_0 to a linear transformation σ of V by the rule

$$(44) \qquad\qquad (\lambda v)\sigma = \lambda(v\sigma_0) \qquad (\lambda \in D, v \in V_0).$$

Given any $u \in V$, there is $\lambda_0 \in D$ such that $\lambda_0 u$ is medial (§1, Theorem 1.7). Then $(\lambda_0 u)\sigma \in V_0$ so that

$$w((\lambda_0 u)\sigma \cdot (\lambda_0 u)\sigma) \geq 0$$

or,

$$w(u\sigma \cdot u\sigma) \geq -2w(\lambda_0) = w(u \cdot u).$$

Thus $\sigma \in L_0$. Hence $\sigma \to \bar{\sigma}$ maps L_0 onto $\bar{L}$, as desired.

Next I will show that each $\phi \in \mathscr{CF}(U)$ maps onto $\bar{\phi} \in \mathscr{CF}(\bar{U})$. Indeed, ϕ maps onto $\bar{\phi} \in \bar{U}$. To say that $\phi \in \mathscr{CF}(U)$ is to say $\phi - 1$ is of finite rank. Equivalently, $\text{Mtrx}(\phi - 1)$ has all its entries zeros but for a finite subset of $I \times I$ (*finite* matrix). Relative to the induced basis ($\bar{f}_i$) it is clear that $\text{Mtrx}(\bar{\phi} - 1)$ is, again, co-finite. Thus $\bar{\phi} \in \mathscr{CF}(\bar{U})$.

It remains to show that each $\bar{\phi} \in \mathscr{CF}(\bar{U})$ lifts to a unitary transformation $\phi \in \mathscr{CF}(U)$.

Let $\bar{V}^{(1)}$ be the image of $\bar{\phi} - 1$. By construction, $\bar{V}^{(1)}$ is a finite-dimensional subspace of $\bar{V}$. By §1 Theorem 1.11, $\bar{V}^{(1)}$ lifts to a finite-dimensional orthogonal summand $V_0^{(1)}$ of V_0. Denote by $\bar{V}^{(2)}$ the orthogonal complement of $\bar{V}^{(1)}$ and by $V_0^{(2)}$ the orthogonal complement of $V_0^{(1)}$ in V_0. By the cited theorem $V_0^{(2)}$ maps onto $\bar{V}^{(2)}$. I distinguish two cases.

Case $\bar{\phi}$ is a symmetry. Here $\bar{\phi}^2 = \bar{\phi}$ implies

$$\bar{\phi} = -1_{\bar{V}^{(1)}} \oplus 1_{\bar{V}^{(2)}}.$$

If $\phi_0 = -1_{V_0^{(1)}} \oplus 1_{V_0^{(2)}}$ then ϕ_0 extends in a unique way to the symmetry ϕ of V. Hence $\bar{\phi}$ has been lifted to the symmetry ϕ in L_0. By construction, ϕ is co-finite.

Case $1 + \bar{\phi}$ is non-singular. Here, $\bar{\phi}$ can be Cayley parametrized. If $\bar{\tau} = (1 - \bar{\phi})/(1 + \bar{\phi})$, then

$$\bar{V}\bar{\tau} = \bar{V}\left(\frac{1}{1 + \bar{\phi}}\right)(1 - \bar{\phi}) = \bar{V}(1 - \bar{\phi}) = \bar{V}^{(1)}.$$

Then $\bar{\tau} = \bar{\tau}/\bar{V}^{(1)} \oplus 0_{\bar{V}^{(2)}}$. Repeating the argument to the effect $L_0(V)$ maps onto $L(\bar{V})$ for V replaced by $DV_0^{(1)}$ one readily gets that $\bar{\tau}/V^{(1)}$ lifts to a skew-symmetric transformation τ_1 of $DV_0^{(1)}$. If $\tau = \tau_1 \oplus 0$ then $\tau = -\tau^* \in L_0$ maps onto $\bar{\tau}$. By construction, $V\tau = DV_0^{(1)}$ is finite-dimensional. If $\phi = (1 - \tau)/(1 + \tau)$ then $\phi \in U$ maps onto $\bar{\phi}$ and

$$V(1 - \phi) = V\left(\frac{1}{1 + \tau}\right)(-2\tau) = V(2\tau) = V\tau = DV_0^{(1)}.$$

Thus ϕ is co-finite.

For general $\bar{\phi} \in \bar{U}$, by standard argument $\bar{\phi} = \bar{\phi}_1 \bar{\phi}_2$, where $\bar{\phi}_1$ is a symmetry and $\bar{\phi}_2$ is such that $1 + \bar{\phi}_2$ is non-singular. One can, further, choose both $\bar{\phi}_1, \bar{\phi}_2$ to be co-finite. It suffices then to lift separately $\bar{\phi}_1, \bar{\phi}_2$ to ϕ_1, ϕ_2 co-finite. Then $\bar{\phi}$ is lifted to the co-finite unitary ϕ_1, ϕ_2. $\square$

COROLLARY 3.24. *If V is as in Theorem 3.23 then* $M^\times \colon U \to \bar{D}_W$ *is a matrix representation of U over the residue division ring* $\bar{D} = R/J$ *with the following features.*

(1) Ker $(M^\times) = U_0^+$

(2) *If* $M^{\mathscr{CF}}$ *is the restriction of* $M^\times$ *to the normal subgroup* $\mathscr{CF}(U)$ *of co-finite unitary transformations in U then*

$$\mathrm{Ker}(M^{\mathscr{CF}}) = \mathscr{CF}(U) \cap (1 + \mathrm{Jac}(L_0)).$$

(3) $\mathscr{CF}(U)$ *maps onto* $\mathscr{CF}(\bar{D}_{I_0 \times I_0}; (\bar{S}_0))$, *the normal subgroup of unitary matrices* $\bar{\Phi}$ *in* $\bar{D}_{I_0 \times I_0}$ *such that* $\bar{\phi} - 1$ *is a finite matrix.*

Question. (Open.) What is the status of Corollary 3.24 in the case of mixed basis? Also, if U maps onto the unitary group of $\bar{D}_W$ must V be finite-dimensional?

4. Normal subgroups of the projective group. Let V be any finite-dimensional non-degenerate elliptic space. Denote by $U^{(0)}$ the unitary group, U, by $U^{(1)} = [U, U]$, the derived unitary group and by $U^{(r)}$, the rth derived unitary group

$$(1) \qquad\qquad U^{(r)} = [U^{(r-1)}, U^{(r-1)}] \qquad (r \geq 1).$$

Let $PU^{(r)}$ stand for the factor group $U^{(r)}/U^{(r)} \cap Z$. I refer to $PU^{(1)}$ as to the projective derived unitary group, in short, the *projective group* of V. Let lat.$(PU^{(r)})$ stand for the lattice of normal subgroups of the group $PU^{(r)}$.

Given $g \in G$, associate to g the normal subgroup U_g of U. To U_g and $r \geq 0$ correspond the normal subgroup $U_g^{(r)} = [U_g^{(r-1)}, U_g^{(r-1)}]$ if $r \geq 1$ and $U_g^{(0)} = U_g$. To $U_g^{(r)}$ corresponds the normal subgroup $PU_g^{(r)} = U_g^{(r)}/\bar{U}^{(r)} \cap Z$ of $PU^{(r)}$. This section is concerned with the nature of the mapping $g \to PU_g^{(r)}$, where g ranges over a suitable open segment of G. Of particular importance is the special case $r = 1$. The work is organized as follows.

4.1: Equalizers of the mapping $g \to U_g$ and of related mappings.
4.2: The mapping $g \to PU_g^{(1)}$ for $\dim_D V > 1$.
4.3: The mapping $g \to PU_g^{(1)}$ for $\dim_D V = 1$.
4.4: The general mapping $g \to PU_g^{(r)}$.

4.1. *Equalizers of the mapping $g \to U_g$ and of related mappings.*

Throughout the rest of this section the mapping $g \to U_g$ and the derived mappings $g \to PU_g^{(r)}$, will all be restricted to the open segment $(w(2), \to)$ of G. To simplify the writing let me make the

DEFINITION 4.1.1. (1) The mapping $g \in (w(2), \to) \to U_g$ from $(w(2), \to)$ into lat.(U) is denoted by π_0

(2) The mapping $g \in (w(2), \to) \to PU_g^{(r)} \in$ lat.$(PU^{(r)})$ is denoted by $P\pi_r$.

(3) The mapping $g \in (w(2), \to) \to U_g^{(r)} \in$ lat.$(U^{(r)})$ is denoted by $P\pi_r$.

The equalizer of, say, π_0 is the subset of pairs (g_1, g_2) such that $\pi_0(g_1) = \pi_0(g_2)$. Facts about the equalizers of π_0 and π_1 will be collected in this subsection with a view to apply the results to subsections 4.2 and 4.3. One more mapping is in order.

DEFINITION 4.1.2. Let Σ_0: $(w(2), \to) \to$ lat.$((V))$ be the mapping sending $g \in (w(2), \to)$ onto the additive subgroup

$$\mathcal{K}_g \left(= \left\{ \phi = -\phi^* \in L \,\middle|\, \phi w \geq g \right\} \right).$$

As observed and established in §2, Theorem 2.16:

Facts 4.1.3. (a) Let $\mathcal{C}$ be the partial operator of $L = L(V)$ which is defined at $\phi \in L$ if and only if $1 + \phi$ is invertible and then

$$\mathcal{C}(\phi) = 2(1 - \phi)/(1 + \phi).$$

Let $\mathcal{C}^{-1}$ be the partial operator of L which is defined at $\phi \in L$ if and only if $1 + \sigma/2$ is invertible and then

$$\mathcal{C}^{-1}(\sigma) = (1 - \sigma/2)/(1 + \sigma/2).$$

Then $\mathcal{C}$, $\mathcal{C}^{-1}$ are one-to-one and $\mathcal{C} \circ \mathcal{C}^{-1} = 1_L = \mathcal{C}^{-1} \circ \mathcal{C}$

(b) $\mathcal{C}$ is entirely defined at $\pi_0(g)$ and $\mathcal{C}(\pi_0(g)) = \Sigma_0(g)$ for every $g > w(2)$

(c) $\mathcal{C}^{-1}$ is entirely defined at $\Sigma_0(g)$ and $\mathcal{C}^{-1}(\Sigma_0(g)) = \pi_0(g)$ for every $g > w(2)$.

Fact 4.1.4. π_0 as well as the derived mappings $\pi_1, p\pi_0, p\pi_1, \ldots$ are dual order homomorphisms of ordered sets.

Facts 4.1.3, points (b) and (c) tell us that π_0 factors through $\mathcal{C}^{-1}$, $\pi_0 = \mathcal{C}^{-1} \circ \Sigma_0$. Hence π_0 and Σ_0 have the same equalizers.

Question. (Open.) If $\Sigma_1: (w(2), \rightarrow) \rightarrow \text{lat}(\mathcal{K})$ is the mapping sending g onto $(\Sigma_0(g), \Sigma_0(g))$, the Lie commutator subgroup of $\Sigma_0(g)$ and if $\pi_1 = g \in (w(2), \rightarrow) \rightarrow [\pi_0(g), \pi_0(g)]$, the commutator subgroup of the unitary subgroup $\pi_0(g)$, does it follow that $\pi_1 = \mathscr{C}^{-1} \circ \Sigma_1$? that π_1 and Σ_1 have same equalizers?

I do not know general answers to the preceding questions. Some partial results tending to affirmative answers will be established.

Recall that if M is the set of medial transformations ϕ of V such that ϕ^* exists and ϕ^* is medial then M is a group (*medial group* of V) all of whose members ϕ are such that

$$(3) \qquad \phi L_g = L_g \phi f = L_g \qquad (g \in G).$$

Equivalently,

$$(4) \qquad \psi \in L \Leftrightarrow g \in G; \quad \psi w \geq g \Leftrightarrow \phi \psi w \geq g \Leftrightarrow \psi \phi w \geq g.$$

DEFINITION 4.1.5. I will say that $\phi_1, \phi_2 \in L = L(V)$ are *equivalent* (notation: $\phi_1 \equiv \phi_2$) if

$$\phi_2 = \mu_1 \phi_1 \mu_2,$$

for some $\mu_i \in M$ $(i = 1, 2)$.

As a corollary to the preceding definition:

Fact 4.1.6. If $\phi_1 \equiv \phi_2$ then for every $g \in G$, $\phi_1 w \geq g \Leftrightarrow \phi_2 w \geq g$.

Fact 4.1.7. If $\sigma, \sigma' \in \pi_0(g)$, then for $[\sigma, \sigma'] = \sigma^{-1}\sigma'^{-1}\sigma\sigma'$ and for $(\phi, \phi') = \phi\phi' - \phi'\phi$ where $\phi = \mathscr{C}(\sigma)$, $\phi' = \mathscr{C}(\sigma')$:
  (1) $\mathscr{C}([\sigma, \sigma']) \equiv [\sigma, \sigma'] - 1$;
  (2) $[\sigma, \sigma'] - 1 \equiv (\phi, \phi')$.

Proof. (1) I will show more generally that if $\sigma \in \pi_0(g)$, then $\sigma - 1 \equiv \mathscr{C}(\sigma)$ or, $\sigma - 1 \equiv \phi$, where $\phi = \mathscr{C}(\sigma)$. By construction,

$$\sigma - 1 = \mathscr{C}^{-1}(\phi) - 1 = \frac{1 - \phi/2}{1 + \phi/2} - 1 = \frac{-\phi}{1 + \phi/2}.$$

By construction, $g > w(2)$. From $\sigma \in \pi_0(g)$ follows $\phi = \mathscr{C}(\sigma) \in \Sigma_0(g)$ and, hence, $\phi \in L_g$ so that $\phi/2 \in L_{g-g(2)} \subset L_0^+$. From §2, Theorem 2.10 follows that $1 + \phi/2$ is medial. Since $\phi/2$ has evidently adjoint which must be in L_0^+ it follows that $(1 + \phi/2)^*$ is medial. Thus $1 + \phi/2 \in M$. Then

$$\sigma - 1 = \left(\frac{-1}{1 + \phi/2} \right) \phi$$

with $-1/(1 + \phi/2) \in M$ for $-1 \in M$ and $1 + \phi/2 \in M$. Hence, $\sigma - 1 \equiv \phi$.

(2) $\qquad [\sigma, \sigma'] - 1 = \sigma^{-1}\sigma'^{-1}(\sigma, \sigma') \equiv (\sigma, \sigma') \qquad (U \subset M)$

$$= \left(\mathscr{C}^{-1}(\phi), \mathscr{C}^{-1}(\phi) \right) = \left(\frac{1 - \phi/2}{1 + \phi/2}, \frac{1 - \phi'/2}{1 + \phi'/2} \right)$$

$$= \left(-1 + \frac{2}{1 + \phi/2}, -1 + \frac{2}{1 + \phi'/2} \right)$$

$$= 4 \left(\frac{1}{1 + \phi/2}, \frac{1}{1 + \phi'/2} \right)$$

$$= 4 \frac{1}{1 + \phi/2} \left(\frac{1}{1 + \phi'/2}, \frac{\phi}{2} \right) \frac{1}{1 + \phi/2}$$

$$= 4 \frac{1}{1 + \phi/2} \left(\frac{1}{1 + \phi'/2}, \phi \right) \frac{1}{1 + \phi/2}$$

$$= 2 \frac{1}{1 + \phi/2} \left(\frac{1}{1 + \phi'/2}, \phi \right) \frac{1}{1 + \phi/2}$$

$$= 2 \frac{1}{1 + \phi/2} \frac{1}{1 + \phi'/2} \left(\phi, \frac{1 + \phi'}{2} \right) \frac{1}{1 + \phi/2} \frac{1}{1 + \phi'/2}$$

$$= \frac{1}{1 + \phi/2} \frac{1}{1 + \phi'/2} (\phi, \phi') \frac{1}{1 + \phi'/2} \frac{1}{1 + \phi/2}.$$

As before, from $\phi, \phi' \in \mathscr{K}_g$, $g > w(2)$, follow $1 + \phi/2, 1 + \phi'/2 \in M$, so that

$$(\sigma, \sigma') \equiv (\phi, \phi').$$

Thus

$$[\sigma, \sigma'] - 1 \equiv (\sigma, \sigma') \equiv (\phi, \phi'),$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Fact 4.1.8. Let (g_1, g_2) be in the equalizer of π_1 (e.g. $[U_{g_1}, U_{g_1}] = [U_{g_2}, U_{g_2}]$). If $\Sigma_1 = g \in (w(2), \rightarrow) \rightarrow (\Sigma_0(g), \Sigma_0(g))(= (\mathscr{K}_g, \mathscr{K}_g))$ then

$$\Sigma_1(g_1) \subset \Sigma_0(2g_2); \qquad \Sigma_1(g_2) \subset \Sigma_0(2g_1).$$

Proof. Let $\phi, \phi' \in \Sigma_0(g_1)$. If $\sigma, \sigma' = \mathscr{C}^{-1}(\phi), \mathscr{C}^{-1}(\phi')$ then $\sigma, \sigma' \in \pi_0(g_1)$. Thus $[\sigma, \sigma'] \in \pi_0(g_1) \in \pi_0(g_1) = \pi_1(g_2) = [U_{g_2}, U_{g_2}] \subset U_{2g_2}$. (§2, Theorem 2.14, (iii)). Equivalently, $[\sigma, \sigma'] - 1w \geq 2g_2$. By Fact 4.1.7,

$(\phi, \phi') \equiv ([\sigma, \sigma'] - 1)w \geq 2g_2$ giving $(\phi, \phi')w \geq 2g_2$, so that $(\phi, \phi') \in \Sigma_0(2g_2)$. As this holds for every pair $\phi, \phi' \in \Sigma_0(g_1)$, it follows that $\Sigma_1(g_1) \subset \Sigma_0(2g_2)$. By symmetry, $\Sigma_1(g_2) \subset \Sigma_0(2g_1)$. $\qquad \square$

Fact 4.1.9. Let $\sigma, \sigma' \in U_{w(2)}^+$. If

$$\gamma = [[\sigma, \sigma'], \sigma']$$

and if $\phi, \phi' = \mathscr{C}(\sigma), \mathscr{C}(\sigma')$, then

$$\gamma - 1 \equiv \mu_1(\phi, \phi')\mu_2(\phi, \phi')\mu_3 + \mu_4(\phi, \phi')\mu_5(\phi, \phi')\mu_6$$
$$+ \mu_7((\phi, \phi'), \phi')\mu_8,$$

for some $\mu_i \in M$.

Proof. Since $[\sigma, \sigma'] \in [U_{w(2)}^+, U_{w(2)}^+] \subset U_{w(2)}^+$ it follows that

$$[\sigma, \sigma'] = \mathscr{C}^{-1}(\mathscr{C}([\sigma,'\sigma'])) = \mathscr{C}^{-1}(t) = \frac{1 - t/2}{1 + t/2},$$

where $t/2 = (1 - [\sigma, \sigma'])/(1 + [\sigma, \sigma'])$. If $t_1 = 1 - [\sigma, \sigma']$, then

$$(5) \qquad\qquad t/2 = \frac{t_1}{2 - t_1}.$$

From Fact 4.1.7 applied to $[\sigma, \sigma']$ and σ' follows

$$\gamma = [[\sigma, \sigma'], \sigma'] - 1 \equiv (t, \phi'),$$

where

$$\phi, \phi' = \mathscr{C}(\sigma), \quad \mathscr{C}(\sigma') = 2\frac{(1 - \sigma)}{(1 + \sigma)}, \quad 2\frac{(1 - \sigma')}{(1 + \sigma')}.$$

Now,

$$(t, \phi') = \left(2\frac{t_1}{2 - t_1}, \phi'\right) = 2\left(\frac{t_1}{2 - t_1}, \phi'\right)$$

$$= 2\left(-1 + \frac{2}{2 - t_1}, \phi'\right) = 2\left(\frac{2}{2 - t_1}, \phi'\right)$$

$$= 4\frac{1}{2 - t_1}(\phi', 2 - t_1)\frac{1}{2 - t_1} = 4\frac{1}{2 - t_1}(t_1, \phi')\frac{1}{2 - t_1}$$

$$= 4\frac{1}{2(1 - t_1/2)}(t_1, \phi')2\left(\frac{1 - t_1}{2}\right)$$

$$= \frac{1}{1 - t_1/2}(t_1, \phi')\frac{1}{1 - t_1/2}.$$

From $t_1 = 1 - [\sigma, \sigma']$ and $[\sigma, \sigma'] \in U^+_{w(2)}$ follows $t_1/2 \in L^+_0$. Since $t_1/2$ has adjoint it follows that $1 - t_1/2 \in M$, giving

$$\gamma \equiv (t, \phi') \equiv (t_1, \phi').$$

Here,

$$
\begin{aligned}
(t_1, \phi') &= \left(1 - [\sigma, \sigma'], \phi'\right) \\
&= \left(\phi', [\sigma, \sigma'] - 1\right) = \left(\phi', \sigma^{-1}\sigma'^{-1}(\sigma, \sigma')\right) \\
&= \left(\phi', \sigma^{-1}\sigma'^{-1}\right)(\sigma, \sigma') + \sigma^{-1}\sigma'^{-1}\left(\phi', (\sigma, \sigma')\right).
\end{aligned}
$$

Since ϕ' evidently commutes with σ'^{-1} it follows that

$$\left(t_1, \phi'\right) = \left(\phi', \sigma^{-1}\right)\sigma'^{-}(\sigma, \sigma') + \sigma^{-1}\sigma'^{-1}\left(\phi', (\sigma, \sigma')\right) = \tau_1 + \tau_2,$$

where $\tau_1 = (\phi', \sigma^{-1})\sigma'^{-1}(\sigma, \sigma')$ and $\tau_2 = \sigma^{-1}\sigma'^{-1}(\phi', (\sigma, \sigma'))$.

Now,

$$
\begin{aligned}
(\phi', \sigma^{-1}) &= \left(\phi', \frac{1 + \phi/2}{1 - \phi/2}\right) = \left(\phi', -1 + \frac{2}{1 - \phi/2}\right) \\
&= \left(\phi', \frac{2}{1 - \phi/2}\right) = 2\frac{1}{1 - \phi/2}(1 - \phi/2, \phi')\frac{1}{1 - \phi/2} \\
&= \frac{1}{1 - \phi/2}(\phi', \phi)\frac{1}{1 - \phi/2} \equiv (\phi', \phi).
\end{aligned}
$$

From Fact 4.1.7, $(\sigma, \sigma') \equiv (\phi, \phi')$ follows. Thus

$$\tau_1 = \frac{1}{1 - \phi/2}(\phi', \phi)\frac{1}{1 - \phi/2}\sigma'^{-1}\mu'_1(\phi, \phi')\mu'_2$$

where $\mu'_1, \mu'_2 \in M$. Since $1/(1 - \varepsilon/2)$ and $\sigma' \in M$ it follows that for some $\mu_1, \mu_2, \mu_3 \in M$,

$$\tau_1 = \mu_1(\phi, \phi')\mu_2(\phi, \phi')\mu_3.$$

Also,

$$
\begin{aligned}
(\phi', (\sigma, \sigma')) &= \left(\phi', \frac{1}{1 + \phi'/2}\frac{1}{1 + \phi/2}(\phi, \phi')\frac{1}{1 + \phi/2}\frac{1}{1 + \phi'/2}\right) \\
&= \frac{1}{1 + \phi'/2}\left(\phi', \frac{1}{1 + \phi/2}(\phi, \phi')\frac{1}{1 + \phi/2}\right)\frac{1}{1 + \phi'/2};
\end{aligned}
$$

$$
\begin{aligned}
& \left(\phi', \frac{1}{1 + \phi/2}(\phi, \phi')\frac{1}{1 + \phi/2}\right) \\
&= \frac{1}{1 + \phi/2}(1 + \phi/2, \phi')\frac{1}{1 + \phi/2}(\phi, \phi')\frac{1}{1 + \phi/2}
\end{aligned}
$$

$$+ \frac{1}{1+\phi/2}\big(\phi',(\phi,\phi')\big)\frac{1}{1+\phi/2} + \frac{1}{1+\phi/2}(\phi,\phi')$$

$$\cdot \frac{1}{1+\phi/2}\left(\frac{1+\phi}{2,\phi'}\right)\frac{1}{1+\phi/2}$$

$$= \frac{1}{2}\frac{1}{1+\phi/2}(\phi,\phi')\frac{1}{1+\phi/2}(\phi,\phi')\frac{1}{1+\phi/2}$$

$$+ \frac{1}{2}\frac{1}{1+\phi/2}(\phi,\phi')\frac{1}{1+\phi/2}(\phi,\phi')\frac{1}{1+\phi/2}$$

$$+ \frac{1}{1+\phi/2}\big(\phi',(\phi,\phi')\big)\frac{1}{1+\phi/2}$$

$$= \frac{1}{1+\phi/2}(\phi,\phi')\frac{1}{1+\phi/2}(\phi,\phi')\frac{1}{1+\phi/2}$$

$$+ \frac{1}{1+\phi/2}\big(\phi',(\phi,\phi')\big)\frac{1}{1+\phi/2}.$$

Therefore $(\phi',(\sigma,\sigma'))$ can be written in the form

$$\big(\phi',(\sigma,\sigma')\big) = \mu_4(\phi,\phi')\mu_5(\phi,\phi')\mu_6 + \mu_7\big(\phi',(\phi,\phi')\big)\mu_8$$

where the $\mu_i \in M$, which completes the proof. $\square$

Recall that if $D_{n\times n}$ is the ring of $n \times n$ matrices over D, where n is a finite integer, then if $\underline{w}\colon D_{n\times n} \to G^{\#}$ is the mapping sending $\Phi = [\phi_{ij}]_{1\le i,j<n}$ onto $\underline{w}(\Phi)$ defined by

$$(6) \qquad \underline{w}(\Phi) = \operatorname*{Min}_{i,j=1,\ldots,n}\big(w(\phi_{ij})\big),$$

then $\underline{w}$ is a vector valuation of $D_{n\times n}$ over D, which verifies the *submultiplicative property*

$$(7) \qquad \underline{w}(\Phi_1\Phi_2) \ge \underline{w}(\Phi_1) + \underline{w}(\Phi_2) \qquad (\Phi_1, \Phi_2 \in D_{n\times n}).$$

Evidently, for every $\Phi \in D_{n\times n}$ and $g \in G^{\#}$,

$$(8) \qquad \Phi\underline{w} \ge g \Leftrightarrow \underline{w}(\Phi) \ge g,$$

where $\cdots \underline{w} \ge \cdots$ is the usual matrix prevaluation on $D_{n\times n}$ ($\Phi\underline{w} \ge g$ means $w(\mathrm{entr}_{i,j}(\Phi)) \ge g$ for every pair $i, j = 1,\ldots,n$). By standard argument $\underline{w}$ verifies the *generalized triangle inequality*

$$(9) \qquad \underline{w}\left(\sum_{i=1}^{r}\Phi_i\right) \ge \operatorname*{Min}_{i=1,\ldots,r}\big(\underline{w}(\Phi_i)\big),$$

and $\underline{w}$ verifies the special triangle equality

$$(10) \qquad \underline{w}(\Phi_1) \ne \underline{w}(\Phi_2) \Rightarrow \underline{w}(\Phi_1 + \Phi_2) = \operatorname{Min}\big(\underline{w}(\Phi_1), \underline{w}(\Phi_2)\big).$$

Fact 4.1.10. Let V be any n-dimensional non-degenerate 0-elliptic space and let (f_i) be any fixed orthogonal basis of V, which is bounded by some ε' in G. Given $\phi \in L$, let

(11) $$\underline{w}(\phi) = \underline{w}(\text{Mtrx}(\phi)).$$

Then:

(1) If $\phi \equiv \phi'$, then $|\underline{w}(\phi) - \underline{w}(\phi')| < 2\varepsilon'$.

(2) If $\sigma, \sigma' \in U_{w(2)}^+$ and if $\phi, \phi' = \mathscr{C}(\sigma), \mathscr{C}(\sigma')$ are such that

$$2\underline{w}(\phi, \phi') > \underline{w}((\phi, \phi'), \phi') + 5\varepsilon'$$

then if $\gamma = [[\sigma, \sigma'], \sigma']$, then

$$\left|\underline{w}(\gamma - 1) - \underline{w}((\phi, \phi'), \phi')\right| \le 4\varepsilon'$$

Proof. 1. Since V is 0-elliptic it follows that if $\tau w \ge g$ then $\text{Mtrx}(\tau)w \ge g - \varepsilon$ and that if $\text{Mtrx}(\tau)w \ge g$ then $\tau w \ge g - \varepsilon$ (§3, Theorem 3.5). In terms of the vector valuation $\underline{w}(\tau)$ this means that

(12) $$\tau w \ge g \Rightarrow \underline{w}(\tau) \ge g - \varepsilon' \quad (g \in G).$$

(13) $$\underline{w}(\tau) \ge g \Rightarrow \tau w \ge g - \varepsilon' \quad (g \in G).$$

Let then $\phi \equiv \phi'$. If $g = \underline{w}(\phi)$, then

$$\underline{w}(\phi) = g \ge g \Rightarrow \phi w \ge g - \varepsilon'.$$

Since $\phi' \equiv \phi$ it follows that $\phi'w \ge g - \varepsilon'$ so that $\underline{w}(\phi') \ge g - 2\varepsilon'$ or, $\underline{w}(\phi') + 2\varepsilon' \ge \underline{w}(\phi)$. By symmetry, $\underline{w}(\phi) + 2\varepsilon' \ge \underline{w}(\phi')$. Thus $|w(\phi) - w(\phi')| \le 2\varepsilon'$, as desired.

(2) From the preceding fact follows

$$\gamma - 1 \equiv (\tau_1 + \tau_2) + \tau_3,$$

where

$$\tau_1 = \mu_1(\phi, \phi')\mu_2(\phi, \psi')\mu_3, \quad \tau_2 = \mu_4(\phi, \phi')\mu_5(\phi, \phi')\mu_6,$$

$$\tau_3 = \mu_7((\phi, \phi'), \phi')\mu_8$$

with all the μ_i in M. Thus from (1) follows

(14) $$\left|\underline{w}(\gamma - 1) - \underline{w}((\tau_1 + \tau_2) + \tau_3)\right| \le 2\varepsilon'.$$

Now,

$$\tau_1 \equiv (\phi, \psi')\mu_2(\phi, \phi').$$

Thus from, again, (1)

$$\underline{w}(\tau_1) \ge \underline{w}(\phi, \phi')\mu_2(\phi, \phi') - 2\varepsilon' \ge 2\underline{w}((\phi, \phi')) + \underline{w}(\mu_2) - 2\varepsilon',$$

and, since, μ_2 is medial and, hence, $\mu_2 w \geq 0$ it follows that $\underline{w}(\mu_2) \geq -\varepsilon'$ so that

$$\underline{w}(\tau_1) \geq 2\underline{w}((\phi, \phi')) - 3\varepsilon' > \underline{w}((\phi, \phi'), \phi') + 5\varepsilon' - 3\varepsilon'$$

$$> \underline{w}((\phi, \phi'), \phi') + 2\varepsilon'.$$

By the same token,

$$\underline{w}(\tau_2) > \underline{w}((\phi, \phi'), \phi') + 2\varepsilon'.$$

By the generalized triangle inequality follows

(15) $$\underline{w}(\tau_1 + \tau_2) > \underline{w}((\phi, \phi'), \phi') + 2\varepsilon'.$$

Now, from

$$\tau_3 = \mu_7((\phi, \phi'), \phi')\mu_8$$

follows

(16) $$\underline{w}(\tau_3) \leq \underline{w}((\phi, \phi'), \phi') + 2\varepsilon' < \underline{w}(\tau_1 + \tau_2).$$

By the special triangle equality follows

$$\underline{w}((\tau_1 + \tau_2) + \tau_3) = \underline{w}(\tau_3).$$

In view of (14)

$$|\underline{w}(\gamma - 1) - \underline{w}(\tau_3)| \leq 2\varepsilon'$$

follows. Thus

$$|\underline{w}(\gamma - 1) - \underline{w}((\phi, \phi'), \phi')|$$

$$\leq |\underline{w}(\gamma - 1) - \underline{w}(\tau_3)| + |\underline{w}(\tau_3) - \underline{w}((\phi, \phi'), \phi')|$$

$$\leq 2\varepsilon + 2\varepsilon = 4\varepsilon,$$

as desired.   □

Fact 4.1.11. (a) If (g_1, g_2) is in the equalizer of $P\pi_0$ then (g_1, g_2) is in the equalizer of π_1.

(b) If (g_1, g_2) is in the equalizer of $P\pi_1$ then:

(i) $[\pi_1(g_1, \pi_0(g_1))] \subset \pi_0(2g_2 + g_1);$

(ii) $[\pi_1(g_2), \pi_0(g_2)] \subset \pi_0(2g_1 + g_2).$

Proof. (a) To say that $P\pi_0(g_1) = P\pi_0(g_2)$ is to say that

$$\pi_0(g_1)(U \cap Z) = \pi_0(g_1)(U \cap Z).$$

Then

$$\pi_1(g_1) = [\pi_0(g_1), \pi_0(g_1)] = [\pi_0(g_1)(U \cap Z), \pi_0(g_1)]$$
$$= [\pi_0(g_2)(U \cap Z), \pi_0(g_1)] = [\pi_0(g_2), \pi_0(g_1)]$$
$$= [\pi_0(g_1), \pi_0(g_2)] = [\pi_0(g_2), \pi_0(g_2)] = \pi_1(g_2).$$

(b) To say that $P\pi_1(g_1) = P\pi_1(g_2)$ is to say

$$\pi_1(g_1)(U^{(1)} \cap Z) = \pi_1(g_2)(U^{(1)} \cap Z).$$

Then

$$[\pi_1(g_1), \pi_0(g_1)] = [\pi_1(g_1)(U^{(1)} \cap Z), \pi_0(g_1)]$$
$$= [\pi_1(g_2)(U^{(2)} \cap Z), \pi_0(g_1)] = [\pi_1(g_2), \pi_0(g_1)]$$
$$\subset [\pi_0(2g_2), \pi_0(g_1)] \subset \pi_0(2g_2 + g_1).$$

Similarly,

$$[\pi_1(g_2), \pi_0(g_2)] \subset \pi_0(2g_1 + g_2).$$

Fact 4.1.12. Let V be any non-degenerate 0-elliptic space and let V' be any orthogonal summand of V. If $\mathcal{F}$ is one of the mappings $\pi_0 = \pi_0(V)$, $P\pi_0 = P\pi_0(V)$, $P\pi_1 = P\pi_1 = (V)$, $\pi_1, \ldots$, and if $\mathcal{F}'$ is the corresponding mapping relative to the space V', then the equalizer of $\mathcal{F}$ is contained in that of $\mathcal{F}'$.

Proof. ($\mathcal{F} = \pi_0$, $P\pi_0$.) Assume that $\pi_0(g_1) = \pi_0(g_2)$. Since V is 0-elliptic it follows by §2, Theorem 2.6 that if $g \in G$, then

$$L_g(V) \supset L_g(V') \oplus 1.$$

Thus

$$U_g(V) \supset U_g(V') + 1.$$

Then

$$U_g(V') \oplus 1 = (U(V') \oplus 1) \cap U_g(V).$$

From $\pi_0(g_1) = \pi_0(g_2)$ follows

$$\pi_0'(g_1) \oplus 1 = U_{g_1}(V') \oplus 1 = (U(V') \oplus 1) \cap U_{g_1}(V)$$
$$= (U(V') + 1) \cap U_{g_2}(V) = \pi_0'(g_2) \oplus 1$$

and, consequently, $\pi_0'(g_1) = \pi_0'(g_2)$.

Assume that $V' \neq V$ and that $P\pi_0(g_1) = P\pi_0(g_2)$. This means that

$$U_{g_1} \cdot (Z \cdot 1 \cap U) = U_{g_2} \cdot (Z \cdot 1 \cap U),$$

where $1 =$ identity mapping of V. Denote by $1'$ the identity mapping of V'. It is to be shown that

$$U_{g_1}(V') \cdot (Z \cdot 1' \cap U(V')) = U_{g_2}(V') \cdot (Z \cdot 1 \cap U(V')).$$

Let $\sigma' \in U_{g_1}(V')$ and let $z \in Z$ be such that $z \cdot 1' \in U(V')$. Equivalently, z is a central unitary in the ground division ring $(D; *)$. If $\sigma_1 = \sigma' \oplus 1''$, where $1'' =$ identity mapping of $V'^{\perp}$, then $\sigma_1 \in U_{g_1}(V') \oplus 1'' \subset U_{g_1}(V)$. Thus $\sigma_1(z \cdot 1) \in U_{g_1}(V) \cdot (Z \cdot 1 \cap U) = U_{g_2}(V) \cdot (Z \cdot \cap U)$. This means that for some $\sigma_2 \in U_{g_2}(V)$ and some z_2, a central unitary element in $(D; *)$,

$$\sigma_1(z \cdot 1) = \sigma_2(z_2 \cdot 1).$$

From the preceding equation follows

$$V'\sigma_2 = z_2 V'\sigma_2 = V'\sigma_2(z_2 \cdot 1) = V'\sigma_1(z_1 \cdot 1) = z_1 V'\sigma_1 = V'$$

and

$$V'^{\perp}\sigma_2 = V'^{\perp}\sigma_2(z_2 \cdot 1) = V'^{\perp}\sigma_1(z \cdot 1) = V'^{\perp}.$$

Thus $\sigma_2 = \sigma_2' \oplus \sigma_2''$, where $\sigma_2' = \sigma_2/V'$ and $\sigma_2'' = \sigma_2/V''$. Moreover, if $V'' \in V'^{\perp}$, then

$$v''\sigma_2'' = v''\sigma_2 = z_2^{-1}v''z_2\sigma_2$$

$$= z_2^{-1}v''z\sigma_1 = z_2^{-1}zv''\sigma_1 = z_2^{-1}zv''.$$

This means that $\sigma_2'' = z_2^{-1}z \cdot 1''$, where $1'' =$ identity mapping of $V'^{\perp}$. Therefore,

$$z\sigma_1 = z(\sigma' \oplus 1'') = z = (\sigma' \oplus 1'')(z \cdot 1)$$

$$= \sigma_2(z_2 \cdot 1) = z_2(\sigma_2' \oplus z_2^{-1}z1'').$$

Then $z\sigma_1' = z_2\sigma_2'$, and, hence

$$\sigma_1' \cdot (z \cdot 1') = \sigma_2'(z_2 \cdot 1'),$$

where $\sigma_2' \in U_{g_2}(V')$ for $\sigma_2 \in U_{g_2}(V)$ and $\sigma_2' = \sigma_2/V'$, and z_2 is a central unitary. Thus

$$\sigma_1'(z \cdot 1') \in U_{g_2}(V')^j \cdot (Z \cap U(V')).$$

As this holds for each pair $\sigma_1' \in U_{g_1}(V')$ and $z \in \mathscr{U}(D; *) \cap Z$ it follows that

$$U_{g_1}(V') \cdot (Z \cap U(V')) \subset U_{g_2}(V') \cdot (Z \cap U(V')).$$

By symmetry,

$$U_{g_2}(V') \cdot (Z \cap U(V')) \subset U_{g_1}(V') \cdot (Z \cap U(V')).$$

Therefore

$$p\pi_0'(g_1) = P\pi_0'(g_2),$$

as desired. □

4.2. *The mapping* $g \to PU_g^{(1)}$ *for* $\dim_D V > 1$.

Generally the mapping $P\pi_1 = g \in (w(2), \to) = PU_g^{(1)} \in \mathrm{lat}(PU^{(1)})$ has a non-trivial equalizer even when $\dim_D V > 1$. Here is an example where $P\pi_1$ is constant.

REMARK 4.2.1. If V is the 2-dimensional orthogonal space then $P\pi_1$ is the constant mapping. In fact, π_1 is the constant mapping.

Proof. Pick any orthogonal basis f_1, f_2 of V. If $s_i = f_i \cdot f_i$, $i = 1, 2$, and $\lambda = s_1 s_2^{-1}$ then by inspection, $\phi \in U \leftrightarrow \underline{\phi} = \mathrm{Mtrx}(\phi)$ is of one of the following types.

(A)
$$\underline{\phi} \begin{vmatrix} a & b \\ -\lambda b & a \end{vmatrix} \quad \left(\det \underline{\phi} = 1 \right);$$

(18)
$$\underline{\phi} = \begin{vmatrix} a & b \\ \lambda b & a \end{vmatrix} \quad \left(\det \underline{\phi} = -1 \right).$$

Thus $1 + \phi$ is invertible if and only if $\det \underline{\phi} = 1$. Since the set of matrices $\underline{\phi}$ as in (17) form an abelian normal subgroup of invertible matrices and since $1 + \phi$ is invertible for every $\phi \in U_g$ where $g > w(2)$, it follows that U_g is abelian so that $\pi_1(g) = 1$ for every $g > w(2)$. □

THEOREM 4.2.2. *Let* V *be any non-degenerate 0-elliptic space with dimension* > 1. *Assume that* V *is of one of the following types.*

(1) *V is a non-orthogonal space containing some non-zero orthogonal vectors f_1 and f_2 such that*

(19)
$$\left| w(f_1 \cdot f_1) - w(f_2 \cdot f_2) \right| \le n\varepsilon,$$

where n is any fixed natural number and ε is either the zero of or else the first positive element in G.

(2) *V is an orthogonal space containing a triple of non-zero orthogonal vectors f_1, f_2, f_3 such that*

(20)
$$\left| w(f_i \cdot f_i) - w(f_j \cdot f_j) \right| \le n\varepsilon \quad (i, j = 1, 2, 3),$$

where n and ε are as in 1.

Then:

(3) *There is an isomorphism γ of the chain G^+ into G^+ such that $\gamma \circ P\pi_2$ $(= g \in G^+ \to [U_{\gamma(g)}, U_{\gamma(g)}]/[U, U] \cap Z)$ is a dual isomorphism from G^+ into* $\mathrm{lat}.(P[U, U])$, *the lattice of normal subgroups of the projective special unitary group.*

Proof. *Case where V is as in* 1. To say that V is not orthogonal is evidently to say that the ground involution $*$ is not the identity mapping. Thus if $0 \neq s = s* \in D$ then the co-gredient involution sending $x \in D$ onto $x^{(s)} = sx*s^{-1}$ is, again, not the identity mapping. Without loss of generality one may assume that if $s_i = f_i \cdot f_i$, $i = 1, 2$, then $w(s_1) \leq w(s_2)$. Since $0 \neq s_1 = s_1^*$ it follows that (s_1) is not the identity mapping. If $0 \neq k \in D$ is such that $k \in \mathcal{K}(D; (s_1))$ then so are $2k$ and k^{-1} in $\mathcal{K}(D; (s_1))$. Hence one can find $k_1 \in \mathcal{K}(D; (s_1))$ such that if $g_0 = w(k_1)$ then $\infty \neq g_0 \geq w(2)$. Let $\gamma: G^+ \to G^+$ be defined by

(20) $\gamma(g) = g_0 + 2mg,$

where $m \geq 1$ is a natural number to be fixed later. If $g \in G$ and $x' \in D$ is such that $w(x') = g$ then

$$g_0 + 2g = w(k_1) + 2w(x')$$
$$= w(k_1) + w(x'x'^{(s_1)}) = w(x'k_1x'^{(s_1)}),$$

where $x'k_1x'^{(s_1)} \cap \mathcal{K}(D; (s_1))$. Thus

$$g_0 + 2g \in w(\mathcal{K}(D; (s_1))).$$

Therefore

$$\gamma(G^+) \subset w(\mathcal{K}(D; (s_1)))$$

and by construction,

$$\gamma(G^+) \subset (w(2), \to).$$

Thus $\gamma \circ P\pi_1$ is a well-defined dual homomorphism from G^+ into lat.$(P[U, U])$. It remains to show that $\gamma \circ P\pi_1$ is one-to-one. In view of Fact 4.1.12 one may assume that V is the subspace that is spanned by f_1 and f_2. Hence (f_1, f_2) is an orthogonal basis of V, which is bounded by $\varepsilon' = n\varepsilon$.

Let then $g_1, g_2 \in \gamma(G^+)$ be such that $P\pi_1(g_1) = P\pi_1(g_2)$. It is to be shown that for m large enough (m depending only on (n, ε)) $g_1 = g_2$ follows. Pick $x \in D$ such that $w(x) = g_1 + n\varepsilon = g_1 + \varepsilon'$ and $\mathcal{K} = -k^{(s)}$ such that $w(k) = g_1 + 2\varepsilon'$ (possible since

$$g_1 + 2\varepsilon' = g_1 + 2n\varepsilon \in \gamma(G^+) + 2G \geq 0 \subset g_0 + 2G \subset w(\mathcal{K}(D; (S_1)))).$$

Define $\phi, \phi' \in L(V)$ by the matrix equations

(21) $\mathrm{Mtrx}(\phi) = \begin{vmatrix} 0 & x \\ -s_2 x' s_1^{-1} & 0 \end{vmatrix}$

(22) $\mathrm{Mtrx}(\phi') = \begin{vmatrix} k & 0 \\ 0 & 0 \end{vmatrix}.$

By construction, ϕ and ϕ' are skew-symmetric transformations and by inspection,

$$\underline{w}(\phi) \quad (= \underline{w}(\text{Mtrx}(\phi))) = g_1 + \varepsilon'$$

$$\underline{w}(\phi') = g_1 + 2\varepsilon'.$$

From the preceding relations follows that

$$\phi, \phi' \in \mathscr{K}_{g_1}(V);$$

in fact, $\phi w \geq (g_1 + \varepsilon') - \varepsilon'$ and $\phi'w \geq (g_1 + 2\varepsilon') - \varepsilon' = g_1 + \varepsilon' \geq g_1$. Since $g_1 > w(2)$ it follows that if $\sigma, \sigma' = \mathscr{C}^{-1}(\phi), \mathscr{C}^{-1}(\phi')$ then σ, σ' are well-defined unitary transformations of V and that $\sigma, \sigma' \in U_{g_1}(V)$.

Now, by inspection

(23) $$\underline{w}((\phi, \phi')) = \underline{w}(\phi) + \underline{w}(\phi') = 2g_1 + 3\varepsilon',$$

(24) $$\underline{w}((\phi, \phi'), \phi') = \underline{w}(\phi, \phi') + \underline{w}(\phi') = 3g_1 + 5\varepsilon'.$$

To ensure the inequality

$$2\underline{w}((\phi, \phi')) > \underline{w}((\phi, \phi'), \phi') + 5\varepsilon'$$

it suffices then to ensure that

$$4g_1 + 6\varepsilon' > 3g_1 + 10\varepsilon'$$

or,

$$g_1 > 4\varepsilon' = 4n\varepsilon.$$

If $\varepsilon = 0$, then $g_1 = g_0 + 2mg > 0$ will follow for $g > 0$ and $g_0 \geq w(2) \geq 0$.

If ε is the first positive element in G then $g_1 = g_0 + 2mg > 4n\varepsilon$ as soon as

$$g_1 + 5\varepsilon' \geq g_2.$$

By symmetry, $g_2 + 5\varepsilon' \geq g_1$. Thus

$$|g_1 - g_2| \leq 5\varepsilon'.$$

If $\varepsilon = 0$ then $\varepsilon' = n\varepsilon = 0$ giving $g_1 = g_2$. If ε is the first positive element in G then since $g_1, g_2 \in \gamma(G^+) = g_0 + 2mG^+$ it follows that $g_1 - g_2 \neq 0$ implies $|g_1 - g_2| \geq 2m\varepsilon \geq 6n\varepsilon = 6\varepsilon'$, contradicting the preceding inequality. This shows that $g_1 = g_2$, as desired. Therefore $\gamma \circ P\pi_1$ is one-to-one as soon as $\gamma = \gamma_m = g \rightarrow g_0 + 2mg$ is such that $m \geq 3n$.

Case V is an orthogonal space as in 2.

Take $\gamma = \gamma_n = g \mapsto w(2) + 2ng$ ($g \in G^+$). This is an isomorphism of G^+ into G^+. It is to be shown that $\gamma \circ P\pi_1$ is one-to-one.

Now, from $g > w(2)$ follows $-1 \notin U_g$ and, consequently, $-1 \notin \pi_1(g) \subset U_g$. Thus $\pi_1(g) \cap ([U, U] \cap Z) \supset \pi_1(g) \cap \{1, -1\} = (1)$. From this π_1 and $P\pi_1$ have same equalizers. Thus $\gamma \circ \pi_1$ and $\gamma \circ P\pi_1$ have same equalizers. It remains then to show that $\gamma \circ \pi_1$ has trivial equalizer. In view of Fact 4.1.12, one may assume that V is spanned by the vectors f_i as in the statement of 2. Without loss of generality if $s_i = f_i \cdot f_i$, then $w(s_1) \le w(s_2) \le w(s_3)$.

Pick any $x \in D$ such that $w(x) = g_1 + \varepsilon'$ ($\varepsilon' = n\varepsilon$). Determine $\phi, \phi' \in L(V)$ by their matrix equations

$$(25) \qquad \mathrm{Mtrx}(\phi) = \begin{vmatrix} 0 & x & 0 \\ -s_2 s_1^{-1} s & 0 & 0 \\ \hline 0 & 0 & 0 \end{vmatrix},$$

$$(26) \qquad \mathrm{Mtrx}(\phi') = \begin{vmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ \hline -s_3 s_1^{-1} z & 0 & 0 \end{vmatrix}.$$

By inspection, ϕ, ϕ' are skew-symmetric transformations of V such that

$$(27) \qquad \underline{w}(\phi) = g_1 + \varepsilon',$$

$$(28) \qquad \underline{w}(\phi') = g_1 + \varepsilon',$$

and

$$(29) \qquad \underline{w}((\phi, \phi')) = \underline{w} \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & -s_2 s_1^{-1} x^2 \\ \hline 0 & s_3 s_1^{-1} x^2 & 0 \end{vmatrix}$$

$$= \underline{w}(\phi) + \underline{w}(\phi') + w(s_2 s_1^{-1}).$$

Since the basis (f_1, f_2, f_3) is bounded by $\varepsilon' = n\varepsilon$ it follows that $\phi, \phi' \in \mathcal{K}_{g_1}(V)$. In view of Fact 4.1.8, $(\phi, \phi')w \ge 2g_2$ follows, and, consequently,

$$\underline{w}((\phi, \phi')) \ge 2g_2 - \varepsilon'.$$

Thus

$$2g_1 + w(s_2 s_1^{-1}) = \underline{w}((\phi, \phi')) \ge 2g_2 - \varepsilon'$$

or,

$$2g_1 + 2\varepsilon' \ge 2g_1 + w(s_2 s_1^{-1}) + \varepsilon' \ge 2g_2; \, g_1 + \varepsilon' \ge g_2.$$

By symmetry, $g_2 + \varepsilon' \ge g_1$. Thus $|g_1 - g_2| \le \varepsilon'$. If $\varepsilon = 0$ then $\varepsilon' = n\varepsilon = 0$ follows giving $g_1 = g_2$. If ε is the first positive element in G then $g_1 \ne g_2$

implies $|g_1 - g_2| \geq 2n\varepsilon = 2\varepsilon' > \varepsilon'$, contradicting the inequality $|g_1 - g_2|$ $\leq \varepsilon'$. This shows that $\gamma_n \circ P\pi_1$ is one-to-one, which concludes the proof. $\square$

COROLLARY 4.2.3. *If V is as in Theorem 4.2.2 then the positive cone G^+ of the underlying value group G can be dually embedded in* lat.$(PU^{(1)})$, *the lattice of normal subgroups of the projective derived unitary group $PU^{(1)} = [U, U]/[U, U] \cap Z$.*

Proof. For $\gamma \circ P\pi_1$ is a dual homomorphism of ordered sets.

COROLLARY 4.2.4. *If V is as in Theorem 4.2.2 then $\gamma \circ P\pi_0$ is a dual one-to-one homomorphism from G^+ into* lat.$(PU^{(0)})$, *the lattice of normal subgroups of the projective unitary group $PU^{(0)} = U/U \cap Z$.*

Proof. For the equalizer of $\gamma \circ P\pi_0$ is contained in the equalizer of $\gamma \circ \pi_1$. The latter is contained in the equalizer of $\gamma \circ P\pi_1$. $\square$

4.3. *The mapping $g \to PU_g^{(1)}$ for* $\dim_D V = 1$.
Will the restriction $\dim_D V = 1$ make the treatment of the mapping $g \to [U_g, U_g]/[U, U] \cap Z$ easier? As it turns out it is the reverse that happens to be true. I suggest as an explanation the following. Theorem 4.2.2 was established by making explicit use of skew-symmetric transformations ϕ and ϕ' with prescribed values relative to the vector valuation $\underline{w}$ and such that

$$(28) \qquad \underline{w}(\phi\phi' - \phi'\phi) = \underline{w}(\phi) + \underline{w}(\phi').$$

(See equations (21)–(26), (29)). In dealing with the simplest instance of 1-dimensional space V, namely the space $_D D$, relative to the form $x \cdot y = xy^*$, (28) means that for certain skew-symmetrics ϕ, ϕ' in $(D; *)$ with prescribed values $w(\phi) = g, w(\phi') = g'$,

$$(29) \qquad w(\phi\phi' - \phi'\phi) = w(\phi) + w(\phi').$$

I do not know of any standard construction of such pairs ϕ, ϕ' for the given *-valued division ring D As a matter of fact there is a clear-cut class of division rings D possibly infinite dimensional, in which, equality (29) *never* occurs even if ϕ, ϕ' are *any* elements in D (see [10] or [4]).
Such division rings D have commutative residue division rings $\overline{D}$. Reversing the trend, I will assume in what will follow that $\dim_{\overline{Z}} \overline{D} > 4$, where $\overline{Z} = \text{center}(\overline{D})$ and $1/2$ exists in $\overline{D}$.

THEOREM 4.3.1. *Let* $(D; *; w)$ *be any $*$-valued division ring. Assume that $1/2$ exists in the valuation ring R and that the residue division ring D is of dimension over its center exceeding 4. For each pair of skew-symmetrics σ and τ one can find symmetric units u and v in R such that if $\sigma' = u\sigma u$ and $\tau' = v\tau v$ then*

$$w(\sigma'\tau - \tau'\sigma') = w(\sigma') + w(\tau').$$

Proof. Deny the conclusion of the theorem. In particular,

$$w(\sigma\tau - \tau\sigma) \neq w(\sigma) + w(\tau).$$

Thus $\sigma \neq 0$ and $\tau \neq 0$. Then

$$w(\sigma\tau - \tau\sigma) = w(\sigma\tau(1 - [\tau, \sigma]))$$
$$= w(\sigma) + w(\tau) + w(1 - [\tau, \sigma]) \neq w(\sigma) + w(\tau)$$

so that $w(1 - [\tau, \sigma]) \neq 0$. Since $[\tau, \sigma]$ is in R, it follows that $w(1 - [\tau, \sigma]) > 0$ and, hence,

(30) $$[\tau, \sigma] \equiv 1 \pmod{J}.$$

I proceed to show that

(31) $$(uu^\sigma)^\tau \equiv uu^\sigma \pmod{J},$$

where u is any symmetric unit in R, and $a^b = b^{-1}ab$ $(a, b \in D)$. By standard properties of group commutators, if u is any symmetric unit in R, then

$$[u\tau u, \sigma] = [u, \sigma]^u[\tau u, \sigma] = [u, \sigma]^{\tau u}[\tau, \sigma]^u[u, \sigma].$$

Since $(\cdot)^u \colon x \in R \to x^u$ is an inner automorphism of the ring R preserving evidently J it follows that

$$[\tau, \sigma]^u \equiv 1^u = 1 \pmod{J},$$

giving

$$[u\tau u, \sigma] \equiv [u, \sigma]^{\tau u}[u, \sigma].$$

By (30) applied to the pair $u\tau u$ and σ it follows that

$$[u, \sigma]^{\tau u}[u, \sigma] \equiv 1 \pmod{J}.$$

Conjugating by u^{-1} the preceding congruence:

$$[u, \sigma]^\tau[u, \sigma]^{u^{-1}} \equiv 1 \pmod{J}.$$

Equivalently,

$$[u, \sigma]^\tau \equiv [\sigma, u]^{u^{-1}}.$$

Now,

$$[\sigma, u]^{u^{-1}} = u[\sigma, u]u^{-1} = u\sigma^{-1}u^{-1}\sigma uu^{-1}$$

$$= u\sigma^{-1}u^{-1}\sigma = [u^{-1}, \sigma].$$

Thus

$$[u, \sigma]^\tau \equiv [u^{-1}, \sigma] \quad (\text{mod } J).$$

Here

$$[u, \sigma]^\tau = [u^\tau, \sigma^\tau] = [u^\tau, \sigma[\sigma, \tau]]$$

$$= [u^\tau, [\sigma, \tau]][u^\tau, \sigma]^{[\sigma\tau]} = [u^\tau, 1][u^\tau, \sigma]^1 = [u^\tau, \sigma]$$

and, consequently,

$$[u^\tau, \sigma] \equiv [u^{-1}, \sigma] \quad (\text{mod } J),$$

or,

$$(u^{-1})^\tau(\sigma^{-1}u^\tau\sigma) \equiv u(\sigma^{-1}u^{-1}\sigma) \quad (\text{mod } J).$$

Left multiply the preceding congruence by u^{-1} and right multiply the resulting congruence by $(\sigma^{-1}u^\tau\sigma)^{-1}$. After replacing u by u^{-1} one gets precisely (31).

I proceed to show that the inner automorphisms (σ) and (τ) induce equal automorphisms on the residue division ring $\overline{D}$. Congruence (31) is evidently true if u is a symmetric in J. Thus (31) holds true for u any symmetric in R and, consequently, (31) holds true if u is replaced by $1 + u$. This substitution gives

$$((1 + u)(1 + u\sigma))^\tau = (1 + u + u^\sigma + uu^\sigma)^\tau$$

$$= 1 + (u + u^\sigma)^\tau + (uu^\sigma)\tau \qquad (1 + u)(1 + u^\sigma)$$

$$= 1 + (u + u^\sigma) + (uu^\sigma)$$

so,

(32) $$u + u^\sigma \equiv (u + u^\sigma)^\tau.$$

Reversing the roles of σ and τ in (32):

(33) $$u + u^\tau \equiv (u + u^\tau)^\sigma.$$

Now, from $[\sigma, \tau] \equiv 1$ follows $u^{\sigma\tau} = u^{\tau\sigma}$. Subtracting (33) from (32) and eliminating congruent terms, it follows that

$$u^\sigma - u^\tau \equiv u^\tau - u^\sigma.$$

Since $1/2$ exists in R, it follows that $u^\sigma - u^\tau \equiv 0$ or $u^\sigma \equiv u^\tau$, for every $u = u^*$ in R. Since $\overline{D}$ is of dimension exceeding 4 over its own center it follows by a well-known theorem of Dieudonné that $\overline{D}$ is generated by its symmetrics. Thus $x^\sigma \equiv x^\tau$ for every $x \in R$.

I will show next that $(\cdot)^\sigma$ is of period 2 (or less) modulo J and that

$$(34) \qquad\qquad uu^\sigma \equiv u^\sigma u \qquad (u = u^* \in R).$$

Indeed, in (33) replacing $(\cdot)^\tau$ by $(\cdot)^\sigma$ this gives

$$u + u^\sigma \equiv u^\sigma + u^{\sigma^2}; \qquad u \equiv u^{\sigma^2}.$$

Same substitution in (31) gives

$$u \cdot u^\sigma \equiv (uu^\sigma)^\sigma = u^\sigma u^{\sigma^2} \equiv u^\sigma \cdot u.$$

I proceed to show that $(\cdot)^\sigma$ is, in fact, the trivial automorphism modulo J.

Replacing u by $u + v$ in (34) where u and v are symmetrics in R and eliminating congruent terms it follows that

$$(35) \qquad\qquad uv^\sigma + vu^\sigma \equiv u^\sigma v + v^\sigma u.$$

Now,

$$(v^\sigma)^* = (\sigma^{-1}v\sigma)^* = \sigma v^* \sigma^{-1} \equiv \sigma^{-1}v^*\sigma = \sigma^{-1}v\sigma = v^\sigma.$$

Thus v^σ is symmetric modulo J. Thus one can replace v by v^σ in (35). Then

$$uv + v^\sigma u^\sigma \equiv u^\sigma v^\sigma + vu;$$

equivalently,

$$(36) \qquad\qquad uv - vu \equiv (uv - vu)^\sigma,$$

for every pair of symmetrics u and v in R. Since the residue division ring $\overline{D}$ is of dimension over its center exceeding 4 it follows by a well-known theorem of I. N. Herstein that $\overline{D}$ is generated by Lie commutators of symmetric elements in $\overline{D}$. From (36) follows that $(\cdot)^\sigma$ is the identity automorphism modulo J.

It is now easy to reach a contradiction. For let u, v be symmetric units in R. By the preceding

$$v^\sigma \equiv v.$$

If σ is replaced by $u\sigma u$ this gives

$$v \equiv v^{u\sigma u} = (v^u)^{\sigma u} = ((v^u)^\sigma)^u (v^u)^u = v^{u^2}.$$

Thus $vu^2 \equiv u^2 v$, for every pair of symmetric units u and v in R. A final linearization on u shows that the symmetrics in $\overline{D}$ commute contradicting thus the dimension of $\overline{D}$ (Dieudonné). $\square$

THEOREM 4.3.2. *Let $(D; *; w)$ be as in Theorem* 4.3.1. *For each pair of non-zero skew symmetrics σ and τ as in the conclusion of Theorem* 4.3.1 *there is some symmetric unit u of the valuation ring R such that if $\tau' = u\sigma u$ or τu and $\sigma' = \sigma\tau - \tau\sigma$, then*

$$w(\sigma'\tau' - \tau'\sigma') = w(\sigma') + w(\tau').$$

Proof. Deny the conclusion of the theorem. Repeating the argument in the proof of (30) for σ replaced by σ' and τ replaced by τ' it follows that

$$[\sigma', \tau'] \equiv 1 \pmod{J}.$$

By construction, τ' can be replaced by σ as well and by $u\tau'u$ for every symmetric unit u in R. Repeating the argument in the proof of (31) for σ replaced by σ' and τ by $u\tau'u$ it follows that

$$(37) \qquad uu^{\sigma'} \equiv (uu^{\sigma'})^{\tau'} \qquad (\tau' = \sigma \text{ or } \tau),$$

for every symmetric u in R. Denote by c the element $u \cdot u^{\sigma'}$ and by $\bar{c}$ the image of c in $\overline{D}$. I proceed to show that $\bar{c} \notin \overline{Z}$, the center of $\overline{D}$, for some u.

For if $\bar{c}$ always belong to $\overline{Z}$ then replacing u by $1 + u$ and eliminating gives $u + u^{\sigma'} \in \overline{Z}$. From the identity

$$(38) \qquad u^2 - u(u + u^{\sigma'}) + uu^{\sigma'} = 0$$

follows that $\bar{u}$ is quadratic over $\overline{Z}$ for every symmetric $\bar{u}$ in $\overline{D}$. By standard argument $\overline{D}$ would be 4-dimensional (at most) over $\overline{Z}$, which is ruled out.

Pick any $c = uu^{\sigma'}$ such that $\bar{c} \notin \overline{Z}$. For each symmetric unit v in R, the inner automorphisms $(\ )^{v\sigma v}$ and $(\ )^{v\tau v}$ preserve c modulo J; it suffices to choose $\tau' = v\sigma v, v\tau v$ successively and to quote (37). It follows that $(v^{-1}\tau^{-1}v^{-1})$, $(v\sigma^{-1}v)$, $(v^{-1}\tau v^{-1})$, $(v\sigma v)$ all preserve c modulo J. Thus the product of these automorphisms preserve c modulo J. Since

$$(v^{-1}\tau^{-1}v^{-1})(v\sigma^{-1}v)(v^{-1}\tau v^{-1})(v\sigma v) = v^{-1}[\tau, \sigma]v$$

it follows that $(\cdot)^{v^{-1}[\tau, \sigma]v}$ preserves c modulo J.

I will show next that $[\tau, \sigma]$ maps onto a central element in $\overline{D}$. For $[\sigma, \sigma]$ is a unit in R. Thus if $\bar{\delta}$ is the image of $[\tau, \sigma]$ in $\overline{D}$ then the relation

$$c^{v^{-1}[\tau, \sigma]v} \equiv c$$

implies

$$[\bar{c}, \bar{v}^{-1}\bar{\delta}\bar{v}] = 1,$$

for every non-zero symmetric $\bar{v}$ in $\overline{D}$. Equivalently, $\bar{\delta}$ commutes with $\bar{c}^{\bar{v}}$ for every non-zero symmetric $\bar{v}$ in $\overline{D}$. By I. N. Herstein's [10, Theorem 6.1.1] follows that $\bar{\delta} \in \overline{Z}$; in view of the fact that $\bar{c} \notin \overline{Z}$.

Now $\bar{\delta} = [\tau, \sigma] \neq 1$, the unity of $\bar{D}$. For if $\bar{\delta}$ were the unity of $\bar{D}$, then $[\tau, \sigma] \equiv 1$, so that

$$w(\sigma') = w(\sigma\tau - \tau\sigma) = w(\sigma\tau(1 - [\tau, \sigma]))$$
$$= w(\sigma\tau) + w(1 - [\tau, \sigma]) > w(\sigma) + w(\tau),$$

contrary to the equality $w(\sigma') = w(\sigma) + w(\tau)$. Thus $1 = [\tau, \sigma]$ is a unit in R and evidently $1 = [\tau, \sigma]$ maps onto a central element in $\bar{D}$. If $x \in R$, $x \notin J$ it is to be shown that

$$x^{1-[\tau, \sigma]} \equiv x.$$

For

$$\overline{x^{1-[\tau,\sigma]}} = \overline{(1 - [\tau, \sigma])^{-1} \times (1 - [\tau, \sigma])}$$

$$= \overline{(1 - [\tau, \sigma])^{-1}\bar{x}(1 - [\tau, \sigma])}$$

$$= \overline{(1 - [\tau, \sigma])^{-1}(1 - [\tau, \sigma])}.$$

Since $1 - [\tau, \sigma] \in \bar{Z}$ it follows that $(1 - [\tau, \sigma])^{-1}\bar{x}(1 - [\tau, \sigma]) = \bar{x}$ giving

$$x^{1-[\tau,\sigma]} = \bar{x};$$

equivalently

(39) $x^{1-[\tau,\sigma]} \equiv x.$

From the preceding congruence follows that for each symmetric unit u in R

$$u^{\sigma'} = u^{\sigma\tau - \tau\sigma} = u^{\sigma\tau(1 - [\tau,\sigma])} = \left(u^{\sigma\tau}\right)^{(1 - [\tau,\sigma])} \equiv u^{\sigma\tau}.$$

In view of (37),

$$uu^{\sigma\tau} \equiv uu^{\sigma'} \equiv \left(uu^{\sigma\tau - \tau\sigma}\right)^{\tau'}$$

$$= u^{\tau'}\left(u^{\sigma\tau - \tau\sigma}\right)^{\tau'} \quad u^{\tau}\left(u^{\sigma\tau}\right)^{\tau'} = \left(uu^{\sigma\tau}\right)^{\tau'}.$$

Since the preceding congruence holds for both cases $\tau' = \sigma$ or $\tau' = \tau$ it follows that if $\gamma = \sigma\tau$ then

(40) $uu^{\gamma} \equiv \left(uu^{\gamma}\right)^{\gamma}.$

In the preceding congruence replace u by $1 + u$ and eliminate congruent terms. This gives $u + u^{\gamma} \equiv (u + u^{\gamma})^{\gamma}$ or, $u \equiv u^{\gamma^2}$. Returning to (40) and replacing u^{γ^2} by u, one has $uu^{\gamma} \equiv u^{\gamma}u$. Exactly as in the proof of Theorem 4.3.1, argument following (36) one deduces from the preceding congruence that γ induces the identity automorphism of $\bar{D}$. Thus from (37) follows

$$u^2 \equiv uu^{\sigma\tau - \tau\sigma} \equiv \left(uu^{\sigma\tau - \tau\sigma}\right)^{\tau'} \equiv \left(uu^{\sigma\tau}\right)^{\tau'} \equiv u^{2\tau'}.$$

By one more linearization argument follows that $u \equiv u^{\tau'}$. Since τ' can be replaced by $v\sigma v$ one deduces that the symmetrics in $\overline{D}$ commute, which is a contradiction. $\qquad \square$

THEOREM 4.3.3. *Let* $(D; *; w)$ *be any* $*$-*valued division ring such that* $1/2$ *exists in the valuation ring* R. *Let* ϕ_1, ϕ_2, ϕ_3 *be three non-zero skew-symmetrics such that*

(i) $w(\phi_1) = w(\phi_2) = w(\phi_3) > 0$, *where if* $g = w(\phi_1)$, *then* $g \neq \infty$;

(ii) $w(\phi_1\phi_2 - \phi_2\phi_1) = 2g$;

(iii) $w(\phi_1\phi_2 - \phi_2\phi_1)\phi_3 - \phi_3(\phi_1\phi_2 - \phi_1\phi_1) = 3g$.

Put $\sigma_i = (1 - \phi_i/2)/(1 + \phi_i/2)$ $(i = 1, 2, 3)$. *Then the* σ_i *are unitaries in* D *such that*

(iv) $w(\sigma_i) = g$ $(i = 1, 2, 3)$;

(v) $w([[\sigma_1, \sigma_2], \sigma_3] - 1) = 3g$.

Proof. (iv) At the risk of repeating slightly the calculations in Fact 4.1.7, point 1, one has

$$\sigma_i - \frac{1 - \phi_i/2}{1 + \phi_i/2} - 1 = \frac{-\phi_i}{1 + \phi_i/2},$$

giving

$$w(\sigma_i - 1) = w(\phi_i) + w(1 + \phi_i/2) = w(\phi_i),$$

for since $w(\phi_i) > 0$ and $w(1/2) = 0$ it follows that $w(1 + \phi_i/2) = 0$.

(v) Represent $[\sigma_1, \sigma_2]$ in the form

$$[\sigma_1, \sigma_2] = \frac{1 - t/2}{1 + t/2}.$$

Indeed, the preceding equation can be solved for $t/2$ provided $[\sigma_1, \sigma_2] \neq -1$. If $[\sigma_1, \sigma_2]$ were central then (iii) would be obviously violated. Thus $[\sigma_1, \sigma_2] \neq -1$. If $t_1 = 1 - [\sigma_1, \sigma_2]$ then

$$t/2 = \frac{t_1}{2 - t_1}.$$

From trivial adaptations of Facts 4.1.7 and 4.1.9 follows that if $\gamma = [[\sigma_1, \sigma_2], \sigma_3]$ then

$$w(\gamma - 1) = w\left(\left(\frac{1}{2 - t_1}, \phi_3\right)\right) = w((t_1, \phi_3)) = w\left(\phi_3, \sigma_1^{-1}\sigma_2^{-1}(\sigma_1, \sigma_2)\right).$$

Now,

$$\left(\phi_3, \sigma_1^{-1}\sigma_2^{-1}(\sigma_1, \sigma_2)\right) = \left(\phi_3, \sigma_1^{-1}\right)\sigma_2^{-1}(\sigma_1, \sigma_2) + \sigma_1^{-1}\left(\phi_3, \phi_2^{-1}\right)(\sigma_1, \sigma_2)$$

$$+ \sigma_1^{-1}\sigma_2^{-1}(\phi_3, (\sigma_1, \sigma_2)) = \tau_1 + \tau_2 + \tau_3,$$

where

$$\tau_1 = \left(\phi_3, \sigma_1^{-1}\right)\sigma_2^{-1}(\sigma_1, \sigma_2), \quad \tau_2 = \sigma_1^{-1}\left(\phi_3, \sigma_2^{-1}\right)(\sigma_1, \sigma_2),$$

$$\tau_3 = \sigma_1^{-1}\sigma_2^{-1}\left(\phi_3, (\sigma_1, \sigma_2)\right).$$

Now

$$w(\tau_1) = w\left(\phi_3, \sigma_1^{-1}\right) + w\left((\sigma_1, \sigma_2)\right)$$
$$= w\left((\phi_3, \phi_1)\right) + w\left((\phi_1, \phi_2)\right) \geq 4g;$$

$w(\tau_2) > 4g$. Also,

$$w(\tau_3) = w\left(\phi_3, (\sigma_1, \sigma_2)\right)$$

and

$$\left(\phi_3, (\sigma_1, \sigma_2)\right) = \left(\phi_3, \; \frac{1}{1 - \phi_1/2} \; \frac{1}{1 - \phi_2/2} (\phi_2, \phi_1) \frac{1}{1 - \phi_2/2} \; \frac{1}{1 - \phi_1/2}\right)$$

$$= \tau_1' + \tau_2' + \tau_3' + \tau_4' + \tau_5',$$

where

$$\tau_3' = \frac{1}{1 - \phi_1/2} \; \frac{1}{1 - \phi_2/2} \left(\phi_3, (\phi_2, \phi_1)\right) \frac{1}{1 - \phi_2/2} \; \frac{1}{1 - \phi_1/2}$$

has value $w(\phi_3, (\phi_2, \phi_1)) = 3g$ and all other terms τ_i', $i \neq 3$, have values at least $4g$. By the special triangle equality follows

$$w\left(\phi_3, \sigma_1^{-1}\sigma_2^{-1}(\sigma_1, \sigma_2)\right) = w(\tau_3') = 3g,$$

so

$$w(\gamma - 1) = 3g = w\left((\phi_1, \phi_2), \phi_3\right). \qquad \square$$

THEOREM 4.3.4. *Let $(D; *; w)$ be any $*$-valued division ring such that $1/2$ exists in the valuation ring R in D and the residue division ring $\overline{D}$ has dimension over its own center exceeding 4. Denote by $\mathcal{U}$ the unitary group of D, $P\mathcal{U}^{(1)}$ the factor group $[\mathcal{U}, \mathcal{U}]/[\mathcal{U}, \mathcal{U}] \cap Z$ and $PU_g^{(1)}$ the subgroup $[\mathcal{U}_g, \mathcal{U}_g]/[\mathcal{U}, \mathcal{U}] \cap Z$, where $\mathcal{U}_g = \{u \in \mathcal{U} \mid w(u - 1) \geq g\}$. Then $g \to P\mathcal{U}_g^{(1)}$ is a dual embedding of the positive cone G^+ of G into the lattice of normal subgroups of the group $P\mathcal{U}^{(1)}$.*

Proof. By inspection, $\mathcal{U}_g$ is a normal subgroup of $\mathcal{U}$ which decreases as g increases. Hence $g \to P\mathcal{U}_g$ is a dual homomorphism from G^+ into lat.($P\mathcal{U}^{(1)}$), the lattice of normal subgroups of the group $\mathcal{U}^{(1)}/\mathcal{U}^{(1)} \cap Z(\mathcal{U}^{(1)} = [\mathcal{U}, \mathcal{U}])$—One can also observe that $\mathcal{U}_g$ is of the form $U_g(V)$ for an appropriate space V.

I proceed to show that if $g > 0$, then the subgroup $[[\mathscr{U}_g, \mathscr{U}_g], \mathscr{U}_g]$ contains some unitary γ such that $w(\gamma - 1) = 3g$. To begin with, I assert that $\mathscr{U}_g$ contains some unitary σ such that $w(\sigma - 1) = g$. Since $g > 0 = w(2)$ it suffices to find some skew-symmetric ϕ in D such that $w(\phi) = g$. For then if $\sigma = (1 - \phi/2)/(1 + \phi/2)$ then $w(\sigma - 1) = w(\phi) = g$. Deny the assertion about ϕ. This means that $w(\phi) \neq g$ for every skew-symmetric ϕ in D. Pick $a \in D$ such that $w(a) = g$. Since $w(a - a^*) = w(a) + w(1 - a^{-1}a^*)$ and since $a - a^*$ is skew-symmetric it follows that $w(1 - a^{-1}a^*) \neq 0$. Since $1 - a^{-1}a^* \in R$, it follows that $a^{-1}a^* \equiv 1$ (mod J), for every $a \in D$ such that $w(a) = g$. If t is any unit in R then $w(at) = w(a) = g$. Hence, $(at)^{-1}(at)^* \equiv 1$ (mod J). Then

$$t^{-1}a^{-1}t^*a^* = (at)^{-1}(at)^* \equiv 1.$$

Left multiplying the preceding congruence by t this gives

$$a^{-1}t^*a^* \equiv t \quad (\text{mod } J).$$

Thus

$$a^{-1}t^*a = a^{-1}t^*a^*(a^{*-1}a) = a^{-1}t^*a^*(a^{-1}a^*)^{-1} \equiv t1 = t.$$

Changing t to t^* this gives $a^{-1}ta \equiv t^*$ or $t^a \equiv t^*$. If a is replaced by at_1, where t_1 is another unit in R then

$$t^* \equiv t^a = t^{at_1} = (t^a)^{t_1} \equiv t^{*t_1}.$$

As this holds for every pair of units t' and t_1 it would follow that $R/J = \overline{D}$ is commutative, which is nonsense. This shows that there is $\phi = -\phi^*$ such that $w(\phi) = g$. Apply Theorem 4.3.1 to the pair $\sigma_1 = \phi_1$, $\tau_1 = \phi_2$. There are u and v symmetric units such that if $\phi_1 = u\phi u$ and $\phi_2 = v\phi v$ then $w((\phi_1, \phi_2)) = w(\phi_1) + w(\phi_2)$. By construction ϕ_1 and ϕ_2 are skew-symmetrics and $w(\phi_1) = w(\phi_2) = w(\phi) = g$. Apply now Theorem 4.3.2 to the pair $\sigma_2 = \phi_1$, $\tau_2 = \phi_2$ to get a third skew-symmetric ϕ_3 such that $w((\phi_1, \phi_2), \phi_3) = 3g$, where $\phi_3 = u'\phi_1 u$ or $v'\phi_2 v'$ for some symmetric units u' and v' in R. Again, $w(\phi_3) = w(\phi_1)$ or $w(\phi_2)$. Since $w(\phi_1) = w(\phi_2) = g$ it follows that $w(\phi_3) = g$. If

$$\sigma_i = (1 - \phi_i/2)/(1 + \phi_i/2)$$

then by Theorem 4.3.3, the σ_i are all unitaries such that $w(\sigma_1) = w(\sigma_2) = w(\sigma_3) = g$ and $w([[\sigma_1, \sigma_2], \sigma_3 - 1]) = 3g$, as desired.

It is now easy to establish the one-to-one requirement about the mapping $g \to P\mathscr{U}_g^{(1)}$. For if $P\mathscr{U}_{g_1}^{(1)} = P\mathscr{U}_{g_2}^{(1)}$ then $[[\mathscr{U}_{g_1}, \mathscr{U}_{g_1}], \mathscr{U}_{g_1}] = [[\mathscr{U}_{g_2}, \mathscr{U}_{g_2}], \mathscr{U}_{g_1}]$. If the σ_i are chosen as in the preceding then since $\gamma = [[\sigma_1, \sigma_2], \sigma_3] \in [[\mathscr{U}_{g_1}, \mathscr{U}_{g_1}], \mathscr{U}_{g_1}]$ it follows that $\gamma \in [[\mathscr{U}_{g_2}, \mathscr{U}_{g_2}], \mathscr{U}_{g_1}] \subset \mathscr{U}_{2g_2 + g_1}$. Thus $w(\gamma - 1) \geq 2g_2 + g_1$ and, consequently, $3g_1 = w(\gamma - 1) \geq 2g_2 + g_1$ giving $2g_1 \geq 2g_2$ or $g_1 \geq g_2$. By symmetry $g_2 \geq g_1$ and, hence, $g_1 = g_2$, which completes the proof. □

Theorem 4.3.4 can be put to work for the considered 1-dimensional spaces V. At no extra cost in the arguments the results apply to *any* 0-elliptic space V. This is the

THEOREM 4.3.5. *Assume that* $(D; *; w)$ *is such that* (i) $1/2 \in D$ *and* (ii) $\dim_{\bar{Z}} \bar{D} > 4$. *If* V *is any* 0-*elliptic non-degenerate space over* D *then* $P\pi_1 = g \in (0, \to) \mapsto [U_t, u_g]/[U, U] \cap Z$ *is a dual isomorphism from* G^+ *into* lat.($PU^{(1)}$), *the lattices of normal subgroups of the projective derived unitary group of* V.

Proof. By Fact 4.1.12, it suffices to establish the theorem in the case $\dim_D V = 1$. As a second reduction, to show the theorem in the latter case it is the same as showing Theorem 4.3.4 for $*$ replaced by the co-gredient involution (s), where s is any non-zero element in D which is represented by the form. Indeed, for such an s it is clear that $V = {}_D D$ (up to isometry), relative to the form $x \cdot y = xsy^*$. Here $L = L_D(D) = D_{\mathscr{R}}$, $x_{\mathscr{R}}^* = x_{\mathscr{R}}^{(s)}$ and $x_{\mathscr{R}} w \geq g \Leftrightarrow w(x) \geq g$. From this $\pi_0(g) = (\mathscr{U}_g(D; (s)))$. Since $x \to x_{\mathscr{R}}$ is an isomorphism from $(D; (s))$ onto $(L; *)$ with correspondence of the subgroups $\pi_0(g)$ and $\mathscr{U}_g(D; (s))$ it follows that the relations $[\pi_1(g), \pi_0(g_1)] = [\pi_1(g_2), \pi_0(g_1)]$, and $[\pi_1(g_2), \pi_0(g_2)] = [\pi_1(g_1), \pi_0(g_2)]$ are equivalent to the corresponding relations where $\pi_0(g)$, $\pi_1(g)$ are replaced respectively by $\mathscr{U}_g(D; (s))$, $[\mathscr{U}_g(D; (s)), \mathscr{U}_g(D; (s))]$. Since the former relations follow from the relation $P\pi_1(g_1) = P\pi_1(g_2)$. (Fact 4.1.11) and since $g_1 = g_2$ follows from the former relations (see the proof of Theorem 4.3.4) it is now evident that $P\pi_1$ is one-to-one provided $(D; (s); w)$ has the same qualification as $(D; *; w)$. Indeed, w remains an (s)-valuation and the residue division ring has not changed. $\qquad\square$

Question. (Open.) Is $g \to PU_g^{(r)}$ again one-to-one where $g > 0$ and $(D; *; w)$ is such that $1/2 \in \bar{D}$ and $\dim_{\bar{Z}} \bar{D} > 4$? If yes, what if $\bar{D} \neq \bar{Z}$?

The assumption $\bar{D} \neq \bar{Z}$ cannot be dropped if one is to establish that $g \to U_g$ is one-to-one—let alone $g \to Pu_g^{(r)}$. This is underscored in the

REMARK 4.3.8. Let D be any finite-dimensional-valued division ring such that $\bar{D}$ is the real field and G is the ordered additive group of integers. Assume, further, that D carries an involution of the first kind. Then $(D; *; w)$ is a $*$-valued division ring for which $g \to \mathscr{U}_h(D; *)$ is not one-to-one on each open segment $(n, \to)$.

Proof. The involution $*$ automatically verifies the assumption $w(x^*) = w(x)$ as this was observed earlier in §1 (Dieudonné). Since $\bar{D}$ is the real

field it follows that the induced involution of $\overline{D}$ is the identity. Equivalently $0 \notin w(\mathscr{K}(D; *))$, where $w(\mathscr{K}(D; (*))) = \{w(x) | x = -x^* \in D\}$. Since $w(\mathscr{K}(D; *)) + 2z \subset w(\mathscr{K}(D; *))$ it follows that $w(\mathscr{K}(D; *))$ consists only of odd integers. If r_0 is the least positive integer then $w(\mathscr{K}(D; *)) = r_0 + 2Z$. If $(n, \rightarrow)$ is any open segment then for some m large enough $r_1 = r_0 + 2m \in w(\mathscr{K}(D; *))$ follows. By construction, if $\mathscr{K}_{r_1} = \mathscr{K}_{r_1}(D; *)$ and $\mathscr{K}_{r_2}(D; *)$ then $\mathscr{K}_{r_1} = \mathscr{K}_{r_2}$, where $x_2 = r_1 + 1$. Hence, $\mathscr{U}_{r_2} = \mathscr{U}_{r_1}$, where $r_1 \neq r_2$ and $r_1, r_2 \in (n, \rightarrow)$. Thus $g \rightarrow \mathscr{U}_g$ is not one-to-one, where $g \in (n, \rightarrow)$. $\qquad \square$

4.4. *The general mapping* $g \in PU_g^{(*)}$. Denote by π_r the mapping sending $g \in (w(2), \rightarrow)$ onto $U_g^{(r)}$ and by $P\pi_r$ the one sending g onto $PU_g^{(r)} = \pi_r(g)/U^{(r)} \cap Z$. Only a scratch of results will be offered here. Yet, I will have to rely on my work [5], which deals with the normal subgroup structure of $PU^{(r)}$, where V is any non-isotropic unitary space over a division ring D containing at least elements. I will begin with a list of exceptional cases labelled cases (A) through (C):

Case (A). V is a unitary space of dimension 1, $\dim_Z D \leq 4$ and $(D; *)$ is commutative.

Case (B). V is 2-dimensional orthogonal space.

Case (C). V is 1-dimensional and $\dim_Z D = 16$.
There follow results established in [4].

Fact 4.4.1 ([4]). Let V be any non-isotropic unitary space over a division ring D containing at least 7 elements and suppose that V is not as in cases (A) or (B). If N is any normal subgroup of the unitary group U of V then:
  (1) If N is abelian then, in fact, $N \subset Z$.
  (2) If N is not abelian then the centralizer of N in L is precisely Z.

Fact 4.4.2 ([4]). Let V be as in Fact 4.4.1. If N is any normal subgroup of U such that $[N, N] \subset Z$ then, in fact, $N \subset Z$.

Fact 4.4.3 ([4]). Let V be as in Fact 4.4.1 and suppose, further, that V is *not* as in Case (C). If N and N' are normal subgroups of U such that $N \cap N' \subset Z$ then $N \subset Z$ or $N' \subset Z$.

Fact 4.4.4 ([**4**]). Let V be as in Fact 4.4.3. If N is any normal subgroup of U *not* contained in Z and if N' is a normal subgroup of a normal subgroup of U such that $[N', N] \subset Z$ then $N' \subset Z$.

If V is any non-degenerate elliptic space it is clear that V is non-isotropic and that the ground division ring D contains at least 7 elements. In fact, since the valuation w is not trivial it follows that D cannot be algebraic over a finite field. Thus Facts 4.4.1 through 4.4.4 will apply. To apply them to the congruence subgroups U_h one is hinted (see Fact 4.4.1) to find the centralizer of U_g in L. From Fact 4.4.1, $C(U_g) = Z$ unless $U_g \subset Z$. By an adaptation of §2, Theorem 2.17, one can show the first part of the

THEOREM 4.4.5. *Let V be any nondegenerate elliptic space, which is not as in Cases* (A) *or* (B). *Then*:

(1) *The mapping* $P\pi_0 = g \in (w(2), \to) \mapsto \mathrm{lat.}(U_g/U \cap Z)$ *assumes non-trivial values.*

(2) *The mapping* $P\pi_r = g \mapsto U_g^{(r)}/U^{(r)} \cap Z$ *is as in 1* $(r \geq 0)$

(3) *The range of $P\pi_r$ is an infinite chain of* $\mathrm{lat.}(PU^{(r)})$.

Proof. (1) To say that $PU_g \neq 1$ is to say that $U_g \not\subset Z$, which holds for every $g \in G$.

(2) To say that $PU_g^{(r)} = 1$ is to say that $U_g^{(r)} \subset Z$. By Fact 4.4.2 follows $U_g^{(r-1)} \subset Z$. Step by step $U_g \subset Z$ follows, which is ruled out.

(3) By construction Range($P\pi_r$) is a chain in $\mathrm{lat.}(PU^{(r)})$. I claim that the infimum of Range($P\pi_r$) is the identity subgroup of $PU^{(r)}$. For let PN be this infimum. Then

$$N = \bigcap_{g > w(2)} U_g^{(r)} \cdot (Z \cap U^{(r)}).$$

Now,

$$[N, N] \subset \bigcap_{g > w(2)} \left[U_g^{(r)}(Z \cap U^{(r)}), U_g^{(r)}(Z \cap U^{(r)}) \right]$$

$$= \bigcap_{g > w(2)} \left[U_g^{(r)}, U_g^{(r)} \right] = \bigcap_{g > w(2)} U_g^{(r+1)} \subset \bigcap_{g > w(2)} U_t$$

$$= \bigcap_{g \in G} U_g = 1.$$

Thus N is abelian. Since N is a normal subgroup it follows from Fact 4.4.1 that $N \subset Z$ so that $PN = N/U^{(r)} \cap Z = 1$. If now Range($P\pi_r$) were finite then since Range($P\pi_r$) is a chain it would follow that this set has a first member, say, $PU_{g_0}^{(r)} = PN = 1$, contradicting 2.   □

THEOREM 4.4.6. *Let V be any non-degenerate elliptic space other than in cases (A), (B) or (C). If $1 \neq PN$ is any member of* lat.$(PU^{(r)})$ *and if PX is any member of* Range(π_r) *then $PN \cap PX \neq 1$.*

Proof. Deny the conclusion of the theorem. For some normal subgroup N of $U^{(r)}$, $N \not\subset Z$, and some $g > w(2)$, $PN \cap PU_g^{(r)} = 1$. Equivalently, $[N, U_g^{(r)}] \subset Z$. Now $C(U_g^{(r)}) = Z$ and $N \vartriangleleft U^{(r)} \vartriangleleft U$. From Fact 4.4.4 follows $N \subset Z$, a contradiction. $\qquad\square$

As an application of the foregoing theorems:

THEOREM 4.4.7. *Let V be any non-degenerate elliptic space other than in cases (A), (B) or (C) and assume that one at least of the following requirements hold true.*
  (1) *$*$ is an involution of the first kind.*
  (2) *The restriction of w on Z is the trivial valuation.*
  (3) *D is finite-dimensional over its center.*
  Then for every natural number $r \neq 0$ the projective group $PU^{(r)}$ contains no torsion normal subgroups $\neq 1$.

Proof. It suffices to find some torsion-free member of Range(π_r) (Theorem 4.4.6). Define g_0 to be the element 0 in G if charac$(D) \neq 0$; if, to the contrary, charac$(D) = 0$, let $g_0 = w(g)$, where $p = $ charac$(\overline{D})$. If $g_1 > \max(g_0, w(2))$ then by §2, Theorem 2.19, U_{g_1} is torsion-free. It is to be shown that $PU_{g_1}^{(r)}$ is torsion-free, where $r \neq 0$. For let $\phi \in U_{g_1}^{(r)}$ be such that $\phi^m \in Z$.

Case 1. Here $U \cap Z = \{1, -1\}$ so that ϕ is a root of unity. Thus $\phi = 1$. Thus $PU_{g_1}^{(r)}$ is torsion-free.

Case 2. If $\phi^m = z \in Z$ then since $\phi^m \in U_{g_1}$ it follows that $(z-1)w \geq g_1 > 0$. Equivalently, $w(z-1) > 0$ (§2, Theorem 2.4.2.). Thus $z - 1 = 0$ or $z = 1$. Thus $\phi = 1$ and, hence, $PU_{g_1}^{(r)}$ is torsion-free.

Case 3. By standard argument $U^{(1)} \cap Z$ consists only of roots of unity. From $\phi^m \in U_{g_1}^{(r)} \subset U^{(1)}$ follows that ϕ is a root of unity. From this $\phi = 1$ so that $Pu_1^{(r)}$ is torsion-free. $\qquad\square$

THEOREM 4.4.8. *Let V be any non-degenerate elliptic space other than in cases (A) or (B). Every torsion normal subgroup of U is central.*

Proof. For let N be any torsion normal subgroup of U. If g_1 is defined as in the proof of Theorem 4.4.7, then U_{g_1} is torison-free. Thus $N \cap U_{g_1} = 1$ and, hence, $N \subset C(U_{g_1}) = Z$. $\square$

Question. (Open.) Does Theorem 4.4.7 carry over for any ground division ring D? What about the case $r = 0$?

To close let me add one more question.

Question (Open.) Is the range of π_r large enough so as for every $Z \not\subset N < U^{(r)}$, $PN \supset PU_g^{(r)}$, some $g > w(2)$ ($r \neq 0$)?

The preceding question can be answered in the negative in case $r = 0$. This uses a counterexample in [3].

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