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**ON THE MINORANT PROPERTIES IN  $C_p(H)$**

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## ON THE MINORANT PROPERTIES IN $C_p(H)$

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**We improve in two directions a recent result of B. Simon about the minorant property in  $C_p(H)$ ; the methods also allow us to extend a result of H. Shapiro and to obtain an apparently new result on matrices with positive entries.**

**Introduction.** Let  $H$  be a complex Hilbert space, which will always be the space  $l^2$  of square summable sequences or the space  $l_n^2$  of all  $n$ -tuples of complex numbers with the hermitian norm, equipped once and for all with an orthogonal basis  $(e_i)_{i \in I}$  ( $I$  finite or countable). Let  $K(H)$  be the set of all compact operators of  $H$ ; if  $C \in K(H)$ , put  $|C| = \sqrt{C^*C}$  and let  $\mu_1(C), \mu_2(C), \dots, \mu_i(C)$  be the eigenvalues of  $|C|$ , rearranged in decreasing order; if  $1 \leq p < \infty$ , put

$$\|C\|_p = \left( \sum_{i \in I} (\mu_i(C))^p \right)^{1/p} = (\text{Tr}|C|^p)^{1/p} = [\text{Tr}(C^*C)^{p/2}]^{1/p}$$

(where for  $A \in K(H)$ ,  $\text{Tr } A \stackrel{\text{def}}{=} \sum_{i \in I} \langle Ae_i, e_i \rangle$  is the trace of  $A$  whenever it exists).

Let  $C_p(H)$  be the set of all  $C \in K(H)$  such that  $\|C\|_p < \infty$ , ( $C_\infty(H) = K(H)$  and  $\|C\|_\infty = \mu_1(C)$  is the usual operator norm of  $C$ ). It is well known that  $C_p(H)$ , with the norm  $\|\cdot\|_p$ , is a Banach space ([11]).

For  $C \in K(H)$ , we put

$$c_{i,j} = \langle C(e_j), e_i \rangle = \text{Tr}(C \cdot (e_i \otimes e_j)) = \hat{C}(i, j).$$

In the last inequality,  $c_{i,j}$  is considered as a Fourier coefficient with respect to the orthonormal (in the Hilbert–Schmidt sense) system  $(e_i \otimes e_j)_{(i,j) \in I \times J}$  and this allows us to keep the analogy with the commutative case ([3], [4]) in the definitions below (recall that  $e_i \otimes e_j$  is the operator of rank one defined by:

$$(e_j \otimes e_j)(x) = \langle x, e_j \rangle e_j).$$

**DEFINITION 1.** If  $A, B \in K(H)$ , we say that  $A$  is a minorant of  $B$  if  $|a_{i,j}| \leq b_{i,j}$  for  $(i, j) \in I \times J$ , that is if  $|\hat{A}| \leq \hat{B}$ . We say that  $C_p(H)$  has the minorant property, and we abbreviate this to  $(m)$ -property, if

$$A, B \in C_p(H) \quad \text{and} \quad |\hat{A}| \leq \hat{B} \Rightarrow \|A\|_p \leq \|B\|_p.$$

We introduce a slightly different definition, the role of which appears in the third part of this paper (Theorem 3).

**DEFINITION 2.** We say that  $C_p(H)$  has the positive minorant property, and we abbreviate this to  $(m^+)$ -property, if

$$A, B \in C_p(H) \quad \text{and} \quad 0 \leq \hat{A} \leq \hat{B} \Rightarrow \|A\|_p \leq \|B\|_p.$$

We will begin by giving a survey a results already known about the  $(m)$ -property. The first questions is: for which  $p$  does  $C_p(H)$  have  $(m)$ ? It is well known, and simple to prove, that if  $p = 2k$ ,  $k$  an integer,  $C_p(H)$  has  $(m)$ : if  $A, B \in C_p(H)$  and  $|\hat{A}| \leq \hat{B}$ , then

$$|\widehat{A^*A}| \leq \widehat{B^*B}, \quad |(\widehat{A^*A})^k| \leq (\widehat{B^*B})^k$$

and

$$\|A\|_p^p = \text{Tr}(A^*A)^k \leq \text{Tr}(B^*B)^k = \|B\|_p^p.$$

It is also obvious that  $C_\infty(H)$  has  $(m)$ .

In ([8]), V. Peller proved that  $1 \leq p < \infty$ ,  $p \neq 2k$ ,  $k$  an integer,  $C_p(l^2)$  does not have  $(m)$ . The answer for  $C_p(H)$  is then analogous to the answer for the commutative case of  $L^p$ -spaces that has already been considered and solved by G. H. Hardy–J. Littlewood ([7]) and R. P. Boas ([2]). It follows from V. Peller’s result that there must exist some  $n$  for which  $C_p(l_n^2)$  does not have  $(m)$ , if  $p \neq 2k$ ,  $k$  an integer; but the proof, which relies on the theory of Hankel operators ([9]) and on ([11]) does not provide an estimate for  $n$ . In ([12]), B. Simon gives a simple proof (which relies only on [2]) that  $(m)$  fails for  $C_p(l^2)$  if  $p \neq 2k$ , and his proof gives an explicit  $n$  for which  $(m)$  fails for  $C_p(l_n^2)$ . More preciscely, B. Simon introduces the following definition.

**DEFINITION 3.** If  $1 \leq p < \infty$ ,  $p \neq 2k$ ,  $N(p)$  denotes the smallest integer  $n$  such that  $C_p(l_n^2)$  does not have  $(m)$ .

B. Simon proved in ([12])

- (i)  $N(p) \leq 2[p/2] + 5$
- (ii)  $N(p) \geq 3$  if  $p \geq 2$

(where  $[x]$  is the greatest integer  $v$  such that  $v \leq x$ ) and he remarked that it would be interesting to know the precise value of  $N(p)$ .

This paper will be divided into three parts. In the first part, we improve (i) to show that

(I) 
$$N(p) \leq [p/2] + 2.$$

(Equivalently, if  $p < 2(n - 1)$ ,  $C_p(l_n^2)$  does not have  $(m)$ ). We give a simple and explicit counter example (Theorem 1,  $\alpha$ ) whereas in ([2]) and ([12]) the methods are variational. Our counter example, in its variational version (Theorem 1,  $\beta$ ) extends to the non commutative case a result of H. Shapiro ([10]). It is also possible to formulate  $\beta$  in terms of the minorant property for some Banach space of operators  $C_\varphi(H)$ , where  $\varphi$  is an Orlicz function: we refer to ([4]) for this.

In the second part, we improve (ii) to show that

$$(II) \quad N(p) \geq 4 \quad \text{if } p > 4$$

(equivalently,  $C_p(l_3^2)$  has  $(m)$  if  $p > 4$ ; because of (I), it has not  $(m)$  if  $1 \leq p < 4, p \neq 2$ ). It follows from (I) and (II) that

$$N(p) = [p/2] + 2 \quad \text{if } 1 < p < 6, p \neq 2, 4.$$

We conjecture that this is the correct value of  $N(p)$  for all  $p, p \neq 2k$ . Equivalently, we conjecture that  $C_p(l_n^2)$  has  $(m)$  if  $p \geq 2(n - 1)$ .

In the third part, using the Gâteaux-differentiability of the  $C_p$ -norm, we prove that the  $(m^+)$  property is equivalent to the following:

$$\text{if } B \in C_p(H) \quad \text{and} \quad \hat{B} \geq 0, \quad \text{then } \widehat{(B^*B)^{p/2-1}B^*} \geq 0 \quad (\text{Theorem 3}).$$

In particular, from this result and from (II) we derive the following fact: if  $B$  is a  $(3 \times 3)$  matrix with positive entries, for  $\alpha \geq 2$  the matrix  $(B^*B)^\alpha$  has positive entries. An analogous result, with a more direct approach, was obtained by B. Viot (Theorem 5).

In view of these results, it would be interesting to know if the  $(m^+)$  property is actually weaker than  $(m)$ . We know no case in which  $(m^+)$  holds and not  $(m)$ . What we know about  $(m^+)$  is collected in Theorem 4.

*Part I.* We shall prove the following theorem.

**THEOREM 1.**

( $\alpha$ ) *Let  $1 \leq p < \infty, p \neq 2k$ . Then  $N(p) \leq [p/2] + 2$  (equivalently, if  $p < 2(n - 1)$ ,  $C_p(l_n^2)$  does not have  $(m)$ ).*

( $\beta$ ) *More generally, let  $\varphi$  be a strictly increasing  $C^\infty$  convex function on  $R^+$ , vanishing at zero and  $\psi(t) = \varphi(\sqrt{t})$ . Suppose that some derivative of  $\psi$  is negative at some point of  $]0, \infty[$ , and let  $n$  be the smallest integer such that this happens; then, there exists  $A$  and  $B$  in  $K(l_n^2)$  such that  $|\hat{A}| \leq \hat{B}$ , but  $\text{Tr}[\varphi(|A|)] > \text{Tr}[\varphi(|B|)]$ .*

Before giving the proof, we shall make some comments: another way to formulate ( $\beta$ ) is the following: if for all  $A, B \in K(l^2)$  such that

$|\hat{A}| \leq \hat{B}$ , we have:  $\text{Tr}[\varphi(|A|)] \leq \text{Tr}[\varphi(|B|)]$  (provided both members are finite) then all the derivatives of  $\psi$  must be positive on  $]0, \infty[$  and so, by the classical result of Bernstein ([13]),  $\varphi$  must be of the form:

$$\varphi(t) = \sum_{n \geq 0} a_n t^{2n}, \quad \text{with } a_n \geq 0 \text{ for all } n \geq 0.$$

So,  $(\beta)$  is the extension in the operator case of the result of Shapiro ([10]).

*Proof of Theorem 1 (α).* Let  $n = [p/2] + 2$ ,  $U$  be the unitary permutation operator of  $l_n^2$  defined by.

$$U(e_1) = e_2, \dots, U(e_{n-1}) = e_n, U(e_n) = e_1.$$

Let  $S$  be the symmetry operator defined by  $S(e_1) = -e_1$ ,  $S(e_i) = e_i$  if  $2 \leq i \leq n$ . (Any operator  $S$  such that  $S(e_i) = \varepsilon_i e_i$ ,  $\varepsilon_i = \pm 1$ ,  $\varepsilon_1 \cdots \varepsilon_n = -1$  would also work.)

Put  $A = I + SU$  and  $B = I + U$ . It is clear that  $|\hat{A}| \leq \hat{B}$  (in fact  $|\hat{A}| = \hat{B}$ ) and we claim that

$$(1) \quad \text{Tr}(A^*A)^q > \text{Tr}(B^*B)^q.$$

It is easy to compute explicitly the proper values of  $A$  and  $B$ , and therefore those of  $A^*A$  and  $B^*B$ . (Observe that  $A$  and  $B$  are normal matrices). In fact, if one puts  $\alpha = e^{i\pi/n}$ ,  $\omega = \alpha^2$ ,  $v_k = \sum_{j=1}^n \omega^j e_j$ ,  $w_k = \sum_{j=1}^n (\omega^k \alpha)^j e_j$  for  $1 \leq k \leq n$ , one has

$$U(v_k) = \bar{\omega}^k v_k, \quad \text{and} \quad SU(w_k) = \bar{\omega}^k \alpha w_k,$$

so that the eigenvalues of  $A^*A$  and  $B^*B$  are respectively  $|1 + e^{(2k+1)i\pi/n}|^2$  and  $|1 + e^{2ki\pi/n}|^2$ , the corresponding eigenvectors being, respectively,  $w_k$  and  $v_k$   $1 \leq k \leq n$ . But we shall use a different presentation.

In order to prove (1), observe that

$$(2) \quad A^*A = 2(I + V) \quad \text{and} \quad B^*B = 2(I + W)$$

$$\text{with } V = \frac{SU + (SU)^*}{2}, \quad W = \frac{U + U^*}{2}.$$

It is easy to check the following relations:

$$(3) \quad \text{Tr } U^k = \begin{cases} 0 & \text{if } k \not\equiv 0(n), \\ n & \text{if } k \equiv 0(n), \end{cases} \quad \text{and}$$

$$\text{Tr}(SU)^k = \begin{cases} 0 & \text{if } k \not\equiv 0(n), \\ (-1)^{\rho} n & \text{if } k \equiv \rho n. \end{cases}$$

Moreover, by the binomial formula, we have:

(4) If  $l \geq 1$ ,

$$V^l = \sum_{\substack{|k| \leq l \\ k \equiv l(2)}} a_{kl} (SU)^k \quad \text{and} \quad W^l = \sum_{\substack{|k| \leq l \\ k \equiv l(2)}} a_{kl} U^k$$

where  $a_{kl}$  are strictly positive coefficients. Put, for  $l \geq 1$ ,

$$C_l = (1/l!)q(q-1) \cdots (q-l+1)$$

and note that:

$$(5) \quad \sum |C_l| < \infty$$

( $|C_{l+1}/C_l| = 1 - (q+1)/l + O(1/l^2)$ , so that  $|C_l| = O(l^{-q-1})$  when  $l$  tends to infinity),

$$(6) \quad C_1 > 0, \dots, C_{n-1} > 0, C_n < 0, \quad \text{and} \quad \text{sign } C_{n+\Gamma} = (-1)^{r+1} \quad \text{if } r \geq 0.$$

Using (5) and the fact that  $\text{Max}(\|V\|_\infty, \|W\|_\infty) \leq 1$ , we can write:

$$(7) \quad (I + V)^q - (I + W)^q = \sum_{l=1}^{\infty} C_l [V^l - W^l].$$

Taking the traces of both members, and taking account of (3) and (4), we get:

$$(8) \quad \begin{aligned} \text{Tr}(I + V)^q - \text{Tr}(I + W)^q &= \sum_{l \geq 1} C_l \sum_{\substack{|k| \leq l \\ k \equiv l(2)}} a_{kl} [\text{Tr}(SU)^k - \text{Tr } U^k] \\ &= \sum_{\substack{r \in \mathbb{Z} \\ k = (2r+1)n}} \sum_{\substack{l \geq |k| \\ l \equiv k(2)}} (-2n) a_{kl} C_l. \end{aligned}$$

The indices  $l$  which appear on the right-side of (8) are all of the form:

$$\begin{aligned} l &= |k| + 2l' = |2r + 1|n + 2l' = (2r' + 1)n + 2l' \\ &= n + 2l'' \quad \text{with } l'' \geq 0. \end{aligned}$$

By (6),  $C_l < 0$ , so the right-hand side of (8) is a sum of positive terms and

$$\text{Tr}(I + V)^q > \text{Tr}(I + W)^q.$$

In view of (2), this implies (1), and  $(\alpha)$  is proved.

$(\beta)$  We shall need the following relations, which are obvious consequences of (3) and (4)

$$(9) \quad \begin{cases} \text{Tr } V^l = \text{Tr } W^l = 0 & \text{if } 1 \leq l \leq n-1, l \text{ odd,} \\ \text{Tr } V^l = \text{Tr } W^l = a_{0l} & \text{if } 1 \leq l \leq n-1, l \text{ even,} \\ \text{Tr } V^n - \text{Tr } W^n = -4n2^{-n} < 0. \end{cases}$$

Let  $n$  be as in the hypotheses of  $(\beta)$ ,  $\xi$  be a positive number such that  $\psi^{(n)}(\xi) < 0$ ,  $U, S, I$  as before,  $a \geq 0$  and  $b \geq 0$  such that  $a^2 + b^2 = \xi$  with

$b \downarrow 0$  and  $a \uparrow \xi$ , and put:

$$A = aI + bSU, \quad B = aI + bU,$$

(so that  $A$  and  $B$  belong to a neighborhood of  $\xi I$ ). It is clear that  $|\hat{A}| \leq \hat{B}$ , (in fact,  $|\hat{A}| = \hat{B}$ ) and we claim that

$$(10) \quad \text{Tr}[\psi(A^*A)] > \text{Tr}[\psi(B^*B)] \quad \text{for } b \text{ small enough.}$$

In order to prove (10), observe that

$$(11) \quad A^*A = \xi I + 2abV \quad \text{and} \quad B^*B = \xi I + 2abW.$$

$V$  and  $W$ , being normal operators, can be diagonalized so that the following symbolic Taylor formulas are valid:

$$\begin{aligned} \psi(A^*A) &= \sum_{l=0}^{n-1} \frac{(2ab)^l}{l!} \psi^{(l)}(\xi) V^l + O(b^{n+1}), \\ \psi(B^*B) &= \sum_{l=0}^{n-1} \frac{(2ab)^l}{l!} \psi^{(l)}(\xi) W^l + \frac{(2ab)^n}{n!} \psi^{(n)}(\xi) W^n + O(b^{n+1}). \end{aligned}$$

Subtracting and taking traces, we get in view of (9):

$$(12) \quad \text{Tr}[\psi(A^*A)] - \text{Tr}[\psi(B^*B)] = -4n2^{-n} \frac{(2ab)^n}{n!} \psi^{(n)}(\xi) + O(b^{n+1}).$$

Since  $\psi^{(n)}(\xi) < 0$ , (12) proves (10) for  $b$  small enough.

*Part 2.* We shall prove the following theorem:

**THEOREM 2.** *Let  $p \geq 4$ ,  $A$  and  $B$  two  $(3 \times 3)$  matrices such that  $|\hat{A}| \leq \hat{B}$ . Then  $\|A\|_p \leq \|B\|_p$ .*

*Equivalently,  $N(p) \geq 4$  if  $p \geq 4$ ,  $p \neq 2k$ .*

We shall need the two following lemmas, the first of which plays a fundamental role in the theory of  $C_p$ -spaces.

**LEMMA 1 ([6], [11]).** *Let  $a_1 \geq \dots \geq a_N \geq 0$  and  $b_1 \geq \dots \geq b_N \geq 0$ . Let  $\varphi$  be an increasing convex function on  $[0, \infty[$ . Suppose that*

$$\sum_1^k a_j \leq \sum_1^k b_j \quad \text{for } k = 1, \dots, N.$$

*Then,  $\sum_1^N \varphi(a_j) \leq \sum_1^N \varphi(b_j)$ .*

LEMMA 2. Let  $0 < \alpha_1 < \dots < \alpha_n$  be positive exponents, and let  $P(x) = a_0 + a_1x^{\alpha_1} + \dots + a_nx^{\alpha_n}$  a "polynomial" with real coefficients  $a_i$ , not all zero. Put

$$V = \#\{i/a_i a_{i+1} < 0\}, \quad Z = \#\{x > 0/P(x) = 0\}.$$

Then  $Z \leq V$ .

Lemma 2 is a generalization of the well-known theorem of Descartes for polynomials; a sketch of the proof can be found in ([1]), but the slavish imitation of Descartes's proof is quicker.

*Proof of Theorem 2.* Let  $A$  and  $B$  be two  $(3 \times 3)$  matrices such that  $|\hat{A}| \leq \hat{B}$ . Let us denote by  $\lambda_1 \geq \lambda_2 \geq \lambda_3$  (resp.  $\mu_1 \geq \mu_2 \geq \mu_3$ ) the eigenvalues of  $A^*A$  (resp.  $B^*B$ ) rearranged in decreasing order. Let  $q = p/2 > 2$  we claim that

$$(13) \quad \sum_1^3 \lambda_i^q \leq \sum_1^3 \mu_i^q.$$

To prove (13), we may as well assume that

$$(14) \quad \sum_1^3 \lambda_i = \sum_1^3 \mu_i.$$

In fact, the extreme points of the parallelotope of  $R^9$  defined by  $|x_{ij}| \leq b_{ij}$  are the points  $(x_{ij})$  such that  $|x_{ij}| = b_{ij}$ ,  $1 \leq i, j \leq 3$ ; so that one has a convex combination  $a_{ij} = \sum_k \lambda_k a_{ij}^{(k)}$ , with  $|a_{ij}^{(k)}| = b_{ij}$  for all  $i, j, k$ . If  $A^{(k)}$  is the element of  $K(I_3^2)$  defined by  $\langle A^{(k)}(e_j), e_i \rangle = a_{ij}^{(k)}$ , we then have  $A = \sum \lambda_k A^{(k)}$ . For the operators  $A^{(k)}$ , (14) holds because, for every  $k$ ,

$$\sum_i \lambda_i^{(k)} = \sum_{i,j} |a_{ij}^{(k)}|^2 = \sum_{i,j} b_{ij}^2 = \sum \mu_i.$$

(If  $\lambda_1^{(k)} \geq \lambda_2^{(k)} \geq \lambda_3^{(k)}$  are eigenvalues of  $A^{(k)*}A^{(k)}$ .) If we are able to deduce the result when (14) holds, we have:  $\|A^{(k)}\|_p \leq \|B\|_p$  for all  $k$ , and then

$$\|A_p\|_p \leq \sum \lambda_k \|A^{(k)}\|_p \leq \sum \lambda_k \|B\|_p = \|B\|_p.$$

So, in the following, we shall assume that (14) holds.

Suppose that (13) is false and consider the continuous function

$$g(r) = \sum \lambda_i^r - \sum \mu_i^r, \quad \text{so that } g(q) > 0.$$

Let  $\nu$  be an integer such that  $\nu > q$ ; since  $C_4(I_3^2)$  and  $C_{2\nu}(I_3^2)$  have  $(m)$ , we have:

$$(15) \quad g(2) \leq 0 \quad \text{and} \quad g(\nu) \leq 0.$$



By the intermediate value theorem, there exists  $q_1 \in [2, q[$  and  $q_2 \in ]q, v]$  such that

$$(16) \quad g(q_1) = g(q_2) = 0 \quad (q_1 < q_2).$$

Notice that

$$(7) \quad \lambda_1 \leq \mu_1.$$

In fact,  $\lambda_1 = \|A\|_\infty^2$ ,  $\mu_1 = \|B\|_\infty^2$ , and  $C_\infty(l_3^2)$  trivially has  $(m)$ . We shall prove that (14) and (16) contradict Lemma 2 for some property chosen polynomial  $P$ . Let us distinguish two cases:

*Case 1.*  $\lambda_3 \geq 0$ .

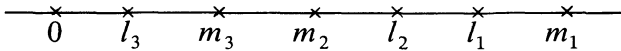
If  $\lambda_1 + \lambda_2 \leq \mu_1 + \mu_2$ , (14), (17) and the application of Lemma 1 to  $\varphi(t) = t^q$  gives us (13). So, we may assume  $\lambda_1 + \lambda_2 > \mu_1 + \mu_2$ . Because of (17) we then have  $\lambda_2 > \mu_2$  and because of (14)

$$\lambda_3 = \mu_3 + (\mu_1 + \mu_2 - \lambda_1 - \lambda_2) < \mu_3.$$

Multiplying all  $\lambda_i$ 's,  $\mu_i$ 's by the same constant, we may assume that  $\lambda_3 \geq 1$ ; so that if

$$l_i = \text{Log } \lambda_i \quad \text{and} \quad m_i = \text{Log } \mu_i$$

we have the following situation



We can write

$$g(r) = P(x) = \sum x^{l_i} - \sum x^{m_i} \quad \text{with } x = e^r.$$

In view of the preceding picture, let us rewrite:

$$P(x) = x^{l_3} - x^{m_3} - x^{m_2} + x^{l_2} + x^{l_1} - x^{m_1}$$

with the notations of Lemma 2, we have  $V = 3$  and  $Z \geq 4$ , in fact, due to (14) and (16),  $P$  vanishes at  $1, e, e^{q_1}, e^{q_2}$ . This proves (13) by contradiction.

*Case 2.*  $\lambda_3 = 0$ .

By (18)  $\mu_3 > 0$  and as before we may assume  $\mu_3 > 1$ , we then consider  $Q(x) = -x^{m_3} - x^{m_2} + x^{l_2} + x^{l_1} - x^{m_1}$ , and have  $V \leq 2, Z \geq 3$ , which again proves (13) by contradiction.

*Part 3.* First, recall the following lemma on the Gâteaux-differentiability of the norm in normed spaces  $E$  with strictly convex dual  $E'$ .

LEMMA 3 ([5]). *Let  $E$  be a complex normed space, and  $b \in E$ ,  $b \neq 0$ ; assume that the norm is smooth at  $b$ , that is to say: there exists a unique  $\tilde{b} \in E'$ ,  $\|\tilde{b}\| = 1$ ,  $\langle \tilde{b}, b \rangle = \|b\|$ . Then, the norm in  $E$  is Gâteaux-differentiable at  $b$ ; more precisely,*

$$\forall c \in E, \quad \lim_{\substack{t \in \mathbb{R} \\ t \rightarrow 0}} \frac{\|b + tc\| - \|b\|}{t} = R[\langle \tilde{b}, c \rangle].$$

Observe that the lemma is applicable when the norm of  $E'$  is strictly convex, a fortiori when it is uniformly convex; if  $E = C_p(H)$ ,  $1 < p < \infty$ ,  $E' = C_{p'}(H)$  is uniformly convex by the Clarkson-McCarthy inequalities ([11]), so that Lemma 3 may be applied to  $C_p(H)$  ( $1 < p < \infty$ ). If  $B \in C_p(H)$ , and  $\|B\|_p = 1$ , we easily compute:

$$(20) \quad \tilde{B} = (B^*B)^{p/2-1} B^*.$$

(This has a meaning even when  $1 \leq p < 2$ , and  $B$  is not injective by putting  $\tilde{B}(x) = 0$  if  $B(x) = 0$ .) It is clear that  $\langle \tilde{B}, B \rangle = \text{Tr } \tilde{B}B = 1\|B\|_p$  and to check that  $\|\tilde{B}\|_{p'} = 1$ , use a polar decomposition of  $B$  and the invariance of the  $C_p$ -norm under multiplication by a unitary operator; by extension, in the following  $\tilde{B}$  will always be given by (20), even when  $\|B\|_p \neq 1$ .

THEOREM 3. *Let  $1 < p < \infty$*

(1) *the following are equivalent:*

(a)  $C_p(H)$  *has*  $(m^+)$

(b)  $B \in C_p(H)$  *and*  $\hat{B} \geq 0 \Rightarrow \hat{\tilde{B}} \geq 0$ . (In particular  $\overline{(B^*B)^{p/2}} \geq 0$ ).

(2) *In particular, if  $B$  is a  $(3 \times 3)$  matrix such that  $\hat{B} \geq 0$ , then*

$$(B^*B)^\alpha \geq 0 \quad \text{for } \alpha \geq 2.$$

*Proof.* (a)  $\Rightarrow$  (b). Let  $B \in C_p(H)$ , with  $\hat{B} \geq 0$ ; we may assume that  $\|B\|_p = 1$ ; let  $C \in C_p(H)$ , with  $\hat{C} \geq 0$ ; first, it is clear that  $\tilde{B}$  has real entries ( $B^*B$  can be diagonalized by the real orthogonal group); besides, if  $t \geq 0$ , we get from (a):

$$\|B\|_p \leq \|B + tC\|_p,$$

so that by Lemma 3

$$\lim_{t \rightarrow 0} \frac{\|B + tC\|_p - \|B\|_p}{t} = \mathcal{R}[\text{Tr } \tilde{B}C] = \text{Tr } \tilde{B} \geq 0.$$

Testing this with the operators of rank one  $C = e_i \otimes e_j$ , we get (b).

(b)  $\Rightarrow$  (a). Let  $B, C \in C_p(H)$  with  $\hat{B} \geq 0$ ,  $\hat{C} \geq 0$ , let us prove that  $\|B\|_p \leq \|B + C\|_p$ . Observe that, if  $0 \leq t_0 \leq 1$

$$\left[ \frac{d}{dt} \|B + tC\|_p \right]_{t=t_0} = \text{Tr}[(\widehat{B + t_0 C})C] \geq 0 \text{ by (b),}$$

so that the function  $t \rightarrow \|B + tC\|_p$  is increasing.

(2) follows trivially from (1) and from Theorem 2.

In view of Theorem 3, the property  $(m^+)$  is of interest: it is quite simple to verify that the properties  $(m)$  and  $(m^+)$  are equivalent in  $C_p(l^2)$  (Theorem 4(c) below), but in the finite-dimensional case and if  $p > 2$ , we do not know if  $(m^+)$  is weaker than  $(m)$ . We shall prove the following results about  $(m^+)$ :

**THEOREM 4.**

(a) If  $1 \leq p < 2$ ,  $C_p(l_n^2)$  does not have  $(m^+)$ .

(b) If  $n$  is even,  $n \geq 4$ ,  $C_p(l_n^2)$  does not have  $(m^+)$  if  $1 \leq p < n - 2$ ,  $p \neq 2k$ .

If  $n$  is odd,  $n \geq 5$ ,  $C_p(l_n^2)$  does not have  $(m^+)$  if  $1 \leq p < n - 3$ ,  $p \neq 2k$ .

(d) If  $p \geq 2$  and  $C_p(l_n^2)$  has  $(m^+)$ , then  $A, B \in K(l_n^2)$  and  $|\hat{A}\hat{\exists} \leq \hat{B}$  implies the following

$$\begin{cases} \|A\|_p \leq (2^p - 1)^{1/p} \|B\|_p & \text{if } \hat{A} \text{ is real,} \\ \|A\|_p \leq 2(2^p - 1)^{1/p} \|B\|_p & \text{if } \hat{A} \text{ is complex.} \end{cases}$$

*Proof.* (a) Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 1 & d \end{bmatrix} \quad \text{for } d > 0.$$

We shall prove that  $\|A\|_p > \|B\|_p$  for  $d$  small enough (depending on  $p$ ); the eigenvalues of  $B$  are:

$$\begin{aligned} \lambda_1 &= \frac{1}{2} (d + 1 + \sqrt{d^2 - 2d + 5}) \quad \text{and} \\ \lambda_2 &= \frac{1}{2} (d + 1 - \sqrt{d^2 - 2d + 5}). \end{aligned}$$

Since  $B$  is symmetric, we have  $\mu_1(B) = |\lambda_1| = \lambda_1$  and  $\mu_2(B) = \|\lambda_2\| = -\lambda_2$  for  $d$  small enough, so that:

$$\begin{aligned} 2^p \|B\|_p^p &= (d + 1 + \sqrt{d^2 - 2d + 5})^p + (-d - 1 + \sqrt{d^2 - 2d + 5})^p \\ &= f(d) \end{aligned}$$

and we claim that  $f'(0) < 0$  if  $p < 2$ . In fact:

$$\frac{1}{p} f'(0) = \left(1 - \frac{1}{\sqrt{5}}\right) (1 + \sqrt{5})^{p-1} - \left(1 + \frac{1}{\sqrt{5}}\right) (\sqrt{5} - 1)^{p-1},$$

so that  $f'(0) < 0$  iff  $(\sqrt{5} + 1)^{p-2} < (\sqrt{5} - 1)^{p-2}$ , which is equivalent to  $p < 2$ .

(b) the proof is similar to a proof of ([10]).

We shall first examine the case  $n$  even; put  $m = n/2$ ,  $q = p/2$ . We then have  $q < m - 1$  and we may clearly assume  $q > m - 2$ . By (b) of Theorem 3, it suffices to find an  $(n \times n)$  matrix  $B$  such that  $\hat{B} \geq 0$ , and such that  $(B^*B)^q$  has some negative entry: this will be the case if  $\text{Tr}[(B^*B)^q C] < 0$  for some  $C$  with positive entries; we shall take:

$$B = \sqrt{1 - r^2} I + rU, \text{ where } U \text{ is as in Theorem 1 and } 0 < r < 1, \\ C = U^m,$$

$B^*B = I + \rho W$  where  $\rho = 2r\sqrt{1 - r^2} \rightarrow 0$  when  $r \rightarrow 0$ , and where  $W$  is as in (2). Observe that if  $C_l = (1/l!)q(q-1)\cdots(q-l+1)$  for  $l \geq 1$ , we have  $C_m < 0$  and

$$(21) \quad (B^*B)^q = I + \sum_{l=1}^{m-1} C_l \rho^l W^l + C_m \rho^m W^m + O(\rho^{m+1}).$$

When one computes  $W^l C$  for  $0 \leq l \leq m - 1$ , only the following powers of  $U$  appear:

$$m, \quad m \pm 1, \quad m \pm 2, \dots, m \pm (m - 1).$$

In view of (3), we then have:

$$(22) \quad \text{Tr } W^l C = 0 \quad \text{if } 0 \leq l \leq m - 1.$$

On the other hand, we have

$$(23) \quad \text{Tr } W^m C = 2^{-m+1} \text{Tr } I = n2^{-m+1} > 0.$$

(21), (22), (23) give:

$$(24) \quad \text{Tr}[(B^*B)^q C] = n2^{-m+1} \rho^m c_m + O(\rho^{m+1})$$

and the right-hand side of (24) is negative for  $r$  small enough. The case  $n$  odd is treated in a similar way: just put  $m = (n - 1)/2$ ,  $B$  as before, and  $C = U^{m+1}$ .

(c) A close examination of ([12]) shows that in fact  $C_p(l^2)$  has  $(m^+)$  iff  $p = 2k$  or  $p = \infty$  so that  $C_p(l^2)$  has  $(m^+)$  iff  $C_p(l^2)$  has  $(m)$ . One can also argue directly as follows: if  $C_p(l^2)$  has  $(m^+)$ , then trivially:  $|\hat{A}| \leq \hat{B}$  implies  $\|A\|_p \leq 4\|B\|_p$ . But it follows from the tensor product argument of ([12]) that if, for a certain constant  $M$

$$|\hat{A}| \leq \hat{B} \quad \text{implies} \quad \|A\|_p \leq M\|B\|_p.$$

then  $C_p(l^2)$  has  $(m)$ ; so we conclude that  $(m^+)$  is equivalent to  $(m)$  for  $C_p(l^2)$ .

(d) It is enough to deal with the case  $\hat{A}$  real; put  $A^+ = (a_{ij}^+)$ ,  $A^- = (a_{ij}^-)$ , we have  $A = A^+ - A^-$  and  $0 \leq A^+ + A^- \leq \hat{B}$ . By the Clarkson-MacCarthy inequality ([1]) for  $p \geq 2$

$$\|A^+ - A^-\|_p^p + \|A^+ + A^-\|_p^p \leq 2^{p-1} (\|A^+\|_p^p + \|A^-\|_p^p).$$

So that using  $(m^+)$  twice we get:

$$\|A^+ - A^-\|_p^p \leq (2^p - 1) \|A^+ + A^-\|_p^p \leq (2^p - 1) \|B\|_p^p.$$

Let us now explain Virot's result, which is as follows:

**THEOREM 5 (B. Virot).** *Let  $E$  be a positive-definite  $(3 \times 3)$  matrix with positive entries,  $f$  a real positive function on  $[0, \infty[$  such that  $f(t)$  is convex and  $f(\sqrt{t})$  concave; then  $f(E)$  has positive entries; in particular,  $E^\alpha$  has positive entries if  $\alpha \geq 1$ .*

*Proof.* By a standard perturbation argument, we may assume that the eigenvalues of  $E$  are distinct, let them be  $\lambda_1 < \lambda_2 < \lambda_3$  one easily computes real numbers  $a_0, a_1, a_2$  such that:

$$(25) \quad f(E) = a_0 I + a_1 E + a_2 E^2.$$

If  $\Delta = (\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)$ , one finds in particular:

$$\Delta a_1 = (f(\lambda_2) - f(\lambda_1))(\lambda_3^2 - \lambda_1^2) - (f(\lambda_2) - f(\lambda_1))(\lambda_2^2 - \lambda_1^2),$$

$$\Delta a_2 = (f(\lambda_3) - f(\lambda_1))(\lambda_2 - \lambda_1) - (f(\lambda_1))(\lambda_3 - \lambda_1).$$

So that  $a_1 > 0$  iff

$$\frac{f(\lambda_2) - f(\lambda_1)}{\lambda_2^2 - \lambda_1^2} > \frac{f(\lambda_3) - f(\lambda_1)}{\lambda_3^2 - \lambda_1^2}.$$

This will be true if  $f(\sqrt{t})$  is concave. In the same way  $a_2 > 0$  iff

$$\frac{f(\lambda_3) - f(\lambda_1)}{\lambda_3 - \lambda_1} > \frac{f(\lambda_2) - f(\lambda_1)}{\lambda_2 - \lambda_1}.$$

This will be true if  $f$  is convex; so that under the hypotheses of Theorem 5, (25) shows that  $(f(E))_{ij} \geq 0$  if  $i \neq j$ ; we know nothing about the sign of  $a_0$ , but it is a priori clear, that  $(f(E))_{ii} \geq 0$  since  $f(E)$  is a positive-definite operator; this can be applied to the function  $f(t) = t^\alpha$  for  $1 \leq \alpha \leq 2$ ; but, then,  $E^\alpha$  has positive entries for all  $\alpha \geq 1$  because  $E^\alpha = E^\mu E^\beta$  with  $\mu$  a positive integer and  $\beta$  a real number such that  $1 \leq \beta \leq 2$ .

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Maurice Chacron, Nonisotropic unitary spaces and modules with Cauchy-Schwarz inequalities .....	1
Myriam Dechamps-Gondim, Françoise Piquard and H. Queffélec, On the minorant properties in $C_p(H)$ .....	89
Klaus Floret and V. B. Moscatelli, On bases in strict inductive and projective limits of locally convex spaces .....	103
Norman Joseph Goldstein, Degenerate secant varieties and a problem on matrices .....	115
Harold Morris Hastings and Stefan Waner, $G$ -bordism with singularities and $G$ -homology .....	125
Takesi Isiwata, Clopen realcompactification of a mapping .....	153
Hisao Kato, Concerning hyperspaces of certain Peano continua and strong regularity of Whitney maps .....	159
Elyahu Katz and Sidney Allen Morris, Free products of topological groups with amalgamation .....	169
Kyewon Koh Park, Nice dense subsets for ergodic flows and Bernoulli flows .....	181
Francis Pastijn and Peter George Trotter, Lattices of completely regular semigroup varieties .....	191
Rae Michael Andrew Shortt, Reticulated sets and the isomorphism of analytic powers .....	215
David A. Stegenga and Kenneth R. Stephenson, Generic covering properties for spaces of analytic functions .....	227
M. V. Subba Rao and R. Sitaramachandra Rao, On some infinite series of L. J. Mordell and their analogues .....	245