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Harold Morris Hastings and Stefan Waner

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HAROLD M. HASTINGS AND STEFAN WANER

The bordism and cobordism theories of singular G-manifolds of specified kinds are used to represent various ordinary G-homology and cohomology theories, and their relationship to each other, as well as their relationship to non-singular G-bordism, is studied.

1. Introduction. Sullivan once pointed out that ordinary homology may be viewed geometrically as a bordism theory with singularities. This has been formally established by Baas in [**1**] and by Buoncristiano, Rourke and Sanderson in [**3**]. Dually, the associated cobordism theories represent ordinary cohomology.

Let G be a finite group. One then has several notions of what is meant by ordinary G-cohomology. The first to be proposed was the functor $X \mapsto H^*(X \times_G EG)$ for a G-space X, where EG denotes the universal contractible free G-space. Subsequently, Bredon [**2**] and Illman [**6**] described a theory of the following type. Let $\mathscr{G}$ denote the category whose objects are the G-spaces G/H for subgroups H and whose morphisms $G/H \to G/K$ are the G-equivariant maps. A contravariant coefficient system is then a contravariant functor T from $\mathscr{G}$ to the category of abelian groups. The associated ordinary G-cohomology theory is a generalized G-cohomology theory (see [**2**]) with dimension axiom of the form

$$H_G^n(G/K; T) = \begin{cases} T(G/K) & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

More recently, it has been shown, [**18**], [**8**], [**9**], that this theory extends to an $RO(G)$-graded theory when the coefficient system T extends to a Mackey functor (in that it admits a transfer). In this theory, the coefficient system A: $G/H \mapsto A(H)$, the Burnside ring of H, then assumes the role played by Z-coefficients nonequivariantly. As yet, no geometric description of cycles in the non-integrally graded part of the dual theory, $H_*^G(X)$ exists; if V is a non-trivial G-module, how does one view the classes in $H_V^G(X; A)$?

It is not clear how to extend Sullivan's ideas to represent these G-cohomology theories as singular cobordism theories. Moreover, two distinct theories would appear to be required. In **[17]** the second-named author relates the two types of ordinary cohomology theory by showing that $H^*(X \times_G EG; Z)$ is a localization of the ($RO(G)$-graded) theory $H_G^*(X; A)$, obtained by inverting an equivariant Chern class in $H_G^{V-v}(\text{point}; A)$, where V denotes the regular representation and $v = \dim V$. This suggests that the associated singular cobordism theories ought to be similarly related, and suggests further the possibility of a relationship between the corresponding non-singular cobordism theories. Further, the cobordism theories Ω_G^* should admit a product, even in the presence of singularities, in order that the required localizations may be carried out. Unfortunately, Baas' model of singular bordism seems to admit no exterior product in general, and his question in **[1]** to that effect has not, as far as we know, been asnwered. There are further technical difficulties regarding an adaptation of his work to the equivariant case arising from the failure in general of a G-transversality theorem. This failure seems also to preclude the use of cone-type singularities in the style of **[3]**.

The purpose of this paper is to

(i) exhibit the required singular bordism and cobordism theories, thereby giving the geometric interpretations referred to above, and to

(ii) show the relationship between the two kinds of theory as a localization obtained from the multiplicative structure by inverting a geometrically described Chern class.

The bordism theories we adapt are the $RO(G)$-graded theories of Pulikowski **[11]** and Kosniowski **[7]** (see also **[14]**), as well as the classical G-bordism theories of Conner/Floyd **[4]** and tom Dieck **[5]**. The results we prove take the following form. (Precise results are stated in §6.)

THEOREM A. *The Bredon-Illman cohomology theory, together with its $RO(G)$-graded extension, is represented as singular $RO(G)$-graded cobordism. Here, the singularities are taken to have codimension* 2. (*See* §4 *for a description of such singularities.*)

THEOREM B. *The cohomology theory $H^*(X \times_G EG; Z)$, viewed as a cohomology theory on X, is represented by "stable" tom Dieck cobordism with singularities of codimension* 2.

THEOREM C. *The singular theories in A and B above are related via inversion of a canonical Chern class in the $RO(G)$ theory, and such a relationship exists for singularities of arbitrary codimension.*

The paper is arranged as follows. In §§2 through 5, the singular theories are introduced and basic results, including closure under products and excision, are proved. In §6, the main results are precisely stated, and their proof occupies the rest of the paper.

We would like to thank R. Stong for helpful conversations.

2. G-manifolds with dimension in $RO(G)$. Let G be a finite group. We recall the following definition of Pulikowski in **[11]**. Let $\gamma \in RO(G)$ be represented by the virtual G-module $V - W$ and let M be a smooth G-manifold. Then M is said to have dimension γ if $M \times D(W)$ is locally of the form $G \times_H D(V)$ for some subgroup $H \in G$. Here, $D(Y)$ denotes the unit disc in the orthogonal G-module Y. It follows that M has the local form $G \times_H D(Y)$, where Y represents γ, considered as an element of $RO(H)$.

The G-bordism of such manifolds is considered in **[11]**, **[7]** and in **[14]**; one has equivariant bordism theories Ω^G_* indexed on $RO(G)$ and possessing suspension maps

$$\sigma\colon \Omega^G_\gamma(X) \to \Omega^G_{\gamma+V}(\Sigma^V X)$$

for based G-spaces X, where $\Sigma^V X = X \wedge S^V$, S^V denoting the one-point compactification of the G-module V. These suspension maps fail, in general, to be isomorphisms, and in **[14]**, Waner defines "stable" G-bordism theories

$$\underline{\Omega}^G_\gamma(X) = \operatorname*{colim}_V \Omega^G_{\gamma+V}(\Sigma^V X)$$

in the spirit of tom Dieck **[5]**, where V runs through all G-invariant submodules of R^∞, R here denoting the regular representation. Due to the lack of G-transversality, classes in $\underline{\Omega}^G_*$ are not in general represented by bordism classes of G-manifolds, but instead by (bordism) classes of G-maps

$$(M, \partial M) \to (D(V), S(V)),$$

where $S(V) = \partial D(V)$ for some G-module V. (See **[14]**.)

As is customary, one may restrict the category of G-manifolds under consideration to be unitary, framed or G-oriented. These theories are described in **[14]**. When referring specifically to one of these theories, we shall use a subscript A = U, Fr or SO, while the unadorned symbol Ω will refer to any one of these theories.

If one drops the requirement that G-manifolds be modelled locally on a fixed virtual representation, one obtains the classical G-bordism theories of Conner/Floyd and tom Dieck [4], [5]. We shall denote these theories by B_*^G and define

$$\underline{B}_n^G(X) = \operatorname*{colim}_V B_{n+v}^G(\Sigma^V X)$$

for based G-spaces X, where $v = \dim V$. Thus $\underline{B}_*^G$ is precisely tom Dieck's stable G-bordism theory described in [5]. Note that B_*^G and $\underline{B}_*^G$ are Z-graded, and we have suspension isomorphisms of the form

$$\sigma\colon \underline{B}_n^G(X) \cong \underline{B}_{n+v}^G(\Sigma^V X).$$

Again, classes in $\underline{B}_*^G$ are not in general represented by bordism classes of G-manifolds, and it is this phenomenon which forces one to use mapping cylinder type singularities for a suitable singular theory.

3. *G*-manifolds with mapping cylinder singularities. Here, we define general classes of G-manifolds with mapping cylinder singularities, and describe the examples of interest. All G-manifolds we consider are compact and smooth, and possibly with boundary, unless otherwise stated.

DEFINITION 3.1. A G-space M is called a G-manifold with mapping cylinder singularities if it admits a decomposition $M = N \cup_K Mf$, where

(i) N is a G-manifold with (possibly empty) boundary ∂N;

(ii) K is a codimension-0 G-submanifold, possibly empty or with boundary, of ∂N, and the closure in ∂N of $N - K$ is a similar codimension-0 G-submanifold;

(iii) Mf denotes the mapping cylinder of a (continuous) G-map of pairs $f\colon (K, \partial K) \to (L, L')$. L' may be empty if ∂K is empty. (See Figure 1.)

DEFINITION 3.2. The boundary ∂M of a G-manifold with singularities is given by $M = \mathrm{cl}(\partial N - K) \cup_K Mf|$, where $\mathrm{cl}(\partial N - K)$ is the closure in

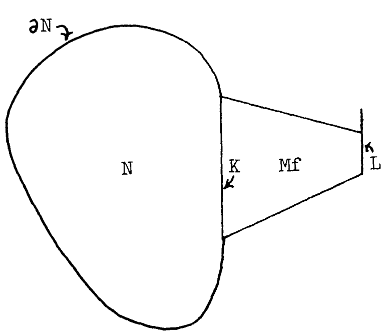

FIGURE 1

M of $\partial N - K$, and where $f|$ denotes the restriction of f to a G-map $\partial K \to L'$.

This definition makes the boundary ∂M of a G-manifold with singularities again a G-manifold with singularities.

DEFINITION 3.3. We refer to L as the singular set of M, and to Mf as the singular neighbourhood of M. On occasion, when the context is clear, we shall refer to each component of L as a singularity, in which case we refer to the singularities of M.

DEFINITION 3.4. We inductively define a G-manifold with mapping cylinder singularities of depth n as follows. A G-manifold without singularities will be said to have depth 0, while a G-manifold with singularities of depth (at most) $n + 1$ is a G-manifold with singularities whose singularities are themselves G-manifolds with singularities of depth (at most) n.

Note that the depth of such a G-singularity is not a well-defined feature of its structure; for example, a smooth G-manifold may be given the structure of a G-manifold with singularities of depth 1 by selecting an arbitrary G-submanifold as its singular set. In this case, the depth of its singularity is no less than 1. We shall refer to a G-manifold with (mapping cylinder) singularities of some depth n as a G-manifold with singularities of finite depth.

All singular G-manifolds we consider will be restricted in the above sense, and we shall further restrict the class of maps used to form the singularities via the following weak pullback requirement.

DEFINITION 3.5. Consider a class of G-maps $f\colon K \to L$ from G-manifolds K to G-manifolds with singularities L which satisfy the following condition. Let C be a closed invariant subset of L, and let U be an open invariant neighborrhood of C in L. Let C' and U' be the pullbacks of C and U respectively under the map f. We shall require the existence of a closed invariant manifold neighbourhood K' of C' in U', and a closed invariant manifold (with singularities) neighbourhood L' of C in U such that the restriction of f maps K' to L'. All maps are required to be G-maps of G-manifolds with boundaries and to preserve boundaries. We shall refer to such a class of maps as closed under thickened weak pullbacks.

In the case of no group action, if f were a submersion of manifolds without singularities, one could simply choose L^1 to be any closed manifold neighbourhood of C in U and K^1 to be $f^{-1}(L')$.

More generally, one needs to choose a closed invariant manifold (with singularities) neighbourhood L^1 of C in U, and then choose a closed invariant manifold neighbourhood K^1 of C^1 in $f^{-1}(\text{int}(L^1))$.

EXAMPLES 3.6. Examples of classes of G-maps closed under thickened weak pullbacks.

(i) The class of all smooth G-fibrations whose fibers are constrained to lie in some fixed "category of fibers". (See [**19**].) This class is closed because $f^{-1}(L^1)$ is a closed invariant manifold neighbourhood K^1 of C^1 in U^1, and the restriction $f|K^1$ is a smooth G-fibration with appropriate fibers.

(ii) The class of all G-maps to G-manifolds of at most a given (nonequivariant) codimension i. Here, "manifolds" may be replaced by "manifolds with singularities (restricted as above, or unrestricted) of finite depth". For closure, let L^1 be a closed invariant manifold (with singularities) neighbourhood of C in U, then let K^1 be a closed invariant manifold neighbourhood of C^1 in $f^{-1}(\text{int } L^1)$ and check that the restricted map has codimension at most i.

(iii) One may restrict the local codimension of the singularities as follows. First define a G-manifold with dimensions $\leq \gamma \in RO(G)$ and (mapping cylinder) singularities of depth 1 to be a G-manifold M with singularities such that M has local dimension $\gamma - n_i$ $(n_i \geq 0)$, away from the singularities, where the singular set $(L, \partial L)$ is a disjoint union of G-manifolds of dimensions $\gamma - m$ for various $m \geq 0$. We also require that each singularity map component $f\colon (K_i, \partial K_i) \to (L_i, \partial L_i)$ maps a $(\gamma - n_i)$-manifold to a $(\gamma - m_i)$-manifold with $n_i \leq m_i$. Proceeding inductively, one now defines a G-manifold with dimensions $\leq \gamma$ and singularities of depth p as in 3.4, insisting at each stage that the singularity map components do not increase dimension on any component. A γ-dimensional G-manfiold with singularities of codimension $i \geq 0$ is then a G-manifold M with singularities such that:

(a) Away from the singular set, M has equivariant dimension γ;

(b) The singular set is a G-manifold with dimensions $\leq \gamma - i$ and singularities of some finite depth p.

For closure, mimic (ii), above.

We shall call such classes nice.

DEFINITION 3.7. Let M be a G-manifold with singularities. A subset W of M is a sub-G-manifold with singularities if it has the structure of a G-manifold with singularities in such a way that, if $M = N \cup_K Mf$, then

$W = N' \cup_{K'} Mf'$, where N' and K' are, respectively G-submanifolds of N and K, and where f' is the restriction of f to K', regarded as a G-map $f'\colon K' \to L'$. Here, L' is a sub-G-manifold with singularities of L and with depth one less.

Since Definition 3.7 makes sense only for G-manifolds with singularities of finite depth, we henceforth restrict attention to these.

4. G-bordism with singularities. Here, we describe the bordism theories associated with singularities of finite depth in a nice class. Proofs of excision and existence of external products are deferred until §5.

Let S be a nice class of G-maps, and let $M(S)$ denote the category of G-manifolds with finite depth mapping cylinder singularities in S. The dimension of a G-manifold (with singularities) in $M(S)$ is then either in Z or in $RO(G)$, it being in either event the dimension of the manifold away from the singular set.

DEFINITION 4.1. Let (X, A) be a pair of G-spaces, and let $\gamma \in Z$ or $RO(G)$ (depending on context). Define $\Omega(S)^G_\gamma(X, A)$ to be the set of all cobordism classes of G-maps $(M, \partial M) \to (X, A)$ from objects in $M(S)$ to (X, A), with $\dim M = \gamma$. Here, two such G-maps are cobordant if one has the following:

(i) a $(\gamma + 1)$-dimensional G-manifold $N \in M(S)$ and a decomposition $\partial N = \partial_1 N \cup \partial_2 N$ of objects in $M(S)$, with $\partial_1 N \cap \partial_2 N$ equal to their common boundary;

(ii) a G-map $h\colon (N, \partial_2 N) \to (X, A)$ with $h|\partial_1 N\colon (\partial_1 N, \partial\partial_1 N) \to (X, A)$ the disjoint union of the two G-maps in question.

The standard gluing arguments from ordinary G-cobordism (cf. Stong **[12]**) now adapt to show the following.

PROPOSITION 4.2. *Each set $\Omega(S)^G_\gamma$ is an abelian group.* □

PROPOSITION 4.3. *The construction of $\Omega(S)^G_\gamma(X, A)$ yields a functor from the homotopy category of pairs of G-spaces to the category of graded abelian groups.* □

We now show that $\Omega(S)^G_*$ forms a generalized G-homology theory. Let $\Omega(S)^G_*(X, \phi)$ be denoted by $\Omega(S)^G_*(X)$.

PROPOSITION 4.4 (*exactness*). *Let (X, A) be a pair of G-spaces. Then there is an exact sequence*

$$\cdots \to \Omega(S)^G_\gamma(A) \to \Omega(S)^G_\gamma(X) \to \Omega(S)^G_\gamma(X, A) \to \Omega(S)^G_{\gamma-1}(A) \to \cdots$$

The proof is similar to the usual proof for ordinary G-bordism and details are omitted. □

A similar result holds for G-triples (X, A, B).

PROPOSITION 4.5 (*excision*). *Let* (X, A) *be a pair of G-spaces and let B be a closed invariant subspace of* $\operatorname{int}(A)$. *Then the inclusion of pairs* $(X - B, A - B) \to (X, A)$ *induces an isomorphism on* $\Omega(S)^G_*$.

The proof is deferred to §5.

PROPOSITION 4.6. *There is an external product*

$$\Omega(S)^G_\gamma(X, A) \otimes \Omega(S)^G_\mu(Y, G) \to \Omega(S)^G_{\gamma+\mu}((X, A) \times (Y, B)).$$

A proof is given in §5.

5. **Proofs of excision and external products.** First, we prove excision (Proposition 4.5) by induction on depth of singularity. Let (S, n) denote G-bordism with singularities of depth at most n in a nice class S. Then

$$\Omega(S, 0) = \Omega,$$

and

$$\Omega(S) = \operatorname*{colim}_n \Omega(S, n),$$

and, as usual, excision holds for $\Omega(S, 0)^G_*$. (See **[12]** for the nonequivariant case.) We shall need some care in formulating excision for $\Omega(S, n)$ in order to prove inductively that it holds for $\Omega(S)$.

PROPOSITION 5.1. *Let* (X, A) *be a pair of G-spaces, and let* $B \subset \operatorname{int}(A)$ *be closed and invariant. Let* $(M, \partial M)$ *be a G-manifold* (*with boundary*) *with singularities in a nice class S of depth at most n. Let* $f\colon (M, \partial M) \to (X, A)$ *be a G-map. Then there is a codimension-*0 *sub-G-manifold* (*with boundary*) $\underline{(M', \partial M')}$ *such that* $f|(M', \partial M')$ *takes values in* $(X - B, A - B)$ *and* $f|M - M'$ *takes values in* $\operatorname{int} A$.

Proof. By induction on n. If $n = 0$, there are no singularities, and the usual proof may be applied (cf. Stong **[12]**).

Now assume that excision holds for $n = n_0$. Write $M = N \cup_K M_\phi$, $\phi\colon K \to L$, as in (3.1), so that $(L, \partial L)$ is a G-manifold with singularities in S of depth at most n_0. By the inductive hypothesis, we may excise open

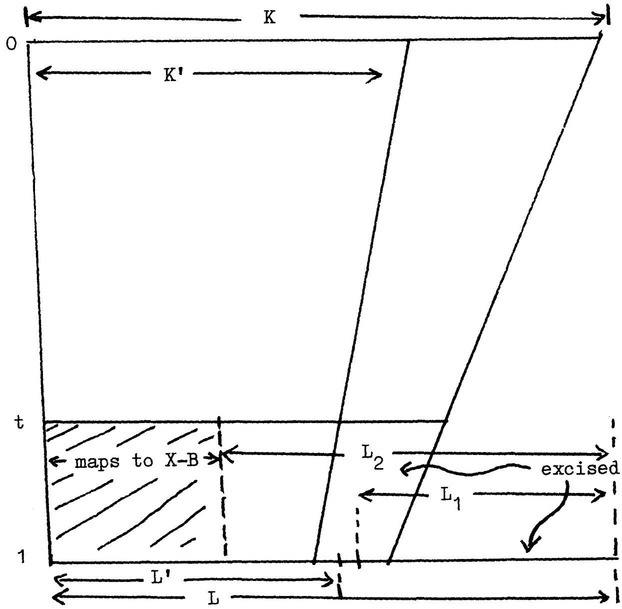

FIGURE 2

sets L_1 or L_2 from L, with $\overline{L}_1 \subset L_2$, in each case satisfying the proposition. Because S is closed under thickened weak pullbacks, there is a pair of codimension-0, closed G-manifolds, $L' \subset L$ and $K' \subset K$ such that

(a) $L - L_2 \subset L' \subset L - L_1$,

(b) ϕ maps K' to L,

(c) for some $t < 1$, f maps $(K', \partial K') \times [t, 1]$ to $(X - B, A - B)$, and

(d) for this t, f maps $K - K' \times [t, 1]$ to int A.

(See Figure 2.)

Now excise from M the complement of $K' \times [t, 1] \cup L'$ in $K \times [t, 1] \cup L$. Further excision of a suitable open set from the remaining manifold with singularities yields the conclusion for $n = n_0 + 1$, as required. □

Since $\Omega(S) = \operatorname{colim}_n \Omega(S, n)$, one may use the above result to show that there is a well-defined excision map in bordism with singularities in a nice class S:

$$\Omega(S)(X, A) \cong \Omega(S)(X - B, A - B).$$

Note that this completes the proof, begun in §4, that $\Omega(S)^G_*$ is a generalized homology theory. We turn now to products.

PROPOSITION 5.2. *The product of two G-manifolds with singularities in S is again a G-manifolds with singularities in S.*

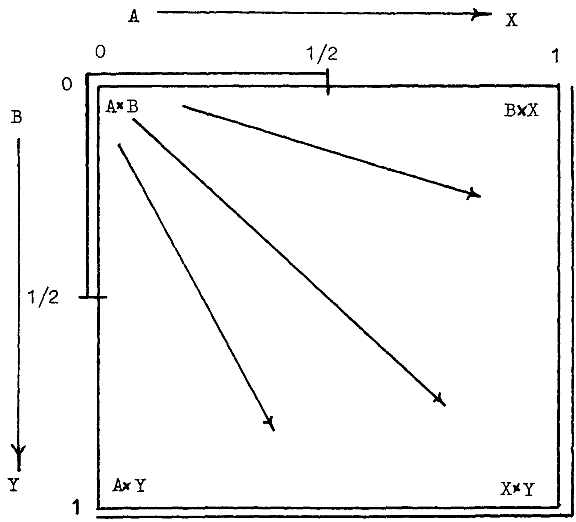

FIGURE 3

The proof requires the following two lemmas.

LEMMA 5.3. *The product of two mapping cylinders admits the structure of a mapping cylinder (although not canonically).*

Proof. Let $f\colon A \to X$ and $g\colon B \to Y$ be G-maps. We shall view $Mf \times Mg$ as a mapping cylinder according to the idea of Figure 3.

For this construction, map $A \times B \times [-\frac{1}{2}, \frac{1}{2}]$ to $X \times Y \times [-1, 1]$ via $h = (f, g,$ multiplication by 2). The mapping cylinder Mh of h may now be identified with $Mf \times Mg$ by identifying $A \times B \times [-\frac{1}{2}, \frac{1}{2}]$ with the upper left emphasized region shown in Figure 3, and by identifying $X \times Y \times [-1, 1]$ with the lower right emphasized region there. This now easily extends to a G-homeomorphism. □

LEMMA 5.4. *Let $M \cup_K Mf$ be a G-manifold with singularities. Then $M \cup_K (K \times I)$ has the structure of a smooth G-manifold, while*

$$M \cup_K (K \times I) \cup_K Mf$$

has the structure of a G-manifold with singularities G-homeomorphic to $M \cup_K Mf$. (Here, the unions are taken over the "0" and "1" ends of $K \times I$ respectively.)

The proof is omitted, being easy. □

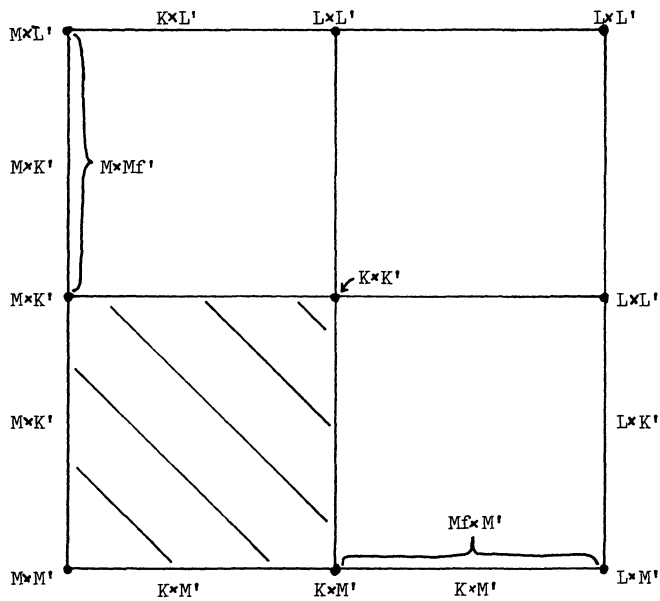

FIGURE 4

Proof of Proposition 5.2. Let $M \cup_K Mf$ and $M' \cup_{K'} Mf'$ be G-manifolds with singularities. Let L denote the singular set of M and L' the singular set of M'. By Lemma 5.4 it suffices to show that the product

$$(M \cup_K K \times I \cup_K Mf) \times (M' \cup_{K'} K' \times I' \cup_{K'} Mf')$$

is itself a G-manifold with singularities. Here, I' is a second copy of Figure 3. This product is represented in Figure 4, where symbols such as "M" indicate fibers over indicated points in the diagram, and where smoothings are not explicitly shown.

The lower left square (shaded) has the structure of a G-manifold whose boundary contains a codimension-0 sub-G-manifold of the form

$$M \times K' \times 0 \cup K \times K' \times [0,1] \cup K \times M' \times 1$$

(with corners smoothed). Using the proof of the Lemma 5.3, define a mapping cylinder structure from this sub-G-manifold to the union of the top and right sides of the square, a copy of

$$M \times L' \times 0 \cup K \times L' \times [0,1] \cup L \times L' \times [1,2]$$

$$\cup L \times K' \times [2,3] \cup L \times M' \times 3.$$

This gives $(M \cup_K K \times I \cup_K Mf) \times (M' \cup_{K'} K' \times I' \cup_{K'} Mf')$ the structure of a G-manifold with singularities, as required. □

COROLLARY 5.5. *If M_1 and M_2 have mapping cylinder singularities in a nice class S, then so does $M_1 \times M_2$, and the depth of $M_1 \times M_2$ is at most* $\mathrm{Max}\{\mathrm{depth}(M_1), \mathrm{depth}(M_2)\} + 1$.

By following straightforward arguments, one can show that products preserve the relation of G-cobordism. Thus $\Omega(S)_*^G$ admits an external product, and the corresponding cohomology theory $\Omega(S)_G^*$ is a ring. More will be said about this theory below.

6. Stable singular G-bordism theories and statement of results. Here, we construct stable, tom Dieck type, theories from the theories $\Omega(S)$, and consider their dual cobordism theories. We then give a precise statement of our results.

Referring to Example 3.6(iii), denote by M_i the (nice) class of G-maps associated with G-manifolds with dimension in $RO(G)$ and with mapping cylinder singularities of codimension at least i. We shall have occasion to specialize to G-oriented (SO), and to G-unitary (U) bordism with singularities. In such cases, the singular G-manifolds under consideration possess the relevant structure away from the singular sets. Thus, for example, $N \cup_K Mf$ is G-oriented if N is G-oriented. (See **[14]** for a treatment of G-oriented bordism). In our notation, a subscript will be used to indicate special structure—for example Ω_{SO} will denote G-oriented bordism, singular or otherwise.

From this point on, we shall deal entirely with reduced bordism and cobordism theories, and will retain the symbol Ω. Thus, $\Omega(S)_*^G(X)$ is the bordism of the pair $(X, *)$ for a based G-space X. One therefore has an $RO(G)$-graded G-homology theory $\Omega(M_i)_*^G$, defined on based G-spaces X, and possessing suspension isomorphisms of the form

$$\sigma\colon \Omega(M_i)_\gamma^G(X) \cong \Omega(M_i)_{\gamma+1}^G(\Sigma X).$$

One passes from one representation to another as follows. If Y is a (finite dimensional) G-module and if M belongs to the class M_i, then so does $M \times (D(Y), S(Y))$. This gives a homomorphism

$$\sigma_Y\colon \Omega(M_i)_\gamma^G(X) \to \Omega(M_i)_{\gamma+Y}^G(\Sigma^Y X).$$

As in §2, one now takes

$$\underline{\Omega}(M_i)_\gamma^G(X) = \operatorname*{colim}_Y \Omega(M_i)_{\gamma+Y}^G(\Sigma^Y X),$$

defined with respect to the σ_Y for Y invariant submodules of R^∞. By *fiat*, one now has suspension isomorphisms

$$\sigma\colon \underline{\Omega}(M_i)_\gamma^G(X) \cong \underline{\Omega}(M_i)_{\gamma+Y}^G(\Sigma^Y X)$$

for arbitrary G-modules Y. This gives $\underline{\Omega}(M_i)_*^G$ the structure of a generalized G-homology theory graded on $RO(G)$ in the sense of Wirthmuller [**22**].

Since the theory $\underline{\Omega}(M_i)_*^G$ satisfies the requisite suspension isomorphisms, it is representable by a G-spectrum in the sense of [**16**], and one therefore has an associated cobordism theory, $\underline{\Omega}(M_i)_G^*$, which also has a ring structure, since $\underline{\Omega}(M_i)_*^G$ retains the external product structure from $\Omega(M_i)_*^G$.

Next, we consider singular versions of the theories B_*^G discussed in §2. For these, we refer to Example 3.6(ii), and denote by N_i the class of G-maps S associated with G-manifolds with mapping cylinder singularities in codimension at least i. Although one has no analogue of SO for such bordism theories, one does have an analogue of U, and the notation conventions above will apply here. Proceeding as above, one obtains theories $B(N_i)_*^G$ and $\underline{B}(N_i)_*^G$, indexed on Z, with

$$\underline{B}(N_i)_m^G(X) = \operatorname*{colim}_Y B(N_i)_{m+y}^G(\Sigma^Y X)$$

where $y = \dim Y$. The theory $\underline{B}(N_i)_*^G$ is then a generalized Z-graded equivariant homology theory with suspension isomorphisms of the form

$$\underline{B}(N_i)_m^G(X) \cong \underline{B}(N_i)_{m+y}^G(\Sigma^Y X).$$

Again, one has a dual multiplicative cobordism theory $\underline{B}(N_i)_G^*$.

Note that one has an inclusion of classes (of nice G-maps)

$$M_i \to N_i$$

which induces a natural transformation

$$F\colon \Omega(M_i)_\gamma^G \to B(N_i)_{|\gamma|}^G,$$

and similarly for the stable theories.

Let T be a covariant coefficient system in the sense of Lewis, May and McClure [**8**], and let $\overline{H}_*^G(-;\, T)$ denote $RO(G)$-graded (reduced) homology with coefficients in T. (This is the theory dual to that described in the introduction.) One may verify that the functor $\underline{\Omega}_0\colon G/H \mapsto \underline{\Omega}_0^G(G/H_+)$ is such a coefficient system in view of the suspension isomorphisms described in §2.

THEOREM 6.1. *There exists a natural isomorphism*

$$\phi\colon \underline{\Omega}(M_2)_*^G(-) \to \overline{H}_*^G(-;\, \underline{\Omega}_0).$$

The theorem is proved in §7. Since $\underline{\Omega}$ may refer to any of the theories $\underline{\Omega}_A$, $A = \mathrm{U}, \mathrm{Fr}, \mathrm{SO}, \ldots$, one has, choosing $A = \mathrm{U}$, SO or Fr (see [**14**]), an isomorphism

$$\underline{\Omega}_A(M_2)^G_*(-) \cong \overline{H}^G_*(-;B),$$

where B denotes the covariant Burnside system, since $\underline{\Omega}_0$ is the Burnside system for these theories. (See [**8**] for the theory of such systems.)

Turning to dual cohomology theories, one has the following

COROLLARY 6.2. *There is a natural isomorphism*

$$\psi\colon \underline{\Omega}(M_2)^*_G(-) \cong \overline{H}^*_G(-;\underline{\Omega}^0),$$

where $\underline{\Omega}^0$ is the contravariant system $G/H \mapsto \underline{\Omega}^0_G(G/H_+)$.

Proof. The cohomology theory dual to $\overline{H}^G_*(-;\ \underline{\Omega}_0)$ is $\overline{H}^*_G(-;\ T)$, where T is the contravariant system defined by $G/H \mapsto \underline{\Omega}^G_0(G/H_+)$ and on a morphism $G/H \to G/K$ by application of $\underline{\Omega}^G_0$ to its G-Spanier-Whitehead dual [**21**]. (The G-spaces G/H_+ are self-dual in this sense). The system T may then be checked to coincide with $\underline{\Omega}^0$, as required. □

THEOREM 6.3. *There is a natural isomorphism*

$$\Psi\colon \underline{B}(N_2)^*_G(X) \cong \overline{H}^*\left(X \wedge_G EG_+;\underline{B}^0\right),$$

where $\underline{B}^0$ is the abelian group $\underline{B}^0(*)$, the lack of subscript G signifying the nonequivariant theory.

Theorem 6.3 is proved in §7.

Turning to the case of codimension-i singularities for general i, we prove the following results.

THEOREM 6.4. *Let α: $EG \to$ point denote the projection. Then the induced map in cohomology,*

$$(1 \wedge \alpha)^*\colon \underline{B}(N_i)^*_G(X) \to \underline{B}(N_i)^*_G(X \wedge EG_+)$$

is an isomorphism for each $i > 0$.

REMARKS. 6.5. No analogous result is known to be true for nonsingular G-bordism, even when G is a p-group and one uses finite coefficients. Löffler [**10**] has, however, shown that $(1 \wedge \alpha)^*$ is an isomorphism in the nonsingular theory when one completes at a certain ideal of $\underline{B}^*_G(*)$.

The relationship between $\underline{\Omega}(M_i)^*_G$ and $\underline{B}(N_i)^*_G$ is given by the following theorem.

THEOREM 6.6. *For each $i \geq 0$, there exists a generalized Chern class $\lambda_i \in \underline{\Omega}(M_i)^{\gamma_i}_G$ where $\gamma_i \in RO(G)$ has the form $V_i - v_i$, $v_i = \dim V_i$, such that the natural transformation $\underline{\Omega}(M_i)^*_G \to \underline{B}(N_i)^*_G$ induces an isomorphism*

$$[\lambda_i^{-1}]\underline{\Omega}(M_i)^n_G \xrightarrow{\cong} \underline{B}(N_i)^n_G.$$

Here, $[x^{-1}]$ denotes inversion of the class x.

Theorems 6.4 and 6.6 are proved in §10.

Finally, we consider what happens as $i \to \infty$.

PROPOSITION 6.7. *Let ι_i: $M_{i+1} \to M_i$ and τ_i: $M \to M_i$ be the natural inclusions, where M denotes the empty class of G-maps (so that the associated singular G-manifolds are singularity-free). Then the τ_i induce an isomorphism*

$$\tau\colon \underline{\Omega}^{\gamma+n}_G(-) \to \lim_i \underline{\Omega}(M_i)^{\gamma+n}_G(-),$$

and similarly for the associated G-bordism theories.

The proposition is proved in §12. An analogous result for $\underline{B}^*_G$ would imply, via Theorem 6.4, that tom Dieck bordism of EG and that of a point are more closely related than Löffler's results indicate, and the matter is still unresolved.

7. Proof of Theorem 6.1. Our strategy will be to use the following uniqueness result on ordinary $RO(G)$-graded G-homology.

PROPOSITION 7.1. *Up to natural isomorphism, there exists a unique $RO(G)$-graded equivariant homology theory h^G_* defined on G-CW complexes and admitting suspension isomorphisms by arbitrary (finite dimensional orthogonal) G-modules such that*

$$h^G_n(-) \cong \overline{H}^G_n(-\,;T),$$

Bredon-Illman homology with coefficients in a given covariant system T with suitable structure.

A proof will appear in [9]. In view of the proposition, it will suffice to show that $\underline{\Omega}(M_2)^G_n(-) \cong \overline{H}^G_n(-;\underline{\Omega}_0)$ for $n \in Z$.

Let $\mathscr{G}$ be the category of G-orbits described in the introduction. In view of the uniqueness theorem for H_n^G [6], it will suffice to show that, as a covariant system,

$$\underline{\Omega}(M_2)_0^G(-): \mathscr{G} \to \mathscr{Ab},$$

the category of abelian groups, agrees with the system $\underline{\Omega}_0$, while $\underline{\Omega}(M_2)_n^G(G/H_+) = 0$ if $n \neq 0$.

We first prove a lemma.

LEMMA 7.2. *Let $(P, \partial P)$ belong to M_i and have dimension $V + n$ for some V, with $n > i - 2$. Then any G-map f: $(P, \partial P) \to (D(V), S(V))$ is null-bordant.*

Proof. Let $(S, \partial S)$ denote the singular set in $(P, \partial P)$, so that S has dimensions $V + r$ for $r \leq n - i$.

We assert first that the strata of S may be assumed to have dimensions $V + s$ with $s \geq 0$. Indeed, write

$$(S, \partial S) = (S_0, \partial S_0) \supset (S_1, \partial S_1) \supset \cdots \supset (S_p, \partial S_p),$$

where $(S_{i+1}, \partial S_{i+1})$ is the singular set in $(S_i, \partial S_i)$. Let j be the largest integer such that $(S_j, \partial S_j)$ has a local dimension $V + s$ with $s < 0$. Then, by choice of j, S_j is a disjoint union of the form $T \cup R$, where T has dimensions $V + s$ with $s < 0$ and no singularities, and R has dimensions $V + t$ with $t \geq 0$. Thus $\dim T^H < \dim(D(V)^H)$ for each $H \subset G$, whence f is G-homotopic to a G-map f' which maps the cylinder neighbourhood U of T in P into $S(V)$. Let $Q = P - U$. Then

$$g = f'|Q: (Q, \partial Q) \to (D(V), S(V))$$

is G-bordant to f as follows. Let $N = P \times [0, \frac{1}{2}] \cup_{P \times 1/2} Q \times [\frac{1}{2}, 1]$ and let h: $(N, \partial N) \to (D(V), S(V))$ be defined via the G-homotopy $f \sim f'$ on the left half, and by $g \times 1$ on the right half. That this bordism is one in M_i now follows from the definitions. Further, we have removed the deepest stratum S_j with $\dim S_j = V + s$ and $s < 0$. Continuing inductively gives the assertion.

Now let F: $Mf \to (D(V), S(V))$ denote the natural extension of f over its mapping cylinder. Then Mf is a G-manifold of dimension $V + n + 1$ with singularities of codimension at least i, stratified as $Mf|(S_0, \partial S_0) \supset \cdots \supset Mf|(S_p, \partial S_p) \supset (D(V), S(V))$. F is now the desired null-G-bordism of f. □

The essential role played by the mapping cylinder construction in the above proof was the motivation for the use of mapping cylinder type singularities.

PROPOSITION 7.3. (a) *The inclusions* $\tau\colon M \to M_2$, (*see Proposition* 6.7), *induce isomorphisms* $T\colon \underline{\Omega}_0^G(X) \to \underline{\Omega}(M_2)_0^G(X)$ *for every based G-space X.*

(b) $\underline{\Omega}(M_2)_n^G(G/H_+) = 0$ *if* $n \neq 0$.

Proof. To show (a), we construct an inverse R of T. If x is a class in $\underline{\Omega}(M_2)_0^G(X)$, then x is represented by a G-map

$$f\colon (P, \partial P) \to (X, *) \times (D(V), S(V))$$

for some G-module V, where $(P, \partial P)$ is V-dimensional and of type M_2. If $(S, \partial S)$ denotes its singular set, then $\dim(S, \partial S) = V - r$ locally, where $r \geq 2$, whence, by the proof of the lemma, one may replace $(P, \partial P)$ by a singularity-free V-manifold, thereby defining a class $R(x)$ in $\underline{\Omega}_0^G(X)$. To show that this class is well-defined up to the bordism class of x, one may perform the same construction on any bordism between x and x', since $i = 2$. By construction, $RT = 1$, and by the proof of Lemma 7.2, $TR = 1$.

For (b), let $f\colon (P, \partial P) \to (G/H_+, +) \times (D(V), S(V))$ represent a class $x \in \underline{\Omega}(M_2)_n^G(G/H_+)$ with $n \neq 0$. If $n > 0$, then P is $(V + n)$-dimensional and one may apply the lemma, which works equally well with $D(V)$ replaced by $D(V) \times G/H_+$. If $n < 0$, then V has the form $W + m$ with $m > 0$ and P of dimension W. By the arguments above, x is represented by a singularity-free bordism class which must be zero since $\underline{\Omega}_n^G = 0$ for $n < 0$. □

Theorem 6.1 now follows.

8. Change of representation and the bottom Chern class. In [**17**], Waner constructs Chern classes $c_i \in H_G^{V-2i}(*; T)$ for arbitrary f.d. unitary G-modules V and coefficient systems T, where $0 \leq i \leq \dim_{\mathbf{C}} V$. Denote by λ the class in dimension $V - v$ (where $v = \dim_{\mathbf{R}} V$). It is proved in [**17**] that, if V contains a free G-orbit, then $[\lambda^{-1}]H_G^*(*;\ T) \cong H^*(BG; T(G/e))$, where $T(G/e)$ is regarded as a G-module via its coefficient system structure. (See for example, [**20**].) In view of Theorem 6.1, it seems natural to seek a geometric representative for the class λ, regarded as an element of $\overline{H}_{v-V}^G(S^0;\ \underline{\Omega}_0)$, in $\underline{\Omega}(M_2)_{v-V}^G(S^0) = \underline{\Omega}(M_2)_v^G(S^V)$. This, together with analogous classes of arbitrary codimension, will emerge from the following.

PROPOSITION 8.1. *Let Y be a smooth G-manifold with fixed sets of codimension $\geq i$. Then Y has the structure of a G-manifold with singularities from class M_i and of dimension in $Z \subset RO(G)$.*

Proof. Assume that Y is connected, and consider first the case that Y contains a free G-orbit. Denote by F the union of its proper fixed subsets. Since Y is free away from F, and since F has an equivariant mapping cylinder neighbourhood in Y, it suffices to show that F is a G-manifold with dimensions (in $RO(G)$) of at most $\dim Y - i$ and finite depth singularities.

F may be given such a structure as follows. Let $(Y_0, \partial Y_0)$ denote the union of subsets of maximal isotropy type. Then Y_0 is a G-manifold with boundary $\partial Y_0 = \partial Y \cap Y_0$, and forms the singularity for the G-space $(Y_1, \partial Y_1)$ consisting of points with maximal isotropy among the remaining subgroups. Continuing inductively, one obtains the required structure.

If Y does not contain a free G-orbit, then Y is a G/K-manifold with K some normal subgroup of G, and the argument thusfar implies that Y is a G/K-manifold of equivariant dimension in Z with singularities. Since Y is now of the form $G/K \times D(n)$ away from the singularities, while the singular set continues to have the form $G/J \times D(m)$ locally, it follows that Y has the required structure. □

One may, in view of the proposition, regard the pair $(D(V), S(V))$ as a singular G-manifold of equivariant dimension in $Z \subset RO(G)$ and with singular set equal to the union of proper fixed subsets. The codimension i is then given by $v - \max_{V^H \neq V}\{\dim V^H\}$. In particular, if V is unitary, then $(D(V), S(V)) \in M_2$. Denote $(D(V), S(V))$ with this structure by $(\underline{D}(V), \underline{S}(V))$. The identity $(\underline{D}(V), \underline{S}(V)) \to (D(V), S(V))$ then represents a class

$$\mu \in \underline{\Omega}(M_2)^G_v(S^V) \cong \underline{\Omega}(M_2)^G_{v-V}(S^0).$$

PROPOSITION 8.2. *Under the isomorphisms of Theorem* 6.1 *and its corollary, μ coincides with the Chern class $\lambda \in \overline{H}^{V-v}_G(S^0; \underline{\Omega}^0)$.*

Before proving the proposition, we establish the following consistency result concerning passage to subgroups.

LEMMA 8.3. *For each $H \subset G$ there is a natural isomorphism*

$$\varepsilon \colon \underline{\Omega}\left(M_i^G\right)^G_\gamma(G_+ \wedge_H X) \cong \underline{\Omega}\left(M_i^H\right)^H_{\gamma|H}(X)$$

for $i \geq 0$, where X is any based G-space, $\gamma \in RO(G)$, and the superscript J in M_i^J indicates ambient group J.

Proof. A class in $\underline{\Omega}(M_i^G)^G(G_+ \wedge_H X)$ is represented by a G-map $f:(M, \partial M) \to G \times_H [(X, *) \times (D(W), S(W))]$ for some G-module W, where $M \in M_i^G$ has dimension $\gamma + W$. It follows that M must have the form $G \times_H (N, \partial N)$, where $N = f^{-1}(\{1\}_+ \wedge_H (X \times D(W)))$ and is H-invariant, giving an object in M_i^H of dimension $\gamma | H + W | H$. The required isomorphism ε is given by assigning to the class $[f]$ the class $[f|(N, \partial N)]$, while its inverse is given by assigning to an H-map g: $(M, \partial M) \to (X, *) \times (D(W), S(W))$ the G-map

$$g \times 1: G \times_H (M, \partial M) \to G \times_H [(X, *) \times (D(W), S(W))]. \qquad \square$$

REMARK 8.4. (i) It follows that the identical result holds for cohomology provided X is a finite G-CW complex, by the self-Spanier-Whitehead duality of G/H. Further, Lemma 9.1 below will guarantee that X may be an infinite G-CW complex with finite skeleta.

(ii) Everything just said applies equally well to $\underline{B}(N_i)_*^G$, with the (possible) exception of the last statement in (i). In practice, consistency results for X infinite are verified by the vanishing of $\lim^1$ terms.

(iii) Note that the theories $\underline{\Omega}(M_i^1)_*$ and $\underline{B}(N_i^1)_*$ coincide.

Proof of Proposition 8.2. By the work in [17], λ is entirely specified by the fact that, if f: $\overline{H}_G^{V-v}(-; \underline{\Omega}^0) \to \overline{H}^0(-; \underline{\Omega}^0(S^0))$ represents the forgetful map (with respect to an orientation of V), then $f(\lambda) = 1$, the unit in cohomology. One now has a commutative diagram

$$\begin{array}{ccc} \underline{\Omega}(M_2^G)_n^G(X) & \to & \overline{H}_n^G(X; \underline{\Omega}_0) \\ \downarrow f & & \downarrow f \\ \underline{\Omega}(M_2^1)_n(X) & \to & \overline{H}_n(X; \underline{\Omega}_0(S^0)), \end{array}$$

where f is the forgetful homomorphism in each case. (In general, f takes the form f: $h_n^G(X) \to h_n^G(G/H_+ \wedge X) = h_n^H(X)$.) Since $f(\mu)$ and the unit both live in $\underline{\Omega}(M_2^1)_0(S^0)$, the result will follow from the following lemma, applied to the case $Y = D(V)$.

LEMMA 8.4. *Let Y be a G-manifold with proper fixed subsets of codimension at least i, and let r: $(Y, \partial Y) \to (D(W), S(W))$ represent a class in $\underline{\Omega}(M_i^G)_\gamma^G$. Let y be the class of $\underline{r}$: $(\underline{Y}, \partial \underline{Y}) \to (D(W), S(W))$ in*

$\underline{\Omega}(M_i^G)^G_{|Y|}$, *where* $\underline{Y} = Y$, *regarded as an integral dimensional G-manifold with singularities as in Proposition* 8.1, *and where* $\underline{r}$ *coincides with* r. *Then* $f(x) = f(y) \in \underline{\Omega}(M_i^1)$.

Proof. To prove the lemma, it suffices to give $(Y, \partial Y) \times I$ the structure of an object in M_i such that $(Y, \partial Y) \times \{0\}$ has its usual manifold structure, while $(Y, \partial Y) \times \{1\}$ has the structure of $(\underline{Y}, \partial \underline{Y})$. This is shown by replacing F in the proof of 8.1 by $F \times [\frac{1}{2}, 1]$, and by applying the rest of the proof verbatim (and nonequivariantly). □

Note that, while $(Y, \partial Y) \times I$ has the structure of an equivariant manifold with singularities and the desired restrictions at the ends of the cylinder, it is not an object of fixed local equivariant dimension away from its singular set. We shall need the following generalization of Lemma 8.1.

PROPOSITION 8.6. *Let Y be a G-manifold with singularities. Then Y has the structure of an object in* M_i *of integral equivariant dimension.*

Proof. We assume Y connected and do induction on the depth of the singularity, thecase of zero depth being settled by Lemma 8.1. Thus assume the result true for depth m singularities, and that Y has depth $m + 1$ singularities. By induction, its singular set S has the structure of a singular G-manifold with equivariant dimension in Z. Further, we inductively assume (as we may in the proof of Lemma 8.1) that S includes the union of its proper fixed subsets, and that the G-orbit of each proper fixed subset is a singularity stratum.

Let $S' = S \cup$ proper fixed subsets of Y. Then S' is a G-manifold with singularities and dimensions in Z. Indeed, adding proper fixed subsets of Y to S as in the proof of 8.1 amounts to attaching G-manifolds of dimension in Z to the strata in S via mapping cylinder constructions. Since all the local structure is now in $Z \subset RO(G)$, the result follows. □

9. Inverting the bottom Chern class. Fix V as the complement of the trivial summand in the complex regular representation, and for each $i > 0$ choose integers $m_i \geq 1$ such that $m_i V$ has proper fixed sets of codimension at least i. One may then regard $(D(m_i V), S(m_i V))$ as a G-manifold of equivariant dimension $m_i v$ and singularities of codimension i. Just as in §8, this gives rise to "Chern" classes

$$\lambda_i \in \underline{\Omega}(M_i)^G_{m_i(v-V)}(S^0)$$

for each $i > 0$.

Consider now the associated cohomology theory $\underline{\Omega}(M_i)^*_G$. One may formally invert the class λ_i in $\underline{\Omega}(M_i)^*_G$ by defining

$$[\lambda_i^{-1}]\underline{\Omega}(M_i)^{\mu}_G(X) = \operatorname*{colim}_n \underline{\Omega}(M_i)^{\mu+n\gamma_i}_G(X),$$

taken with respect to multiplication by λ_i, where $\gamma_i = m_i(V - v)$. (As usual, v is the real dimension of V.)

LEMMA 9.1. *Let $\gamma \in RO(G)$, and let X be a G-CW complex. Then, for N sufficiently large, inclusion $X^N \to X$ of skeleta induces an isomorphism $\underline{\Omega}(M_i)^{\gamma}_G(X) \cong \underline{\Omega}(M_i)^{\gamma}_G(X^N)$.*

Proof. It suffices to show that $\underline{\Omega}(M_i)^{\gamma}_G(G/H_+ \wedge S^N) = 0$ for N large enough. Write $\gamma = W - U$. Then

$$\begin{aligned}\underline{\Omega}(M_i)^{\gamma}_G(G/H_+ \wedge S^N) &\cong \underline{\Omega}(M_i)^{\gamma|H}_H(S^N)\\ &\cong \underline{\Omega}(M_i)^H_{N-\gamma|H}(S^0)\\ &\cong \underline{\Omega}(M_i)^H_{N+U}(S^W),\end{aligned}$$

this bordism group having contributions of the form

$$\underline{\Omega}(M_i)^H_{U+N}(H/K_+ \times S^j)$$

for $K \subset H$ and $j \leq \dim W$. Since these vanish for N large enough (by Lemma 7.2), we are done. □

Of course, Lemma 9.1 eliminates the need for considering $\lim^1$ terms when considering infinite G-complexes in singular G-cobordism.

LEMMA 9.2. *Let X be a free connected G-space with finite skeleta. Then the localization*

$$\underline{\Omega}(M_i)^{W+r}_G(X) \to [\lambda_i^{-1}]\underline{\Omega}(M_i)^{W+r}_G(X)$$

is an isomorphism for every G-module W.

Proof. By the five lemma and cofibration exact sequence arguments, it suffices to prove the lemma in the special case $X = G_+ \wedge S^n$, where it is immediate, since λ_i, regarded as a class in nonequivariant singular bordism, coincides with the unit. □

If $X = EG_+$, then the natural projection α: $EG_+ \to S^0$ induces a (localized) homomorphism

$$\omega^*\colon [\lambda_i^{-1}]\underline{\Omega}(M_i)^*_G(S^0) \to \underline{\Omega}(M_i)^*_G(EG_+)$$

by the lemma.

THEOREM 9.3. *The map*

$$\alpha^*\colon [\lambda_i^{-1}]\underline{\Omega}(M_i)^n_G(S^0) \to \underline{\Omega}(M_i)^n_G(EG_+)$$

is an isomorphism for every $i > 0$ and $n \in Z$.

Proof. Fix $m = m_i$ and $\gamma = \gamma_i$, and let $W = mV$. (Thus $\gamma = W - w$.) Let $J \subset G$ be such that W^J is a proper fixed subspace of W having maximum dimension r, and let N be any integer with $N(w - r) > w + n + i$. It will then suffice to show that

$$\alpha^*\colon \underline{\Omega}(M_i)^{n+N\gamma}_G(S^0) \to \underline{\Omega}(M_i)^{n+N\gamma}_G(EG_+)$$

is an isomorphism for all such N.

Let P_N be the free G-manifold obtained from $S(NW)$ by deleting a neighbourhood of the union of proper fixed subsets. Then

$$EG \simeq \operatorname{colim}_N P_N,$$

via the inclusions induced by $NW \to (N + 1)W$. If $j\colon P_N \to S(NW)$ is the inclusion, we assert that

$$j^*\colon \underline{\Omega}(M_i)^{n+N\gamma}_G(S(NW)_+) \to \underline{\Omega}(M_i)^{n+N\gamma}_G((P_N)_+)$$

is an isomorphism. Indeed, it suffices to show that

$$\underline{\Omega}(M_i)^{n+N\gamma+t}_G(S(NW)/P_N) = 0 \quad \text{if } t = 0 \text{ or } 1.$$

Write $P_N = S(NW) - Q_N^0$, where Q_N is a regular neighbourhood of the union of proper fixed subsets of $S(NW)$, closed and invariant under the action of G. Then $S(NW)/P_N = Q_N/\partial Q_N$. Since $Q_N/\partial Q_N$ is equivariantly Spanier-Whitehead dual to the Thom space, $T(Q_N)$, of Q_N by **[21]**, and since the latter may be taken to be $\Sigma(Q_N)_+$, one has a duality isomorphism

$$\underline{\Omega}(M_i)^{n+N\gamma+t}_G(Q_N/\partial Q_N) \cong \underline{\Omega}(M_i)^G_{(NW-1)-(n+N\gamma+t)}((Q_N)_+),$$

$(NW - 1)$ being the dimension of Q_N. Here,

$$NW - 1 - n - N\gamma - t = Nw - t - n - 1.$$

Since Q_N is G-equivalent to the union of proper fixed subsets in $S(NW)$, it is also G-equivalent to a G-CW complex with G-cells of type $G/H \times D^s$ with $s \leq \dim NW^H - 1$. For such G-cells,

$$\underline{\Omega}(M_i)^G_k(G/H_+ \wedge S^s) \cong \underline{\Omega}(M_i)^H_k(S^s) \cong \underline{\Omega}(M_i)^H_{k-s}(S^0),$$

where $k = NW - 1 - n - t$, and $k - s > i$ provided that $Nw - 1 - n - 1 > NW^H - 1 + i$. But, by definition of r, one has

$$Nw - NW^H \geq N(w - r) > w + n + i, \text{ by definition of } N,$$

$$> W^H + 1 + n + i.$$

Thus, $k - s > i$, whence $\underline{\Omega}(M_i)^H_{k-s}(S^0) = 0$, proving the assertion.

One now has a commutative diagram

$$\begin{array}{ccc} \underline{\Omega}(M_i)_G^{n+N\gamma}(S(NW)_+) & \underset{\cong}{\overset{j^*}{\to}} & \underline{\Omega}(M_i)_G^{n+N\gamma}((P_N)_+) \\ \uparrow \eta & & \uparrow \sigma \\ \underline{\Omega}(M_i)_G^{n+N\gamma}(S^0) & \overset{\alpha^*}{\to} & \underline{\Omega}(M_i)_G^{n+N\gamma}(EG_+). \end{array}$$

It therefore remains to show that η and σ are isomorphisms.

For σ, one observes that EG_+ may be obtained from P_N by attaching G-cells of the form $G \times D^s$ with s large, since the connectivity of $P_N \to \infty$ as $N \to \infty$. (P_N is at least as connected as $Q_N/\partial Q_N$ and

$$H_t(Q_N, \partial Q_N) \cong H^{Nw-1-t}(Q_N) = 0 \quad \text{for } t < N(w-r) - 1.)$$

For such cells,

$$\underline{\Omega}(M_i)_G^{n+N\gamma+\varepsilon}(G_+ \wedge D^s) \cong \underline{\Omega}(M_i)^{n+\varepsilon-s}(S^0),$$

since $|\gamma| = 0$, where $\varepsilon = 0$ or 1. These groups vanish for large enough s, and we may assume N having been chosen sufficiently large to guarantee this.

That η is an isomorphism follows from the G-cofibration exact sequence

$$\begin{array}{l} \cdots \to \underline{\Omega}(M_i)_G^{n+N\gamma}(S^{NW}) \to \underline{\Omega}(M_i)_G^{n+N\gamma}(S^0) \to \underline{\Omega}(M_i)_G^{n+N\gamma}(S(NW)_+) \\ \qquad \wr\| \\ \qquad \to \cdots \ \underline{\Omega}(M_i)_G^{n-Nw}(S^0), \end{array}$$

the isomorphism on the left following from the definition of γ. Since $n - Nw$ and $n - Nw - 1$ are $< i$ by choice of N, η is an isomorphism. □

10. Relationship with Z-graded singular bordism. Let m_i, V and λ_i be as in §9 (for each $i > 0$), and let μ_i be the class $f_*(\lambda_i)$, where

$$f_*: \underline{\Omega}(M_i)_*^G \quad \to \underline{B}(N_i)_*^G$$

denotes the forgetful homomorphism described in §6. Thus μ_i is represented by the identity map

$$(\underline{D}(m_iV), \underline{S}(m_iV)) \to (D(m_iV), S(m_iV)),$$

where the pair on the left expresses the disc as a singular G-manifold with integral equivariant dimension as above.

LEMMA 10.1. *The class μ_i coincides with the unit in cohomology.*

Proof. It suffices to give $D(m_iV)\times I$ the structure of an object in N_i such that the zero-end of the cylinder has its usual G-manifold structure, while its one-end is given the singular structure of $\underline{D}(m_iV)$. This may be done by replacing F in the proof of 8.1 by $F\times[\frac{1}{2},1]$, and by repeating the argument there. $\square$

Proof of Theorem 6.6. Define homomorphisms

$$\psi\colon [\lambda_i^{-1}]\underline{\Omega}(M_i)_*^G \quad\to \underline{B}(N_i)_{|*|}^G,$$

and

$$\phi\colon \underline{B}(N_i)_*^G \quad\to [\lambda_i^{-1}]\underline{\Omega}(M_i)_*^G$$

as follows. ψ is taken to be the localized version of f_* above; $\underline{B}(N_i)_*^G$ being already local by the lemma. To define ϕ, let $x\in\underline{B}(N_i)_n^G(S^0)$ be represented by $f\colon (M,\partial M)\to(D(sV),\,S(sV))$ for some $s\geq 0$, where $\dim M = n+sv$. By Proposition 8.1, we may regard M as an object in M_i of integral dimension, since one may assume $(M,\partial M)$ having been suspended so that its proper fixed subsets have arbitrarily high codimension. Thus f defines an element $y(f)$ in $\underline{\Omega}(M_i)_{n+sv}^G(S^{sV})\cong\underline{\Omega}(M_i)_{n+s(v-V)}^G(S^0)$, and hence in $[\lambda_i^{-1}]\underline{\Omega}(M_i)_n^G(S^0)$. We therefore define $\phi(x)$ to be $y(f)$. To check that ϕ is well-defined, one first observes that $\phi(x)$ is independent of the choice of representative of x in the geometric theory $B(N_i)_{n+sv}^G(S^{sV})$, as one may include the proper fixed sets of any bordism in the singular set. It therefore remains to check that $f\colon (M,\partial M)\to(D(sV),\,S(sV))$, and

$$f\times 1\colon (M,\partial M)\times(D(m_iV),\,S(m_iV))\to(D((s+m_i)V),\,S((s+m_i)V))$$

are mapped to bordant classes in $[\lambda_i^{-1}]\underline{\Omega}(M_i)_n^G(S^0)$. Here, $\lambda_i y(f)$ and $y(f\times 1)$ are both represented by $f\times 1$, but with different singularity sets; $\lambda_i y(f)$ has singularity

$$S_1=[(S\cup F(M))\times D(m_iV)]\cup[M\times F(D(m_iV))],$$

while $y(f\times 1)$ has singularity

$$S_2=[S\times D(m_iV)]\cup F(M\times D(m_iV)),$$

where $F(-)$ denotes the union of proper fixed subsets. (Note that $S_2\subset S_1$.) An explicit G-bordism between these representatives is given by endowing $N=M\times D(m_iV)\times I$ with singular set $T=S_1\times[0,\frac{1}{2}]\cup S_2\times[\frac{1}{2},1]$, and noting that N and T have integral equivariant dimension as singular G-manifolds. An argument similar to that in the proof of Lemma 10.1 now shows that $\psi\phi=1$. To show that $\phi\psi=1$, we use a less direct argument.

Consider the following commutative diagram, induced by the G-map $\psi\colon EG_+ \to *$.

$$\begin{array}{ccc} [\lambda_i^{-1}]\underline{\Omega}(M_i)_G^{-n}(S^0) & \xrightarrow{\tilde{\psi}} & \underline{B}(N_i)_G^{-n}(S^0) \\ \beta^* \downarrow & & \alpha^* \downarrow \\ \underline{\Omega}(M_i)_G^{-n}(EG_+) & \xrightarrow{f^*} & \underline{B}(N_i)_G^{-n}(EG_+), \end{array}$$

where $\tilde{\psi}$ is ψ in cohomological form. Since $\tilde{\psi}$ is now a (split) epimorphism, and since β^* is an isomorphism by Theorem 9.3, it follows that, if one can show f^* to be an isomorphism, it will then follow that both $\tilde{\psi}$ and α^* are isomorphisms, (and also that ϕ and ψ are inverses of each other).

By Lemma 9.1, one may replace EG by one of its skeleta EG^N. Further, since the G-cells in EG are free, the argument of that lemma, with H replaced by the trivial subgroup, shows the same to be true for $\underline{B}(N_i)_G^{-n}(EG_+)$. On the other hand,

$$f^*\colon \underline{\Omega}(M_i)_G^m(G_+ \wedge S^r) \to \underline{B}(N_i)_G^m(G_+ \wedge S^r)$$

is an isomorphism for any $m \in Z$, since both its domain and target coincide with the nonequivariant groups $\underline{\Omega}(M_i)^{m-r} \cong \underline{B}(N_i)^{m-r}$. The result now follows by the five-lemma and cofiber sequence arguments. □

Note that, during the course of the above proof, we have shown the following counterpart to 9.3.

THEOREM 10.3. *The G-map $\alpha\colon EG \to *$ induces an isomorphism*

$$\alpha^*\colon \underline{B}(N_i)_G^*(S^0) \to \underline{B}(N_i)_G^*(EG_+). \qquad \square$$

Note that this is an equivalent formulation of 6.4, and thus completes the proof of that theorem. Finally, we deduce Theorem 6.3. Indeed, one has isomorphisms

$$\begin{aligned} \underline{B}(N_2)_G^*(X) &\cong \underline{\Omega}(M_2)_G^*(EG_+ \wedge X) && \text{(diagram in proof of 6.6)} \\ &\cong \overline{H}_G^*(EG_+ \wedge X; \underline{\Omega}^0) && (6.2) \\ &\cong \overline{H}_G^*(EG_+ \wedge X; \underline{B}_0) \end{aligned}$$

for any G-CW complex X. Also note that

$$\overline{H}_G^*(EG_+ \times X; \underline{B}^0) = \overline{H}^*(EG_+ \times X; \underline{B}^0(G/e)),$$

this being a well-known fact from Bredon cohomology.

11. Proof of Proposition 6.7. Consider the homomorphism

$$\tau\colon \underline{\Omega}_\gamma^G(X) \to \varprojlim{}_i \underline{\Omega}(M_i)_\gamma^G(X).$$

By suspending if necessary, we may assume that $|\gamma| \geq \gamma^H$ for each $H \subset G$. An element of $\lim_i \underline{\Omega}(M_i)_\gamma^G(X)$ is represented by a sequence of bordism classes of G-maps

$$f_i\colon (P_i, \partial P_i) \to (X, *) \times (D(V_i), S(V_i))$$

such that f_i and f_{i+1} are G-bordant after suspension by some G-module, with $(P_i, \partial P_i) \in M_i$ and of dimension $\gamma + V_i$. Thus the singularity S_i in $(P_i, \partial P_i)$ has dimensions $\leq (\gamma + V_i - i)$. If i is chosen $> |\gamma| + 2$, then $\dim S_i \leq \gamma + V_i - |\gamma| - 2$, whence

$$\dim S^H \leq \gamma^H - |\gamma| + V_i^H - 2 \leq V_i^H - 2$$

for $H \subset G$, so that $f_i|S_i$ is null bordant. Let U_i denote the mapping cylinder neighbourhood of S_i, and let $(Q_i, \partial Q_i) = (P_i, \partial P_i) - U_i$. Then f_i and $f_i|Q_i$ are G-bordant via the restriction of f_i to $(P_i \times [0, \frac{1}{2}]) \cup (Q_i \times [\frac{1}{2}, 1])$, showing that τ is epic.

On the other hand, given a G-map

$$f\colon (M, \partial M) \to (X, *) \times (D(V), S(V))$$

representing an element of $\ker \tau$. One then has a null-bordism of $\tau[f]$ in $\underline{\Omega}(M_i)_\gamma^G(X)$ for each i. Denote this null-bordism by F_i: $(X_i, \partial X_i) \to (X, *) \times (D(V_i), S(V_i))$. Fix $i \geq |\gamma| + 2$ as before. The singularity set S_i of $(X_i, \partial X_i)$ then has fixed set dimensions $\leq \gamma^H - |\gamma| + V_i^H - 1 \leq V_i^H - 1$, so that, again, $F_i|S_i$ is null-bordant, and one may exchange the singularity S_i for a boundary as before. □

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HOFSTRA UNIVERSITY
HEMPSTEAD, NY 11550

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