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## **CLOPEN REALCOMPACTIFICATION OF A MAPPING**

TAKESI ISIWATA

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## CLOPEN REALCOMPACTIFICATION OF A MAPPING

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In this note, we give a necessary and sufficient condition on  $\varphi$ :  $X \rightarrow Y$  for  $v\varphi$  to be an open perfect mapping of vX onto vY and other related results.

Throughout this paper, by a space we mean a completely regular Hausdorff space and mappings are continuous and we assume familiarity with [1] whose notation and terminology will be used throughout. We denote by  $\varphi: X \to Y$  a map of X onto Y, by  $\beta X(\nu X)$  the Stone-Čech compactification (Hewitt realcompactification) of X and by  $\beta \varphi$  ( $\nu \varphi = (\beta \varphi) | \nu X$ ) the Stone extension (realcompactification) over  $\beta X(\nu X)$  of  $\varphi$ .

Concerning clopenness of  $v\varphi$  of a clopen map  $\varphi: X \to Y$  the following results are known.

THEOREM A (Ishii [4]). If  $\varphi: X \to Y$  is an open quasi-perfect map, then  $v\varphi$  is an open perfect map of vX onto vY.

THEOREM B (Morita [8]). If  $\varphi: X \to Y$  is a clopen map such that the boundary of each fiber is relatively pseudocompact, then  $v\varphi$  is also a clopen map of vX onto vY.

In §2, concerning Theorem A we give a necessary and sufficient condition on  $\varphi$  for  $v\varphi$  to be an open perfect map of vX onto vY without using the theory of hyper-spaces (Theorem 2.3 below) and a necessary and sufficient condition on  $\varphi$  for  $v\varphi$  to be an open *RC*-preserving map of vXonto vY under some condition (Theorem 2.6 below).

We use the following notation and abbreviation: C(X) is the set of real-valued continuous functions defined on X,  $C(X; \varphi) = \{ f \in C(X); f \text{ is } \varphi \text{-bounded} \}$ , Bd A = the boundary of A, usc = upper semicontinuous, lsc = lower semicontinuous and  $\omega(\omega_1)$  = the first infinite (uncountable) ordinal, clopen = closed and open.

## 1. Definitions and Lemmas.

1.1. DEFINITION. Let  $\varphi: X \to Y$ .  $f \in C(X)$  is said to be  $\varphi$ -bounded if  $\sup\{|f(x)|; x \in \varphi^{-1}(y)\} < \infty$  for every  $y \in Y$ . Whenever f is  $\varphi$ -bounded,

we put

$$f^{s}(y) = \sup\{f(x); x \in \varphi^{-1}(y)\} \text{ and}$$
$$f^{i}(y) = \inf\{f(x); x \in \varphi^{-1}(y)\} \text{ for each } y \in Y.$$

A subset A of X is *relatively pseudocompact* if f|A is bounded for each  $f \in C(X)$ .  $\varphi: X \to Y$  is said to be

(1) WZ if  $\operatorname{cl}_{\beta X} \varphi^{-1} y = (\beta \varphi)^{-1} y$  for each  $y \in Y[5]$ .

(2)  $W_r N$  if  $\operatorname{cl}_{\beta X} \varphi^{-1} R = (\beta \varphi)^{-1} (\operatorname{cl}_{\beta Y} R)$  for every regular closed set R of Y [3].

(3) \*-open (W\*-open) if  $int(cl \varphi U) \supset \varphi U$  ( $int(cl \varphi U) \neq \emptyset$ ) for every open set U of X [2, 7].

(4)  $\beta$ -open if  $\varphi$  is \*-open and  $W_r N$ .

(5) a  $d^*$ -map if  $\bigcap cl \varphi Z_n = \emptyset$  for any decreasing sequence  $\{Z_n\}$  of zero sets of X with empty intersection [6].

(6) *RC-preserving* (an *RC-map*) if  $\varphi R$  is regular closed(closed) for every regular closed set R of X [2].

We note that (1) a closed map is a Z-map and a Z-map is WZ [5], (2) an open map is \*-open and a \*-open map is  $W^*$ -open [7], (3) a space Y is cb \* iff any d\*-map onto Y is hyper-real, i.e.,  $v\varphi$  is a perfect map onto vY[6], (4) an RC-preserving map is RC and (5) an open WZ-map is  $\beta$ -open by 1.2 (1, 5) below. Thus it is easy to see that if  $\varphi$  is  $\beta$ -open, then  $(\beta\varphi)|Z$ :  $Z \to (\beta\varphi)Z$  is  $\beta$ -open for each Z with  $X \subset Z \subset \beta X$ .  $Y \supset B$  is said to be  $\varphi$ -d\* if  $(\beta\varphi)^{-1}B \subset vX$ . By 1.2(4) below,  $\varphi$  is a d\*-map iff Y is  $\varphi$ -d\*.

LEMMA 1.2. Let  $\varphi: X \to Y$ .

(1) If  $\varphi$  is WZ, then  $\varphi$  is open iff  $\beta \varphi$  is open [5].

(2) if  $\varphi$  is open (WZ), then f' is usc (lsc) and  $f^s$  is lsc (usc) for every  $f \in C(X; \varphi)$  (for example, see [5]).

(3) If  $\varphi$  is open WZ, then  $f^i$  and  $f^s \in C(Y)$  for every  $f \in C(X; \varphi)$  [5].

(4)  $\varphi$  is a d\*-map iff  $(\beta \varphi)^{-1} Y \subset \nu X[\mathbf{6}]$ .

(5)  $\varphi$  is  $\beta$ -open iff  $\beta \varphi$  is open [7].

(6) If  $\varphi$  is an RC-map, then  $\varphi$  is WZ [3].

(7)  $\varphi$  is *RC*-preserving iff  $\varphi$  is a *W*\*-open *RC*-map [2].

## 2. Main Theorems.

LEMMA 2.1. Let  $\varphi: X \to Y$ . Then the following are equivalent: (1)  $\varphi$  is WZ (open). (2)  $f^i$  is lsc (usc) for every  $f \in C(X; \varphi)$ (3)  $f^s$  is usc (lsc) for every  $f \in C(X; \varphi)$ . *Proof.* (2)  $\Leftrightarrow$  (3) is evident. (1)  $\Rightarrow$  (2). From 1.2(2).

We will prove  $(2) \Rightarrow (1)$ . Suppose that  $\varphi$  is not WZ. Then there are  $y \in Y$  and  $p \in \beta X$  with  $p \in (\beta \varphi)^{-1} y - \operatorname{cl}_{\beta X} \varphi^{-1} y$ . Since  $p \notin \operatorname{cl}_{\beta X} \varphi^{-1} y$ , there is  $g \in C(\beta X)$  such that  $p \in \operatorname{int}_{\beta X} Z(g)$  and g = 1 on  $\operatorname{cl}_{\beta X} \varphi^{-1} y$ . Let us put f = g | X. Then  $f \in C(X)$ ,  $f^i(y) = 1$ ,  $A = Z(f) \neq \emptyset$  and  $p \in \operatorname{cl}_{\beta X} A$ . On the other hand,  $\operatorname{cl}_{\beta Y} \varphi A = \operatorname{cl}_{\beta Y} (\beta \varphi) A = (\beta \varphi) \operatorname{cl}_{\beta X} A \ni (\beta \varphi) p$ = y. This shows  $y \in \operatorname{cl} \varphi A$  and hence for each neighborhood V of y, there is  $z \in V$  with  $f^i(z) = 0$ , i.e.,  $f^i$  is not lsc.

Now suppose that  $\varphi$  is not open. Then there are a point x and an open set  $U \ni x$  such that  $V - \varphi U \neq \emptyset$  for every open set  $V \ni y = \varphi(x)$ . Let  $f \in C(X; \varphi)$  such that  $x \in \text{int } Z(f) \subset U$  and f = 1 on X - U. Obviously  $f^{i}(y) = 0$  and  $f^{i} = 1$  on  $V - \varphi U$ . This shows that  $f^{i}$  is not usc.

Using 2.1, it is easy to see the following:

THEOREM 2.2.  $\varphi: X \to Y$  is open WZ iff  $f^i$  and  $f^s \in C(Y)$  for every  $f \in C(X; \varphi)$  equivalently,

$$C(Y) = \left\{ f^i; f \in C(X; \varphi) \right\} = \left\{ f^s; f \in C(X; \varphi) \right\}.$$

THEOREM 2.3.  $\varphi: X \to Y$  is a  $\beta$ -open  $d^*$ -map iff  $v\varphi$  is an open perfect map of vX onto vY.

*Proof.* ⇐) From 1.2(1, 4, 5) and  $(\beta \varphi)^{-1}Y \subset (\beta \varphi)^{-1}vY = vX. ⇒)$  By 1.2(5),  $\beta \varphi$  is open. We will prove that  $v\varphi$  is a perfect map onto vY. To do this, it suffices to show that  $(\beta \varphi)p = q \in \beta Y - vY$  for every  $p \in \beta X - vX$ . Let  $p \in \beta X - vX$ . Then there is  $f \in C(\beta X)$  with  $p \in Z(f) \subset \beta X - vX$ .  $\beta \varphi$  being open WZ by 1.2(5), it follows from 2.2 that  $f^i \in C(\beta Y)$ ,  $f^i(q) = 0$  and  $f^i > 0$  on Y. This shows  $q \in \beta Y - vY$ , so  $v\varphi$  is a perfect map onto vY. Since  $\beta(v\varphi) = \beta \varphi$  and  $\beta \varphi$  is open,  $v\varphi$  is open by 1.2(1). Thus  $v\varphi$  is an open perfect map of vX onto vY.

2.4. EXAMPLE. Let  $X = [0, \omega_1]^2 - \{(\omega_1, \alpha); \omega \le \alpha \le \omega_1\}, Y = [0, \omega_1]$ and  $\varphi$  the projection of X onto Y. It is obvious that  $\varphi$  is not WZ and hence not closed and  $\varphi^{-1}(\omega_1)$  is not compact. On the other hand  $\beta\varphi$ :  $\beta X = \nu X = [0, \omega_1]^2 \rightarrow Y = \nu Y = \beta Y$  is open perfect (compare with the assumption of Theorem A).

2.5. LEMMA. If  $\varphi: X \to Y$  is a \*-open RC-map, then  $\varphi$  is open.

*Proof.* Let U be open in X and  $x \in U$ . Take a regular closed set R with  $x \in \text{int } R \subset R \subset U$ . Since  $\varphi$  is a \*-open RC-map, we have  $y = \varphi(x) \in \text{int}(\operatorname{cl} \varphi(\operatorname{int} R)) \subset \varphi R \subset \varphi U$ , so  $y \in \operatorname{int} \varphi U$ . Thus  $\varphi$  is open.

In the following we put

 $Y_d = \{ y \in Y; \varphi^{-1}y \text{ is open but not relatively pseudocompact} \},$  $Y_e = X - Y_d.$ 

THEOREM 2.6.  $\varphi: X \to Y$  is a  $\beta$ -open map such that  $Y_e$  is  $\varphi$ -d\* iff  $v\varphi$  is an open RC-preserving map of vX onto vY such that  $cl_{vY}Y_e$  is  $(v\varphi)$ -d\*.

*Proof.*  $\Leftarrow$  ) Since  $v\varphi$  is open WZ by 1.2(6, 7),  $\beta\varphi$  is open by 1.2(1) and  $\varphi$  is a  $\beta$ -open map by 1.2(5). The fact that  $cl_{vY}Y_e$  is  $(v\varphi)-d^*$  implies that  $Y_e$  is  $\varphi-d^*$ .

⇒) (1) We will first prove that if  $p \in \beta X - vX$  and  $(\beta \varphi)p = q \in vY$ , then there is a clopen subset *D* of *Y* such that  $q \in cl_{vY}D$ ,  $D \subset Y_d$  and  $cl_{vY}D \cap cl_{vY}Y_e = \emptyset$ . There is  $f \in C(\beta X)$  with  $p \in Z(f) \subset \beta X - vX$ . By 1.2(5),  $\beta \varphi$  is open. Thus  $f^i \in C(\beta Y)$ . Since  $Y_e$  is  $\varphi - d^*$ ,  $f^i > 0$  on  $Y_e$  and hence  $Z(f^i) \cap Y_e = \emptyset$ . Since  $f^i(q) = 0$ ,  $q \in vY$  and  $Z(f^i)$  is closed.  $D = Z(f^i) \cap Y_d = Z(f^i) \cap Y$  is a non-empty clopen discrete subset of *Y* contained in  $Y_d$ .  $Cl_{vY}D = Z(f^i) \cap vY$  implies  $q \in cl_{vY}D$  and  $cl_{vY}D \cap$  $cl_{vY}Y_e = \emptyset$ .

(2) Let us put  $\mathcal{D} = \{ D \subset Y_d; D \text{ is a clopen subset of } Y \}$  and  $\operatorname{cl}_{vY} \mathcal{D} = \bigcup \{ \operatorname{cl}_{vY} D; D \in \mathcal{D} \}$ . Then it is easy to see the following

 $vY = \mathrm{cl}_{vY} \mathscr{D} \cup \mathrm{cl}_{vY} Y_e, \qquad \mathrm{cl}_{vY} \mathscr{D} \cap \mathrm{cl}_{vY} Y_e = \varnothing$ 

and

$$(\beta \varphi)^{-1} \mathrm{cl}_{\nu Y} Y_e \subset \nu X.$$

(3)  $v\varphi$  is onto vY. Let  $q \in cl_{vY}D$ ,  $D \in \mathcal{D}$ . For each  $y \in D$ , let us pick a point p(y) from  $\varphi^{-1}y$  and put  $A = \{ p(y); y \in D \}$ . Then A is a discrete closed C-embedded subset of X. Thus  $vA = cl_{vX}A$  is homeomorphic to  $cl_{vY}D$  under the map  $v\varphi$ . Thus we have  $v\varphi(vX) = vY$ .

(4)  $v\varphi$  is an RC-map. Let F be regular closed in vX and  $E = (v\varphi)F$ . Suppose that there is  $q \in cl_{vY}E - E$ . By (2) and the clopenness of  $\varphi^{-1}y$ ,  $y \in Y_d$ , we have  $q \notin Y_d \cup cl_{vY}Y_e$ . Thus there is  $D \in \mathscr{D}$  with  $q \in cl_{vY}D$  and  $cl_{vY}D \cap cl_{vY}Y_e = \emptyset$  by (2). Since  $\beta\varphi$  is open by 1.2(5),  $v\varphi$  is also \*-open and we have that  $E \supset (v\varphi)int_{vX}F$  is dense in  $cl_{vY}E$  because F is regular closed. Let  $M = E \cap D \cap Y_d$ . Then  $q \in cl_{vY}M$ . Let us pick a point p(y) from  $\varphi^{-1}(y) \cap F$ ,  $y \in M$ .  $A = \{p(y); y \in M\}$  is a discrete closed C-embedded subset of X and hence  $vA = cl_{vX}A \subset F$  and vA is homeomorphic to  $vM = cl_{vY}M$ , so  $q \in E$  a contradiction.

(5)  $v\varphi$  is open RC-preserving. Since  $v\varphi$  is an RC-map,  $v\varphi$  is WZ by 1.2(6). Thus the openness of  $\beta\varphi$  implies that  $v\varphi$  is open by 1.2(1) and RC-preserving by 1.2(7).

As a direct consequence of the above theorem, we have the following corollary which is a generalization of the result obtained in [5] if X is realcompact and  $\varphi: X \to Y$  is an open WZ map with Bd  $\varphi^{-1}y =$  compact for each  $y \in Y$ , then Y is also realcompact.

COROLLARY 2.7. If X is realcompact and  $\varphi: X \to Y$  is a  $\beta$ -open map such that  $Y_e$  is  $\varphi$ -d<sup>\*</sup>, then Y is also realcompact.

THEOREM 2.8. Let  $\varphi: X \to Y$  and  $Z = (\beta \varphi)^{-1} Y_d \cup vX$ . Then the following are equivalent:

(1) Z is a real compact and  $\varphi$  is a  $\beta$ -open map such that  $Y_e$  is  $\varphi$ -d\*.

(2)  $\varphi' = (\beta \varphi) | Z$  is an open perfect map of Z onto vY.

(3)  $v\varphi$  is a clopen map of vX onto vY such that  $Bd(v\varphi)^{-1}q$  is compact for every  $q \in vY$ .

(4)  $v\varphi$  is a clopen map of vX onto vY such that  $(vY)_e$  is  $(v\varphi)-d^*$ .

*Proof.* (1)  $\Rightarrow$  (2) If  $Z = \beta X$ , then  $\varphi' = \beta \varphi$  and  $\varphi'$  is an open perfect map onto  $\nu Y$ . Let  $p \in \beta X - Z$  and  $q = (\beta \varphi) p$ . Then  $Z = \nu Z$ ,  $\beta Z = \beta X$  and there is  $f \in C(\beta X)$  such that  $p \in Z(f) \subset \beta X - Z$  and  $0 \le f \le 1$ . Since  $\beta \varphi$  is open WZ and  $Y_e$  is  $\varphi \cdot d^*$ , it is easy to see that  $f^i \in C(\beta Y)$ ,  $f^i(q) = 0$  and  $f^i > 0$  on Y. Thus  $q \in \beta Y - \nu Y$ , so  $\varphi'$  is a perfect map onto  $\nu Y$ . The openness of  $\varphi'$  follows from 1.2(1, 5).

 $(2) \Rightarrow (3)$  We shall show that  $v\varphi$  is closed. Let F be closed in vX and  $q \in cl_{vY}(v\varphi)F - (v\varphi)F$ . Since  $\varphi'$  is perfect and every point of  $Y_d$  is isolated, we have  $q \notin Y_d$ , so  $(\beta\varphi)^{-1}q = (v\varphi)^{-1}q$  is disjoint from  $cl_ZF$ , and hence  $q \notin \varphi'(cl_ZF)$ , a contradiction. Thus  $v\varphi$  is closed. The verifications of other parts are easy. (3)  $\Rightarrow$  (4) Evident.

(4)  $\Rightarrow$  (1) Since  $v\varphi$  is clopen,  $\beta(v\varphi) = \beta\varphi$  is open by 1.2(1) and hence  $\varphi$  is  $\beta$ -open by 1.2(5). Since  $vY = (vY)_e \cup Y_d$ , the  $(v\varphi)$ -d\*-ness of  $(vY)_e = vY - Y_d$  implies the  $\varphi$ -d\*-ness of  $Y_e$ . Since  $Y_d = (vY)_d$  and  $(vY)_e$  is  $(v\varphi)$ -d\*, we have  $Z = (\beta\varphi)^{-1}vY$ , and hence  $\varphi': Z \rightarrow vY$  is an open perfect map which shows that Z is realcompact.

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