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JOHN MORSE DELAURENTIS AND BORIS G. PITTEL

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Consider the cycles of the random permutation of length n . Let $X_n(t)$ be the number of cycles with length not exceeding n^t , $t \in [0, 1]$. The random process $Y_n(t) = (X_n(t) - t \ln n) / \ln^{1/2} n$ is shown to converge weakly to the standard Brownian motion $W(t)$, $t \in [0, 1]$. It follows that, as a process, the empirical distribution function of "loglengths" of the cycles weakly converges to the Brownian Bridge process. As another application, an alternative proof is given for the Erdős-Turán Theorem: it states that the group-order of random permutation is asymptotically $e^{\mathcal{Q}}$, where \mathcal{Q} is Gaussian with mean $\ln^2 n/2$ and variance $\ln^2 n/3$.

1. Introduction. Results. Consider S_n , the symmetric group of permutations of a set $\{1, \dots, n\}$ endowed with the uniform distribution, $P(\sigma) = 1/n!$ for each $\sigma \in S_n$. Since a pioneering work by Goncharov [10], [11], a considerable attention has been paid to the asymptotic study of the order sequence of cycles lengths for the random permutation (r.p.), and of components sizes for the random mapping (Kolchin, et al. [13], [14], Shepp and Lloyd [20], Balakrishnan, et al. [1], Stephanov [21], Vershik and Schmidt [22]). Let $X_{ns} = X_{ns}(\sigma)$ designate the random number of cycles of length s in the r.p. σ . It is known [11] that X_n , the total number of cycles, is asymptotically normal with mean and variance $\ln n$. A similar result holds true for the total number of cycles whose lengths are divisible by a given number, [4], [20]. In this paper, we study the asymptotical behavior of the *joint* distribution of X_{n1}, \dots, X_{nn} .

For each $t \in [0, 1]$, consider

$$(1.1) \quad X_n(t) = \sum_{1 \leq s \leq n^t} X_{ns}, \quad Y_n(t) = (X_n(t) - t \ln n) / \ln^{1/2} n;$$

so, $X_n(t)$ is the total number of cycles of the r.p. with lengths not exceeding n^t . Clearly, each sample function of $Y_n(\cdot)$ belongs to $D[0, 1]$ the space of functions on $[0, 1]$ which are right-continuous at each $t \in [0, 1]$ and have left limits at each $t \in (0, 1]$. Introduce $W(t)$, $t \in [0, 1]$, the standard Brownian motion defined on a complete probability space with continuous sample paths. Let \mathcal{H} be a class of functionals on $D[0, 1]$ continuous in the sup-norm metric.

THEOREM. $Y_n(\cdot)$ converges to $W(\cdot)$ in terms of finite dimensional distributions. Moreover, for each $H \in \mathcal{H}$, the random variable $H(Y_n(\cdot))$ converges weakly to $H(W(\cdot))$; in short, $Y_n \Rightarrow W$.

Notes. Since $(X_n - \ln n)/\ln^{1/2} n = Y_n(1)$, the Goncharov result is a direct corollary of the theorem.

(2) To each cycle of the r.p., let us assign its “loglength” which is the logarithm of the cycle length with base n . Clearly, all the loglengths are in $[0, 1]$. Introduce the empirical distribution function (e.d.f.) $F_n(t)$, $t \in [0, 1]$, of the loglengths, that is, $F_n(t) = X_n(t)/X_n$. The theorem yields, after simple manipulations, that, as a process, $\ln^{1/2} n(F_n(t) - t)$, $t \in [0, 1]$, converges weakly (\Rightarrow) to $W(t) - tW(1)$, $t \in [0, 1]$. Thus, the asymptotical behavior of the loglengths is very nearly the same as of that for a sequence of $[\ln n]$ independent random variables each uniformly distributed on $[0, 1]$, [9].

(3) Consider Z_n and P_n respectively the order and the product of the cycle lengths of the r.p. Erdős and Turán [5] proved that $\ln P_n$ and $\ln Z_n$ are relatively close in probability, as $n \rightarrow \infty$. Later [6], they established, via very complicated argument, asymptotic normality of $\ln P_n$, whence of $\ln Z_n$. Best [4] found a simpler proof of closeness of $\ln P_n$ and $\ln Z_n$, but his proof that $\ln P_n$ is nearly normal remains rather technical. We are aware of, but have not seen, two other published proofs (Kolchin [15], Pavlov [18]) of the Erdős-Turán theorem.

Let us show how this theorem follows from our result.

First, we prove that, for each $\alpha > 2$,

$$(1.2) \quad P(\Delta_n \geq \ln n (\ln \ln n)^\alpha) \rightarrow 0, \quad n \rightarrow \infty,$$

where $\Delta_n = \ln P_n - \ln Z_n$. (Our proof resembles the Best argument, but is much simpler.) Introduce

$$D_{nk} = \sum_{s=1}^n \theta_s(k) X_{ns}, \quad \theta_s(k) = \begin{cases} 1 & \text{if } k|s, \\ 0 & \text{otherwise.} \end{cases}$$

Since $E(X_{ns}) = 1/s$, $E(X_{ns}(X_{ns'} - \delta_{ss'})) = 1/ss'$, $s + s' \leq n$, a simple computation leads to

$$(1.3) \quad \begin{aligned} E(D_{nk}) &= \sum_{s=1}^n \theta_s(k)/s = O(\ln n/k), \\ E(D_{nk}(D_{nk} - 1)) &\leq \left(\sum_{s=1}^n \theta_s(k)/s \right)^2 = O(\ln^2 n/k^2), \end{aligned}$$

the estimates being uniform in $k \leq n$.

Denote $D_{nk}^* = \min(1, D_{nk})$. Since the multiplicity of a prime factor p in P_n (resp. Z_n) is $\sum_{s \geq 1} D_{np^s}$, (resp. $\sum_{s \geq 1} D_{np^s}^*$), we have

$$\ln P_n = \sum_p \sum_{s \geq 1} D_{np^s} \ln p, \quad \ln Z_n = \sum_p \sum_{s \geq 1} D_{np^s}^* \ln p,$$

so that

$$\Delta_n \leq \sum_{k \geq 1} (D_{nk} - D_{nk}^*) \ln k = \sum_{k \geq 1} \Delta_{nk} \ln k.$$

As $\Delta_{nk} \leq D_{nk}$, $\Delta_{nk} \leq D_{nk}(D_{nk} - 1)/2$, we obtain (see (1.3)),

$$E(\Delta_n) \leq c \left(\ln n \sum_{k=1}^{[\ln n]} \ln k/k + \ln^2 n \sum_{k > [\ln n]} \ln k/k^2 \right) = O(\ln n (\ln \ln n)^2).$$

Since $\Delta_n \geq 0$, the last estimate implies (1.2).

Second, we prove that $\ln P_n$ is asymptotically normal with mean $2^{-1} \ln^2 n$ and variance $3^{-1} \ln^3 n$. (Then, in view of (1.2), $\ln Z_n$ has the same limiting distribution.) Introducing $t_{ns} = \ln s / \ln n$, $1 \leq s \leq n$, and summing up by parts, we have (see (1.1))

$$\begin{aligned} \ln P_n &= \sum_{1 \leq s \leq n} X_{ns} \ln s = \ln^2 n \left[1 - \sum_{1 \leq s \leq n-1} t_{ns} (t_{n,s+1} - t_{ns}) \right] \\ &\quad + \ln^{3/2} n \left[Y_n(1) - \sum_{1 \leq s \leq n-1} Y_n(t_{ns}) (t_{n,s+1} - t_{ns}) \right] \\ &= \ln^2 n [2^{-1} + O(\ln^{-2} n)] + \ln^{3/2} n \left[Y_n(1) - \int_0^1 Y_n(t) dt \right]. \end{aligned}$$

So, by the theorem,

$$(\ln P_n - 2^{-1} \ln^2 n) / \ln^{3/2} n \Rightarrow \int_0^1 (W(1) - W(t)) dt \stackrel{\mathcal{D}}{=} \int_0^1 W(t) dt.$$

It remains to observe that the last integral is normal with zero mean and variance 3^{-1} .

(4) For $\sigma \in S_n$, let $i \leq i_1 < \dots < i_\nu \leq n$, $\nu = \nu(\sigma)$, be the locations of all the (upper forward) record values in σ . Consider the inter-record times $\Delta_j = i_{j+1} - i_j$, $1 \leq j \leq \nu$, $\Delta_{\nu+1} = n + 1 - i_\nu$, and let $R_{ns} = R_{ns}(\sigma)$ stand for the number of Δ 's equal to s , $1 \leq s \leq n$. Since there exists a one-to-one mapping T of S_n onto itself such that

$$\{X_{ns}(\sigma)\}_{s=1}^n = \{R_{ns}(T(\sigma))\}_{s=1}^n,$$

([12], [16]), the sequences $\{R_{ns}\}_{s=1}^n$ and $\{X_{ns}\}_{s=1}^n$ are equidistributed. Thus, with no other proof needed, we could have formulated the analogues of the theorem, and the statement in (2), in terms of the inter-record times. The corresponding results appear to be new, though the

(inter)record times have been studied by many authors, [2], [8]. (For example, Neuts [17] proved asymptotic normality of the n th interrecord time in the (infinite) r.p. associated with a sequence of independent random variables with a common continuous distribution function.)

2. Proof of the theorem. The joint distribution of X_{ns} , $1 \leq s \leq n$, is given by Cauchy’s formula [3]:

$$(2.1) \quad P(X_{ns} = \alpha_s, 1 \leq s \leq n) = \begin{cases} \prod_{s=1}^n ((1/s)^{\alpha_s}/\alpha_s!), & \text{if } \sum_{s=1}^n s\alpha_s = n, \\ 0 & \text{otherwise.} \end{cases}$$

Introduce a bounded sequence $z = \{z_s\}_{s=1}^\infty$ and the sequence of generating functions (g.f.) $f_n(z) = E(\prod_{1 \leq s \leq n} z_s^{X_{ns}})$, $n \geq 1$, $f_0(z) \equiv 1$. It follows from (2.1) that, for $|t| < 1$,

$$(2.2) \quad \sum_{n \geq 0} t^n f_n(z) = \exp \left[\sum_{s \geq 1} z_s t^s / s \right],$$

[19], (cf. [20]). Fix the positive integers r, l_1, \dots, l_r , and introduce the r -dimensional g.f. $g_n(y) = E(\prod_{\nu=1}^r y_\nu^{X_{n l_\nu}})$, ($X_{n l} = 0$, for $l > n$). Choosing in (2.2) $z_s = y_\nu$, if $s = l_\nu$ ($1 \leq \nu \leq r$), and $z_s = 1$ otherwise, we obtain that ($g_0(y) \equiv 1$)

$$(2.3) \quad \begin{aligned} \sum_{n \leq 0} t^n g_n(y) &= \exp \left[\sum_{\nu=1}^r y_\nu t^{l_\nu} / l_\nu + \sum_{s \neq l_1, \dots, l_r} t^s / s \right] \\ &= \exp \left[\sum_{\nu=1}^r (y_\nu - 1) t^{l_\nu} / l_\nu \right] / (1 - t). \end{aligned}$$

Hence, by Cauchy’s integral formula,

$$(2.4) \quad g_n(y) = (2\pi i)^{-1} \int_C \exp \left[\sum_{\nu=1}^r (y_\nu - 1) z^{l_\nu} / l_\nu \right] / ((1 - z) z^{n+1}) dz,$$

where C is any circle with radius less than one surrounding the origin in the complex plane. It is important that (2.4) holds for each set of positive integers n, r, l_1, \dots, l_r .

Introduce a process

$$(2.5) \quad Y_n^*(t) = \left(\sum_{1 \leq s \leq [n^t]} (X_{ns} - 1/s) \right) / \ln^{1/2} n, \quad t \in [0, 1].$$

Since $\sum_{1 \leq s \leq \nu} 1/s - \ln \nu = 0(1)$, $\nu \rightarrow \infty$, it suffices to prove that $Y_n^* \Rightarrow W$. (Centering of X_{ns} by $1/s$ is natural since $E(X_{ns}) = 1/s$, $1 \leq s \leq n$.)

LEMMA 1. For each fixed k and $0 = t_0 < \dots < t_k = 1$, the random vector $\{Y_n^*(t_j)\}_{j=1}^k$ converges to $\{W(t_j)\}_{j=1}^k$ in distribution.

Proof. For $1 \leq j \leq k$, let $n_j = [n^{t_j}]$ so that $n_0 = 1, n_k = n$. Denote

$$\mathcal{X}_{n_j} = \sum_{s=n_{j-1}+1}^{n_j} X_{ns}, \quad \bar{\mathcal{X}}_{n_j} = \sum_{s=n_{j-1}+1}^{n_j} 1/s.$$

We have to show that $\{(\mathcal{X}_{n_j} - \bar{\mathcal{X}}_{n_j})/\ln^{1/2} n\}_{j=1}^k$ converges weakly to the Gaussian vector with k independent components having parameters $(0, t_j - t_{j-1}), 1 \leq j \leq k$.

Introduce $x_j = \exp(u_j/\ln^{1/2} n)$, $u_j > 0$ and is fixed, $1 \leq j \leq k$. Setting $r = n, l_s = s$ for each s , and $y_\nu = x_j$ for $n_{j-1} + 1 \leq \nu \leq n_j$ in (2.4), and choosing the radius of C equal to $\rho = 1 - n^{-1}$, we have

$$(2.6) \quad h_n(x) = E \left(\prod_{j=1}^k x_j^{\mathcal{X}_{n_j}} \right) \\ = (2\pi\rho^n)^{-1} \exp \left[\sum_{j=1}^k (x_j - 1) \sum_{s=n_{j-1}+1}^{n_j} \rho^s/s \right] \cdot I,$$

where

$$(2.7) \quad I = \int_{[-\pi, \pi]} e^{-in\phi} b_n(\phi) d\phi, \quad b_n(\phi) = (1 - \rho e^{i\phi})^{-1} \exp[a_n(\phi)],$$

$$(2.8) \quad a_n(\phi) = \sum_{j=1}^k (x_j - 1) \sum_{s=n_{j-1}+1}^{n_j} (\rho^s/s)(e^{is\phi} - 1).$$

To estimate I , we proceed as follows. Break $[-\pi, \pi)$ into $[-\phi_0, \phi_0]^c, [-\phi_0, \phi_0], \phi_0 = n^{-3/4}$; let the corresponding integrals be I_1, I_2 . First, we estimate I_1 . In I_2 , we replace $b_n(\phi)$ by $\tilde{b}_n(\phi)$, which is close to $b_n(\phi)$ for $\phi \in [-\phi_0, \phi_0]$, and nicely manageable if $\phi \in (-\infty, \infty)$. The resulting integral \tilde{I}_2 is a difference of two integrals J_1 and J_2 , over respectively $(-\infty, \infty)$ and $(-\infty, \infty) - [-\phi_0, \phi_0]$. We estimate J_2 . J_1 , whose contribution in the value of I is dominant, is asymptotically evaluated by means of the inversion formula for an L_1 -integrable characteristic function.

The proof follows.

(1) Show that

$$(2.9) \quad I_1 = O(n^{-1/4}).$$

Integrating once by parts, we have

$$\left| \int_{[\phi_0, \pi]} e^{-in\phi} b_n(\phi) d\phi \right| \leq n^{-1} \left[|b_n(\pi)| + |b_n(\phi_0)| + \int_{[\phi_0, \pi]} |b'_n(\phi)| d\phi \right].$$

Here, (see (2.7), (2.8)),

$$|b_n(\pi)| = O(\exp(\operatorname{Re} a_n(\pi))) = O(1),$$

since

$$\operatorname{Re}(a_n(\phi)) \leq \sum_{j=1}^k (x_j - 1) \sum_{s=n_{j-1}+1}^{n_j} (\rho^s/s)(\cos s\phi - 1) \leq 0,$$

($x_j \geq 1, j = 1, \dots, k$). Also,

$$(2.10) \quad |b_n(\phi_0)| \leq |1 - \rho e^{i\phi_0}|^{-1} = [(1 - \rho)^2 + 2\rho(1 - \cos \phi_0)]^{-1/2} \\ = O[(1 - \cos \phi_0)^{-1/2}] = O(\phi_0^{-1}).$$

Further, since

$$b'_n(\phi) = \rho i(1 - \rho e^{i\phi})^{-2} \exp[a_n(\phi)] + i(1 - \rho e^{i\phi})^{-1} \\ \times \exp[a_n(\phi)] \sum_{j=1}^k (x_j - 1) \sum_{s=n_{j-1}+1}^{n_j} (\rho e^{i\phi})^s,$$

estimate

$$|b'_n(\phi)| \leq |1 - \rho e^{i\phi}|^{-2} + |1 - \rho e^{i\phi}|^{-2} \sum_{j=1}^k (x_j - 1) |1 - (\rho e^{i\phi})^{n_j - n_{j-1}}| \\ = O(|1 - \rho e^{i\phi}|^{-2}) = O[(1 - \cos \phi)^{-1}] = O(\phi^{-2}),$$

for $\phi \in [\phi_0, \pi]$. Therefore

$$\int_{[\phi_0, \pi]} |b'_n(\phi)| d\phi = O(\phi_0^{-1}),$$

and, together with (2.10), it yields

$$\left| \int_{[\phi_0, \pi]} e^{-in\phi} b_n(\phi) d\phi \right| = O((n\phi_0)^{-1}) = O(n^{-1/4}).$$

The case of $[-\pi, -\phi_0]$ is similar.

(2) To estimate I_2 , compare it with

$$(2.11) \quad \tilde{I}_2 = \int_{[-\phi_0, \phi_0]} e^{-in\phi} \tilde{b}_n(\phi) d\phi,$$

$$\tilde{b}_n(\phi) = [1 - \rho(1 + i\phi)]^{-1} (1 - i\phi)^{-1} \exp[a_n(\phi)].$$

Since $\rho = 1 - n^{-1}$ and $|e^{i\phi} - (1 + i\phi)| \leq 2^{-1}\phi^2$,

$$\begin{aligned} & |(1 - \rho e^{i\phi})^{-1} - [1 - \rho(1 + i\phi)]^{-1}(1 - i\phi)^{-1}| \\ & \leq 2^{-1}n^2\phi^2 + |1 - \rho(1 + i\phi)|^{-1}|1 - (1 - i\phi)^{-1}| \\ & \leq 2^{-1}n^2\phi^2 + n|\phi|. \end{aligned}$$

Subsequently ($\phi_0 = n^{-3/4}$),

$$\begin{aligned} (2.12) \quad |I_2 - \tilde{I}_2| & \leq \int_{[-\phi_0, \phi_0]} |b_n(\phi) - \tilde{b}_n(\phi)| d\phi \\ & \leq 2^{-1}n^2 \int_{[-\phi_0, \phi_0]} \phi^2 d\phi + n \int_{[-\phi_0, \phi_0]} |\phi| d\phi \\ & = O(n^2\phi_0^3 + n\phi_0^2) = O(n^{-1/4}). \end{aligned}$$

Thus, it suffices to estimate \tilde{I}_2 . Notice first that $\tilde{b}_n(\cdot) \in L_1(-\infty, \infty)$; (one reason why the factor $(1 - i\phi)^{-1}$ is included in $\tilde{b}_n(\phi)$ is to have this happen). If so,

$$\begin{aligned} (2.13) \quad \tilde{I}_2 & = \int_{(-\infty, \infty)} e^{-in\phi} \tilde{b}_n(\phi) d\phi - \int_{[-\phi_0, \phi_0]^c} e^{-in\phi} \tilde{b}_n(\phi) d\phi \\ & = n^{-1} \int_{(-\infty, \infty)} e^{-iu} \tilde{b}_n(u/n) du - n^{-1} \int_{|u| \geq n\phi_0} e^{-iu} \tilde{b}_n(u/n) du \\ & = J_1 - J_2. \end{aligned}$$

(2a) Evaluate J_1 . By (2.8), (2.11), we have

$$\begin{aligned} (2.14) \quad n^{-1}\tilde{b}_n(u/n) & = (1 - i\alpha_1 u)^{-1}(1 - i\alpha_2 u)^{-1} \exp[a_n(u/n)] \\ & = (1 - i\alpha_1 u)^{-1}(1 - i\alpha_2 u)^{-1} \prod_{j=1}^k \prod_{s=n_{j-1}+1}^{n_j} \exp[\Lambda_{js}(e^{isu/n} - 1)] \end{aligned}$$

where

$$(2.15) \quad \alpha_1 = \rho = 1 - n^{-1}, \quad \alpha_2 = n^{-1},$$

$$(2.16) \quad \Lambda_{js} = (x_j - 1)\rho^s/s.$$

Notice that

$$(1 - iu)^{-1} = E[\exp(iuV)], \quad \exp[\Lambda(e^{iu} - 1)] = E[\exp(iu\mathcal{P}(\Lambda))],$$

where $V \geq 0$ is exponentially distributed with parameter 1, and $\mathcal{P}(\Lambda)$ is Poisson distributed with parameter Λ . Hence, a crucial observation:

$$(2.17) \quad n^{-1}\tilde{b}_n(u/n) = E[\exp(iuM_n)],$$

$$(2.18) \quad M_n = \alpha_1 V_1 + \alpha_2 V_2 + \sum_{j=1}^k \sum_{s=n_{j-1}+1}^{n_j} (s/n) \mathcal{P}(\Lambda_{js}),$$

where $V_1, V_2, \{\mathcal{P}(\Lambda_{js})\}_{j,s}$ are all independent. Since $n^{-1}\tilde{b}_n(u/n) \in L_1(-\infty, \infty)$, M_n has a (bounded) continuous density f_{M_n} (of course, it is seen directly from (2.18)). Moreover, by the inversion formula [7], for each x

$$f_{M_n}(x) = (2\pi)^{-1} \int_{(-\infty, \infty)} e^{-iux} E[\exp(iuM_n)] du,$$

so

$$J_1 = n^{-1} \int_{(-\infty, \infty)} e^{-iu} \tilde{b}_n(u/n) du = 2\pi f_{M_n}(1). (!)$$

The density of $\alpha_1 V_1$ is $\alpha_1 \exp(-\alpha_1 x)$, $x \geq 0$; denote F_n the distribution function of

$$\tilde{M}_n = \alpha_2 V_2 + \sum_{j=1}^k \sum_{s=n_{j-1}+1}^{n_j} (s/n) \mathcal{P}(\Lambda_{js}).$$

Then

$$(2.19) \quad f_{M_n}(1) = \int_{(-\infty, 1]} \alpha_1 \exp(-\alpha_1(1-x)) dF_n(x).$$

Now (see (2.18)),

$$\begin{aligned} E(\tilde{M}_n) &= \alpha_2 + \sum_{j=1}^k \sum_{s=n_{j-1}+1}^{n_j} (s/n) \Lambda_{js} \\ &\leq n^{-1} + \left[\max_{1 \leq j \leq k} (x_j - 1) \right] n^{-1} \sum_{s \geq 1} \rho^s \\ &= O\left(n^{-1} + \max_{1 \leq j \leq k} (x_j - 1) \right) = O(\ln^{-1/2} n). \end{aligned}$$

Therefore, for each $\epsilon > 0$,

$$(2.20) \quad \lim_{n \rightarrow \infty} [F_n(\epsilon) - F_n(-\epsilon)] = 1.$$

Since $\alpha_1 \rightarrow 1$ as $n \rightarrow \infty$, we get from (2.19), (2.20) that

$$\lim_{n \rightarrow \infty} f_{M_n}(1) = \lim_{n \rightarrow \infty} \alpha_1 \exp(-\alpha_1) = e^{-1}.$$

Thus, $J_1 \rightarrow 2\pi e^{-1}$. More precisely, since $\alpha_1 = 1 - n^{-1}$ and $E(\tilde{M}_n) = O(\ln^{-1/2} n)$,

$$(2.21) \quad J_1 = 2\pi e^{-1} + O(\ln^{-\delta} n), \quad \forall \delta \in (0, 1/2).$$

(2b). Estimate J_2 . Integrating by parts, we have ($B_n(u) = n^{-1}\tilde{b}_n(u/n)$)

$$\left| \int_{u \geq n\phi_0} e^{-iu} B_n(u) du \right| \leq |B_n(n\phi_0)| + \int_{u \geq n\phi_0} |B'_n(u)| du.$$

Here (see (2.8), (2.14)),

$$(2.22) \quad |B_n(n\phi_0)| = O(|1 - i\alpha_1 n\phi_0|^{-1}) = O((n\phi_0)^{-1}) = O(n^{-1/4}),$$

$$|B'_n(u)| = O(u^{-2} + n^{-1}u^{-1}|1 - i\alpha_2 u|^{-1} + n^{-1}|a'_n(u/n)|nu^{-2})$$

$$= O(u^{-2} + |a'_n(u/n)|u^{-2}),$$

and

$$(2.23) \quad |a'_n(u/n)| = \left| \sum_{j=1}^k (x_j - 1) \sum_{s=n_{j-1}+1}^{n_j} (\rho e^{iu/n})^s \right|$$

$$\leq 2k \max_{1 \leq j \leq k} (x_j - 1) |1 - \rho e^{iu/n}|^{-1},$$

$$(2.24) \quad |1 - \rho e^{iu/n}|^{-1} \leq [n^{-2} + 1 - \cos(u/n)]^{-1/2}$$

$$= \begin{cases} O(nu^{-1}), & \text{if } u/n \leq \pi, \\ O(n), & \text{always.} \end{cases}$$

Putting together (2.22)–(2.24), we obtain

$$\int_{u \geq n\phi_0} |B'_n(u)| du = O\left(\int_{u > n\phi_0} u^{-2} du \right)$$

$$+ O\left(\max_{1 \leq j \leq k} (x_j - 1) \left(n \int_{[n\phi_0, n\pi]} u^{-3} du + n \int_{u \geq \pi n} u^{-2} du \right) \right)$$

$$= O\left((n\phi_0)^{-1} + \max_{1 \leq j \leq k} (x_j - 1) \right) = O(\ln^{-1/2} n).$$

Therefore (the case $u \leq -n\phi_0$ is similar),

$$(2.25) \quad J_2 = O(\ln^{-1/2} n).$$

(3) Combining (2.9), (2.12), (2.13), (2.21) and (2.25), we can conclude:

$$I = J_1 + [I_1 + (I_2 - \tilde{I}_2) - J_2] = 2\pi e^{-1} + O(\ln^{-\delta} n), \quad \delta \in (0, 1/2).$$

Hence, by (2.6) and $\rho^n = e^{-1}(1 + O(n^{-1}))$,

$$\begin{aligned} h_n(x) &= E\left(\prod_{j=1}^k x_j^{\mathcal{X}_{n_j}}\right) \\ &= \exp\left[\sum_{j=1}^k (x_j - 1) \sum_{s=n_{j-1}+1}^{n_j} \rho^s/s\right] (1 + O(\ln^{-\delta} n)). \end{aligned}$$

What remains is to evaluate the first factor on the right. Since

$$x_j - 1 = \exp(u_j/\ln^{1/2} n) - 1 = u_j/\ln^{1/2} n + u_j^2/2 \ln n + O(\ln^{-3/2} n),$$

$$(2.26) \quad \left| \sum_{s=n_{j-1}+1}^{n_j} (\rho^s/s - 1/s) \right| \leq 1,$$

$$(2.27) \quad \left| \sum_{s=n_{j-1}+1}^{n_j} 1/s - (\ln n_j - \ln n_{j-1}) \right| \leq 1,$$

$$(2.28) \quad |(\ln n_j - \ln n_{j-1}) - (t_j - t_{j-1}) \ln n| \leq 1,$$

we have

$$\begin{aligned} &E\left\{\exp\left[\left(\sum_{j=1}^k u_j \mathcal{X}_{n_j}\right)/\ln^{1/2} n\right]\right\} \\ &= \exp\left[\left(\sum_{j=1}^k u_j \sum_{s=n_{j-1}+1}^{n_j} 1/s\right)/\ln^{1/2} n\right] \\ &\quad \times \exp\left[2^{-1} \sum_{j=1}^k u_j^2 (t_j - t_{j-1})\right] (1 + O(\ln^{-\delta} n)). \end{aligned}$$

So, by definition of \mathcal{X}_{n_j} ,

$$\begin{aligned} \lim_{n \rightarrow \infty} E\left\{\exp\left[\left(\sum_{j=1}^k u_j (\mathcal{X}_{n_j} - \bar{\mathcal{X}}_{n_j})/\ln^{1/2} n\right)\right]\right\} \\ = \prod_{j=1}^k \exp\left[2^{-1} u_j^2 (t_j - t_{j-1})\right]. \end{aligned}$$

It follows from this relation that $\{(\mathcal{X}_{n_j} - \bar{\mathcal{X}}_{n_j})/\ln^{1/2} n\}_{j=1}^k$ converges in distribution to $\{W(t_j) - W(t_{j-1})\}_{j=1}^k$. Lemma 1 is proven.

To complete the proof of the Theorem, it suffices to show [9] that the processes $Y_n^*(\cdot)$ are equicontinuous, or more precisely, that for each $\varepsilon > 0$,

$$\lim_{c \downarrow 0} \limsup_{n \rightarrow \infty} P\left\{\sup_{|t''-t'|\leq c} |Y_n^*(t'') - Y_n^*(t')| \geq \varepsilon\right\} = 0.$$

A method we shall use to prove it is inspired by a proof of equicontinuity of the e.d.f. processes $\xi_n(\cdot)$ on $[0, 1]$ for a sequence of n independent random variables uniformly distributed on $[0, 1]$ (see Introduction), which is given in [9].

By definition of $Y_n^*(\cdot)$ (see (2.5), (2.6)), $Y_n^*(t) + (\sum_{s=1}^{\lfloor n^t \rfloor} 1/s) / \ln^{1/2} n$ is a nondecreasing function of t . Hence, for $0 \leq t_1 \leq t_2 \leq t_3 \leq t_4 \leq 1$,

$$(2.29) \quad \begin{aligned} -\Delta_n(t_1, t_4) &\leq Y_n^*(t_3) - Y_n^*(t_2) \\ &\leq Y_n^*(t_4) - Y_n^*(t_1) + \Delta_n(t_1, t_4), \\ \Delta_n(t_1, t_4) &= \left(\sum_{s=\lfloor n^{t_1} \rfloor + 1}^{\lfloor n^{t_4} \rfloor} 1/s \right) / \ln^{1/2} n, \end{aligned}$$

where (see (2.27), (2.28)),

$$(2.30) \quad \Delta_n(t_1, t_4) \leq (t_4 - t_1) \ln^{1/2} n + 2 \ln^{-1/2} n.$$

The proof quoted above is based only on (2.29) (with $\xi_n(\cdot)$ instead of $Y_n^*(\cdot)$, of course), where

$$(2.31) \quad \Delta_n(t_1, t_4) \leq (t_4 - t_1) n^{1/2},$$

and an inequality

$$(2.32) \quad E \left[(\xi_n(t+h) - \xi_n(t))^4 \right] \leq ch^2, \quad \text{for } h \geq n^{-1}.$$

No changes would have been necessary, had the inequality (2.31) contained on its right-hand side an extra term $o(1)$, which is present in (2.30). Thus, in our case it would be sufficient, (compare (2.30) with (2.31), (2.32)), to prove an inequality analogous to (2.32) with restriction on h of the form: $h \geq \ln^{-1} n$. Fortunately, it is exactly the case here.

LEMMA 2. *There is*

$$(2.33) \quad E \left[(Y_n^*(t+h) - Y_n^*(t))^4 \right] \leq 174h^2, \quad \text{if } h \geq \ln^{-1} n.$$

Proof. Fix $1 \leq \nu_1 \leq \nu_2$. Introduce $\sum_{s=\nu_1}^{\nu_2} X_{ns}$, the total number of cycles with lengths from ν_1 to ν_2 . Denote it just C_n , for simplicity of subsequent expressions. We shall prove

$$(2.34) \quad E \left[(C_n - E(C_n))^4 \right] \leq 15E^2(C_n) + 13E(C_n),$$

where

$$(2.35) \quad E(C_n) = \sum_{\{s \leq n: \nu_1 \leq s \leq \nu_2\}} 1/s.$$

But let us show first how (2.34), (2.35) lead to (2.33). We have:

$$(2.36) \quad Y_n^*(t+h) - Y_n^*(t) = \left(\sum_{s=[n^t]+1}^{[n^{t+h}]} (X_{ns} - 1/s) \right) / \ln^{1/2} n \\ = (C_n - E(C_n)) / \ln^{1/2} n,$$

with $\nu_1 = [n^t] + 1$, $\nu_2 = [n^{t+h}]$. Then, (see (2.27), (2.28)),

$$E(C_n) = \sum_{s=[n^t]+1}^{[n^{t+h}]} 1/s \leq h \ln n + 2 = \ln n (h + 2 \ln^{-1} n) \leq 3h \ln n,$$

if $h \geq \ln^{-1} n$. Since (2.34), (2.36), we conclude that

$$E \left[(Y_n^*(t+h) - Y_n^*(t))^4 \right] \leq (135h^2 \ln^2 n + 39h \ln n) / \ln^2 n \\ = 135h^2 + 39h \ln^{-1} n \leq 174h^2.$$

In order to prove (2.34), notice first that by (2.3),

$$\sum_{n \geq 0} t^n E(y^{C_n}) = \exp \left[\sum_{s=\nu_1}^{\nu_2} (y-1)t^s/s \right] / (1-t).$$

Taking the j th order derivative of both sides of this relation at $y = 1$, we obtain

$$(2.37) \quad \sum_{n \geq 0} t^n m_n^{(j)} = \left(\sum_{s=\nu_1}^{\nu_2} t^s/s \right)^j / (1-t),$$

where

$$m_n^{(j)} = E [C_n(C_n - 1) \cdots (C_n - j + 1)]$$

is the j th order factorial moment of C_n . Equating coefficients by the same powers of t on both sides of (2.37) yields

$$(2.38) \quad m_n^{(j)} = \sum_{s_1 + \cdots + s_j \leq n} \prod_{\mu=1}^j 1/s_\mu;$$

(here and everywhere below, the restrictions $\nu_1 \leq s_\mu \leq \nu_2$ ($1 \leq \mu \leq j$), are silently assumed; the same goes for $s_\mu \leq n$ ($1 \leq \mu \leq j$), though in this case these restrictions are redundant). In case $j = 1$, (2.38) gives (2.35). A direct corollary of (2.38) is

$$(2.39) \quad m_n^{(j)} \leq (m_n^{(1)})^j = E^j(C_n),$$

or, more generally,

$$(2.40) \quad m_n^{(j_2)} \leq m_n^{(j_1)} (m_n^{(1)})^{j_2 - j_1}, \quad j_2 \geq j_1 \geq 1.$$

Now, a simple argument shows that

$$E[(C_n - E(C_n))^4] = E_1 + E_2 + E_3 + E_4,$$

$$(2.41) \quad E_1 = m_n^{(1)}, \quad E_2 = 7m_n^{(2)} - 4(m_n^{(1)})^2,$$

$$(2.42) \quad E_3 = 6m_n^{(3)} - 12m_n^{(2)}m_n^{(1)} + 6(m_n^{(1)})^3,$$

$$(2.43) \quad E_4 = m_n^{(4)} - 4m_n^{(3)}m_n^{(1)} + 6m_n^{(2)}(m_n^{(1)})^2 - 3(m_n^{(1)})^4.$$

Estimate E_2, E_3, E_4 . By (2.39), (2.41),

$$(2.44) \quad E_2 \leq 3(m_n^{(1)})^2.$$

Then, by (2.40), (2.42),

$$\begin{aligned} E_3 &= 6(m_n^{(3)} - m_n^{(2)}m_n^{(1)}) + 6m_n^{(1)}[(m_n^{(1)})^2 - m_n^{(2)}] \\ &\leq 6m_n^{(1)}[(m_n^{(1)})^2 - m_n^{(2)}]; \end{aligned}$$

here (see (2.38)),

$$\begin{aligned} (m_n^{(1)})^2 - m_n^{(2)} &= \sum \frac{1}{s_1 s_2} - \sum_{s_1 + s_2 \leq n} \frac{1}{s_1 s_2} \\ &= \sum_{s_1 + s_2 > n} \frac{1}{s_1 s_2} = \sum_{s_1 + s_2 > n} (s_1 + s_2)^{-1} \left(\frac{1}{s_1} + \frac{1}{s_2} \right) \\ &\leq \left(\frac{2}{n} \right) \sum_{s_1 + s_2 \geq n} \frac{1}{s_1} \leq \left(\frac{2}{n} \right) \sum_s \left(\frac{1}{s} \right) s \leq 2, \end{aligned}$$

so

$$(2.45) \quad E_3 \leq 12m_n^{(1)}.$$

Consider finally E_4 . Write (see (2.40), (2.43))

$$\begin{aligned} (2.46) \quad E_4 &= (m_n^{(4)} - m_n^{(3)}m_n^{(1)}) + 3(m_n^{(1)})^2[m_n^{(2)} - (m_n^{(1)})^2] \\ &\quad + 3m_n^{(1)}(m_n^{(2)}m_n^{(1)} - m_n^{(3)}) \\ &\leq 3m_n^{(1)}(m_n^{(2)}m_n^{(1)} - m_n^{(3)}) = 3m_n^{(1)} \sum. \end{aligned}$$

Here (see (2.38))

$$\begin{aligned}
 (2.47) \quad \sum &= \sum_{s_1+s_2 \leq n} \frac{1}{s_1 s_2 s_3} - \sum_{s_1+s_2+s_3 \leq n} \frac{1}{s_1 s_2 s_3} \\
 &= \sum_{\substack{s_1+s_2 \leq n \\ s_1+s_2+s_3 > n}} (s_1 + s_2 + s_3)^{-1} \left[\frac{1}{s_1 s_2} + \frac{1}{s_1 s_3} + \frac{1}{s_2 s_3} \right] \\
 &\leq n^{-1} \sum_{\substack{s_1+s_2 \leq n \\ s_1+s_2+s_3 > n}} \frac{1}{s_1 s_2} + 2n^{-1} \sum_{\substack{s_1+s_2 \leq n \\ s_1+s_2+s_3 > n}} \frac{1}{s_1 s_3} \\
 &= \sum' + \sum''.
 \end{aligned}$$

Since in Σ' , for each (s_1, s_2) , s_3 can assume at most $s_1 + s_2$ values,

$$\begin{aligned}
 (2.48) \quad \sum' &\leq n^{-1} \sum_{s_1+s_2 \leq n} \frac{s_1 + s_2}{s_1 s_2} = 2n^{-1} \sum_{s_1+s_2 \leq n} \frac{1}{s_1} \\
 &\leq 2n^{-1} \sum_s \frac{n}{s} = 2m_n^{(1)};
 \end{aligned}$$

similarly,

$$(2.49) \quad \sum'' \leq 2n^{-1} \sum \frac{s_3}{s_1 s_3} = 2n^{-1} \sum_s \frac{n}{s} = 2m_n^{(1)}.$$

Hence, (see (2.46)–(2.49)),

$$(2.50) \quad E_4 \leq 12(m_n^{(1)})^2.$$

Collecting together (2.44), (2.45), and (2.50), we arrive at (2.34) (remember, $E(C_n) = m_n^{(1)}$).

The theorem is proven.

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REFERENCES

[1] V. Balakrishnan, G. Sankaranarayanan, and C. Suyambulingom, *Ordered cycle lengths in a random permutation*, *Pacif. J. Math.*, **36** (1971), 603–613.
 [2] D. E. Barton and C. L. Mallows, *Some aspects of the random sequence*, *Ann. Statist.*, **36** (1965), 236–260.
 [3] C. Berge, *Principles of Combinatorics*, Acad. Press, New York, 1971.

- [4] M. R. Best, *The distribution of some variables on symmetric groups*, Proc. Kon. Ned. Akad. Wet. Ser. A **73** (1970), 385–402.
- [5] P. Erdős and P. Turán, *On some problems of a statistical group theory*, I. Zeitschrift für Wahrsch. und Verw. Geb. **4**, H.2, (1965), 175–186.
- [6] ———, *On some problems of statistical group theory*, II, III, IV. Acta Math. Acad. Sci. Hung., **18** (1967, 1968), 1–2, 151–163; 3–4, 309–320; 19, 3–4, 413–435.
- [7] W. Feller, *An introduction to Probability Theory and its Applications*, II, Wiley and Sons, New York, 1966.
- [8] J. Galambos, *The Asymptotic Theory of Extreme Order Statistics*, Wiley and Sons, New York, 1978.
- [9] I. I. Gihman and A. V. Skorohod, *The Theory of Stochastic Processes*, I. Springer-Verlag, New York, 1974.
- [10] V. L. Goncharov, *Sur la distribution des cycles dans les permutations*, C. R. (Doklady), Acad. Sci. URSS (N.S.), **35** (1942), 267–269.
- [11] ———, *Some facts from combinatorics*, Izvestia Akad. Nauk SSSR, Ser. Mat. **8** (1944), 3–48; (see also (1962) Translations of AMS, Ser. 2, 19, 1–46).
- [12] D. E. Knuth, *The art of computer programming*, I, Addison Wesley, Reading, Massachusetts, 1968.
- [13] V. F. Kolchin, B. A. Sevast'yanov and V. P. Chistyakov, *Random Allocations*, Wiley and New York, 1978.
- [14] ———, *A problem of the allocation of particles in cells and random mappings*, Theory Probability Appl., **21** (1976), 48–63.
- [15] ———, *A new proof of asymptotical lognormality of the order of a random permutation*. Proceedings “Combinatorial and asymptotical analysis”, Krasnoyarsk State University Press, (1977), 82–93. (In Russian)
- [16] L. Lovasz, *Combinatorial Problems and Exercises*, North-Holland, New York, 1979.
- [17] M. F. Neuts, *Waiting times between record observations*, J. Appl. Probability, **4** (1967), 206–208.
- [18] A. I. Pavlov, *On a theorem by Erdős and Turán*, Problems of Cybernetics, (Moscow), **64** (1980), 57–66. (In Russian)
- [19] V. N. Sachkov, *Probabilistic methods in combinatorial analysis*, Nauka, Moscow, 1978. (In Russian)
- [20] L. A. Shepp, and S. P. Lloyd, *Ordered cycle lengths in a random permutation*, Trans. Amer. Math. Soc., **121** (1966), 340–357.
- [21] V. E. Stepanov, *Limit distributions of certain characteristics of random mappings*, Theory Probability Appl., **14** (1969), 612–626.
- [22] A. M. Vershik and A. A. Shmidt, *Limit measures arising in the asymptotic theory of symmetric groups*, I, II, Theory Probability Appl., **22** (1977, 1978), 70–85; **23**, 36–49.

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