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Bounds are established for the number of generators of the graded homogeneous ideal of a set of points in generic or in uniform position in the projective plane. For $n \leq 11$, n points in uniform position must have the "general" number of generators. It is shown by example that this fails for $n = 12$.

Introduction. Let Z be a set of points in $\mathbf{P}_k^2$, k algebraically closed. We say the points of Z lie in generic position if Z imposes independent conditions on curves containing it. If this holds for all subsets of Z we say that Z lies in uniform position. Given a set Z in one of these types of "general position", one would like to count the number of equations needed to cut out Z, or more precisely, the minimal number of generators ν of the graded homogeneous ideal $I(Z)$.

This question has arisen most recently in calculations of the Cohen-Macaulay-type of singularities. For example, it is shown in [7] that if A is the local ring at a curve singularity P in $\mathbf{A}_k^3$, and if the lines of the tangent cone at P correspond to a set of distinct points Z in generic position in $\mathbf{P}_k^2$, then the Cohen-Macaulay-type of A is equal to $\nu(I(Z)) - 1$. It is then natural to look for geometric conditions on Z which will allow the Cohen-Macaulay-type to be computed.

Let s denote the number of points belonging to Z, d the integer such that $\binom{d+1}{2} \leq s < \binom{d+2}{2}$, and define

$$N(s) = \begin{cases} d + 1 - s + \binom{d+1}{2} & \text{if } \binom{d+1}{2} \leq s \leq \dfrac{d(d+2)}{2} \\ d + 2 + s - \binom{d+2}{2} & \text{if } \dfrac{d(d+2)}{2} \leq s < \binom{d+2}{2}. \end{cases}$$

Geramita and Maroscia [6] have shown that almost all sets of s points in $\mathbf{P}^2$ are defined by exactly $N(s)$ equations. We give a new proof of this fact (1.7). However, $\nu(I(Z))$ is not constant on the sets of s points in generic position. It follows from a theorem of Dubreil [4] that the best one can say is that if Z lies in generic position, then $N(s) \leq \nu(I(Z)) \leq d + 1$.

Uniform position was introduced in [7] as a more stringent condition on Z; it is known that $\nu(I(Z)) = N(s)$ if Z is a set of s points in uniform position and $s \leq 11$. If $s = 12$, then $N(s) = 3$, but we construct in §2 a set Z of 12 points in uniform position for which $\nu(I(Z)) = 4$. Therefore, in a sense, some sets of points in uniform position are "more general" than others. We also establish an upper bound for $\nu(I(Z))$, where Z lies in uniform position:

$$N(s) \leq \nu(I(Z)) \leq d \qquad \text{if } \binom{d+1}{2} < s < \binom{d+2}{2} - 1$$

$$N(s) = \nu(I(Z)) = d + 1 \quad \text{if } s = \binom{d+1}{2} \text{ or } \binom{d+2}{2} - 1.$$

For example, if $s = 12$, this result shows that $3 \leq \nu \leq 4$.

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1. Bounds on the number of generators.

DEFINITION. Let Z be a set of s points contained in $\mathbf{P}^2 = \mathbf{P}_k^2$, k an algebraically closed field, and $\mathscr{I}_Z$ its sheaf of ideals. We say Z lies in *generic position* if for every nonnegative integer m, $\dim H^0(\mathscr{I}_Z(m)) = \max\{0, \binom{m+2}{2} - s\}$, where $\mathscr{I}_Z(m)$ denotes $\mathscr{I}_Z$ twisted m times by the hyperplane line bundle. We say Z lies in *uniform position* if each subset of Z (including Z itself) lies in generic position.

REMARK 1.0.1. Roughly speaking, Z lies in generic position if Z imposes independent conditions on curves containing it. The sets of s points in generic position in $\mathbf{P}^2$ form a Zariski-open subset of the Hilbert scheme $\mathrm{Hilb}^s(\mathbf{P}^2)$ parametrizing subschemes of $\mathbf{P}^2$ of length s. The sets of s points in uniform position form an open subset of the sets of s points in generic position.

Let Z be a zero-dimensional subscheme of $\mathbf{P}^2$ of length s. Because the projective dimension of an ideal sheaf in $\mathbf{P}^n$ is at most $n - 1$, I_Z has a minimal projective resolution

$$(1) \qquad 0 \to \sum_{i=1}^{\nu-1} \mathscr{O}_{\mathbf{P}^2}(-t_i) \xrightarrow{A} \sum_{j=1}^{\nu} \mathscr{O}_{\mathbf{P}^2}(-r_j) \xrightarrow{(f_1,\ldots,f_\nu)} \mathscr{I}_Z \to 0$$

where A is a $(\nu - 1) \times \nu$ "relations matrix" of homogeneous forms of degrees $\alpha_{ij} = t_i - r_j$, and we arrange $t_1 \leq \cdots \leq t_{\nu-1}$ and $r_1 \leq \cdots \leq r_\nu$. A standard Chern class calculation shows $\sum_{i=1}^{\nu-1} t_i = \sum_{j=1}^{\nu} r_j$ and $2s = \sum_{i=1}^{\nu-1} t_i^2 - \sum_{j=1}^{\nu} r_j^2$. Further, the minimality of the resolution implies that the entries of A are in the irrelevant ideal, so we have $t_1 > r_1$ and $t_{\nu-1} > r_\nu$.

LEMMA 1.1 (*Burch*, [3]). *If I is an ideal of projective dimension one in a regular local ring $(R, \mathscr{M})$, then given a minimal resolution*

$$0 \to R^{n-1} \xrightarrow{A} R^n \xrightarrow{(f_1,\ldots,f_n)} I \to 0$$

there exists $r \in R$ such that $f_i = r\Delta_i$, $i = 1,\ldots,n$, where the Δ_i are the maximal minors of the matrix A.

The lemma applies in our case with $R = k[x_0, x_1, x_2]_{(x_0,x_1,x_2)}$ and $I(Z) = \sum_{m=0}^{\infty} H^0(\mathscr{I}_Z(m))$ localized at (x_0, x_1, x_2) for I. Since ht $I = 2$, r must be a unit, and we may assume in the following that $f_i = \Delta_i$.

EXAMPLE 1.1.1. If Z is a complete intersection of curves of degrees r_1 and r_2, then the minimal resolution is the familiar Koszul complex

$$0 \to \mathscr{O}_{\mathbf{P}^2}(-r_1 - r_2) \to \mathscr{O}_{\mathbf{P}^2}(-r_1) \oplus \mathscr{O}_{\mathbf{P}^2}(-r_2) \xrightarrow{(f_1,f_2)} \mathscr{I}_Z \to 0$$

and the relations matrix is $(-f_2, f_1)$. It is easy to check that the complete intersection of f_1 and f_2 lies in generic position if and only if $\deg f_i \leq 2$. More generally:

PROPOSITION 1.2. (a) *A subscheme Z of length s in $\mathbf{P}^2$ lies in generic position if and only if $0 \leq \alpha_{ij} \leq 2$ for a minimal resolution* (1). (b) *Moreover, in this situation, $r_1 \leq r_i \leq r_1 + 1$ for all $1 \leq i \leq \nu$.*

Proof. (a) Consider the long exact sequence of cohomology

$$0 \to H^0(\mathscr{I}_Z(m)) \to H^0(\mathscr{O}_{\mathbf{P}^2}(m)) \to H^0(\mathscr{O}_Z(m)) \to H^1(\mathscr{I}_Z(m)) \to 0.$$

Denote dim H^i by h^i. Because $h^0(\mathscr{O}_{\mathbf{P}^2}(m)) = \binom{m+2}{2}$ and $h^0(\mathscr{O}_Z(m)) = s$, it is clear that Z lies in generic position if and only if $\mathscr{I}_Z$ has "seminatural cohomology", i.e. for each $m \geq 0$, at most one of $H^i(\mathscr{I}_Z(m))$, $i = 0, 1, 2$, is nonzero. Since $H^0(\mathscr{I}_Z(m)) \neq 0$ if and only if $m \geq r_1$, $\mathscr{I}_Z$ has seminatural cohomology if and only if $H^1(\mathscr{I}_Z(m)) = 0$ for $m \geq r_1$. By (1) and the fact that $t_{\nu-1} > r_\nu$, this condition is equivalent to $t_{\nu-1} \leq r_1 + 2$.

(b) In this situation, since $t_{\nu-1} > r_i$ for all i, $r_1 \leq r_i < t_{\nu-1} \leq r_1 + 2$ for all i.

REMARK 1.2.1. Fact (b) was proved in [7].

One of the advantages of generic position is that it is preserved under linkage. We say that two closed subschemes Z, Z' of $\mathbf{P}^n$, equidimensional and without embedded components, are *linked* via the complete intersection X containing Z and Z' if $I(Z) = I(X)\colon I(Z')$ and $I(Z') = I(X)\colon I(Z)$, where $I(Z)$ denotes the homogeneous graded ideal of Z, and so forth. The next proposition was essentially known to Apéry (see [**5**]) and has an elementary proof, which is provided in an appendix for lack of a reference.

PROPOSITION 1.3. *Let Z be a projectively Cohen-Macaulay subscheme of $\mathbf{P}^n$ of codimension two, and let*

$$0 \to \sum_{i=1}^{\nu-1} \mathcal{O}_{\mathbf{P}^n}(-t_i) \xrightarrow{A} \sum_{j=1}^{\nu} \mathcal{O}_{\mathbf{P}^n}(-r_j) \xrightarrow{(f_1,\ldots,f_\nu)} \mathcal{I}_Z \to 0$$

be a minimal projective resolution for the ideal sheaf, where f_i is the ith maximal minor of A. Suppose $\{f_i, f_j\}$ forms a regular sequence of length two in $k[X_0,\ldots,X_n]$, and let Z' be the (projectively Cohen-Macaulay) subscheme of $\mathbf{P}^n$ linked to Z via $\{f_i, f_j\}$. Then a relations matrix for a minimal resolution for $\mathcal{I}_{Z'}$ is obtained by deleting columns i and j from A and transposing.

REMARK 1.3.1. The liaison theorem ([**9**, Thm. 3.2]) follows directly from (1.3) and the fact that if I is an ideal in $k[x_0,\ldots,x_n]$, a graded quotient of a polynomial ring, and I contains a non-zero-divisor, then there exists a non-zero-divisor $x \in I$ belonging to a set of minimal generators for I.

REMARK 1.3.2. If we assume $i = \nu - 1$, $j = \nu$ in (1.3), then $\alpha'_{kl} = \alpha_{lk}$ for $1 \le k \le \nu - 2$, $1 \le l \le \nu - 1$.

COROLLARY 1.4. *Let Z be a set of points in $\mathbf{P}^2$ lying in generic position and f_1, f_2 two homogeneous forms with no common factor which belong to a minimal set of generators of the ideal $I(Z) = \sum_{m=0}^{\infty} H^0(\mathcal{I}_Z(m))$. Let Z' be the scheme linked to Z via $\{f_1, f_2\}$. Then Z' lies in generic position.*

Proof. Follows from (1.2) and (1.3.2).

Next we find a lower bound on the number of generators ν of the graded ideal $I(Z)$, where Z is a length s subscheme of $\mathbf{P}^2$ lying in generic position. A set of generators must contain a basis for $H^0(\mathcal{I}_Z(r))$, where r

is the least degree of a curve containing Z. Generic position implies that r is the least integer such that $\binom{r+2}{2} > s$, and we have $\nu \geq k$ where $k = h^0(\mathcal{I}_Z(r)) = \binom{r+2}{2} - s$. Using (1.2) we see that a minimal resolution for $\mathcal{I}_Z$ has the form

$$0 \to \sum_{i=1}^{\nu-l-1} \mathcal{O}_{\mathbf{P}^2}(-r-1) \oplus \sum_{i=1}^{l} \mathcal{O}_{\mathbf{P}^2}(-r-2)$$

$$\to \sum_{i=1}^{k} \mathcal{O}_{\mathbf{P}^2}(-r) \oplus \sum_{i=1}^{\nu-k} \mathcal{O}_{\mathbf{P}^2}(-r-1) \to \mathcal{I}_Z \to 0$$

where $l = r + 1 - k$. Since $\nu - k$ and $\nu - l - 1$ are nonnegative, $\nu \geq \max\{k, r - k + 2\}$. Set $N(s) = \max\{k, r - k + 2\}$ (clearly r and k depend only on s). Then we can compute

$$\begin{aligned} N(s) &= \max\{k, r - k + 2\} \\ &= \max\left\{\binom{r+2}{2} - s, r - \binom{r+2}{2} + s + 2\right\} \\ &= \begin{cases} r + 1 - s + \binom{r+1}{2} & \text{if } \binom{r+1}{2} \leq s \leq \dfrac{r(r+2)}{2} \\ r + 2 + s - \binom{r+2}{2} & \text{if } \dfrac{r(r+2)}{2} \leq s < \binom{r+2}{2}. \end{cases} \end{aligned}$$

Therefore, $N(s)$, a number depending only on s, is a lower bound on the number of generators for the ideal $I(Z)$. To show this bound is sharp, we will exhibit for each integer s a zero-dimensional scheme of length s in generic position with exactly $N(s)$ ideal generators. By the following Lemma 1.5, having $N(s)$ ideal generators is an open condition on the set Z. It follows (1.7) that the general set of s distinct points in $\mathbf{P}^2$ has $N(s)$ ideal generators.

LEMMA 1.5. *Let $\mathcal{U}$ be the open dense subset of* $\mathrm{Hilb}^s(\mathbf{P}^2)$ *corresponding to schemes in generic position. For any integer N, the subset of $\mathcal{U}$ corresponding to schemes defined by at most N equations is open in* $\mathrm{Hilb}^s(\mathbf{P}^2)$.

Proof. The sheaf of differentials $\Omega_{\mathbf{P}^2}$ fits in the following exact sequence:

$$0 \to \Omega_{\mathbf{P}^2} \to 3\mathcal{O}_{\mathbf{P}^2}(-1) \xrightarrow{(x,y,z)} \mathcal{O}_{\mathbf{P}^2} \to 0$$

where $\mathbf{P}^2 = \operatorname{Proj} k[x, y, z]$. Let $Z \in \mathrm{Hilb}^s(\mathbf{P}^2)$, lying in generic position. Tensor the above sequence with $\mathcal{I}_Z(r+1)$, where r is the least integer such that $\binom{r+2}{2} > s$. The long exact sequence of cohomology yields

$$H^0(3\mathscr{I}_Z(r)) \stackrel{(x,y,z)}{\to} H^0(\mathscr{I}_Z(r+1)) \to H^1(\Omega_{\mathbf{P}^2} \otimes \mathscr{I}_Z(r+1)) \to 0$$

since $H^1(3\mathscr{I}_Z(r)) = 0$ by the seminatural cohomology of $\mathscr{I}_Z$. The dimension of $H^1(\Omega_{\mathbf{P}^2} \otimes \mathscr{I}_Z(r+1))$ measures the number of generators of $I(Z)$ of degree $r+1$. Thus

$$\nu(I(Z)) = \binom{r+2}{2} - s + h^1(\Omega_{\mathbf{P}^2} \otimes \mathscr{I}_Z(r+1)).$$

By the semicontinuity theorem ([**8**], Thm. III.12.8), $\nu(I(Z))$ is an upper semicontinuous function of Z in $\mathscr{U}$.

LEMMA 1.6 (*Buchsbaum, Eisenbud* [**2**]). *Let R be a commutative noetherian ring, F_j free R-modules. The complex*

$$0 \to F_n \stackrel{\varphi_n}{\to} F_{n-1} \stackrel{\varphi_{n-1}}{\to} F_{n-2} \to \cdots \to F_0$$

is exact if and only if for $j = 1, 2, \ldots, n$,

(a) rank φ_j + rank φ_{j-1} = rank F_{j-1},

(b) *the ideal of maximal minors of φ_j contains a regular sequence of length j.*

PROPOSITION 1.7 (*Geramita, Maroscia* [**6**]). *If Z is a set of s points in generic position in $\mathbf{P}^2$, then $\nu(I(Z)) \geq N(s)$. Moreover, equality holds for all Z in an open dense subset of* $\mathrm{Hilb}^s(\mathbf{P}^2)$.

Proof. The first statement is proved above. For the second, it remains to construct for each s a length s scheme Z in generic position with $\nu(I(Z)) = N(s)$.

Case 1. Suppose $\binom{r+1}{2} \leq s \leq r(r+2)/2$ for some integer r. Set $m = r(r+2) - 2s$, $N = N(s) = r + 1 - s + \binom{r+1}{2} > 0$. An easy calculation shows $0 \leq m \leq N - 1$. Consider the $(N-1) \times N$ matrix

$$A = \begin{pmatrix} l_{11} & \cdots & l_{1n} \\ \vdots & & \vdots \\ l_{m1} & \cdots & l_{mN} \\ c_{m+1,1} & \cdots & c_{m+1,N} \\ \vdots & & \vdots \\ c_{N-1,1} & \cdots & c_{N-1,N} \end{pmatrix}$$

where $\deg l_{ij} = 1$ and $\deg c_{ij} = 2$. If the entries of the matrix are chosen generally it is clear that at least two of the maximal minors $f_1, \ldots, f_N$ will share no common factor, and grade $\langle f_1, \ldots, f_N \rangle \geq 2$. Now (1.6) applies to show that the corresponding sequence (1) is exact with $\nu = N$, so that we have constructed a zero-dimensional scheme Z with A as relations matrix.

Counting degrees, we have $r_1 = \cdots = r_n = 2N - m - 2$, $t_1 = \cdots = t_m = 2N - m - 1$, and $t_{m+1} = \cdots = t_{N-1} = 2N - m$. A calculation shows the length of Z is s, and by (1.2) Z lies in generic position. The resolution is minimal, so $\nu(I(Z)) = N(s)$.

Case 2. Suppose $r(r+2)/2 \leq s < \binom{r+2}{2}$ for some integer r. Set $m = \binom{r+2}{2} - s$ and $N = N(s) = s - \binom{r+2}{2} + r + 2 > 0$. Note that $0 \leq m \leq N$. Consider the $(N-1) \times N$ matrix

$$A = \begin{pmatrix} c_{11} & \cdots & c_{1m} & l_{1,m+1} & \cdots & l_{1N} \\ \vdots & & \vdots & \vdots & & \vdots \\ c_{N-1,1} & \cdots & c_{N-1,m} & l_{N-1,m+1} & \cdots & l_{N-1,N} \end{pmatrix}$$

where $\deg l_{ij} = 1$ and $\deg c_{ij} = 2$. As before, if the entries are chosen generally there exists a minimal projective resolution of an ideal sheaf I_Z with A as relations matrix. We have

$$r_1 = \cdots = r_m = N + m - 2, \; r_{m+1} = \cdots = r_N = N + m - 1,$$

$t_1 = \cdots = t_{N-1} = N + m$, showing that Z consists of s points in generic position, and $\nu(I(Z)) = N(s)$.

PROPOSITION 1.8. *Let Z be a set of s points in generic position in $\mathbf{P}^2$, r the least integer such that $\binom{r+2}{2} > s$. Then $\nu(I(Z)) \leq r + 1$.*

Proof. Let Z be a length s subscheme of $\mathbf{P}^2$, and r the least degree of a curve containing Z. Let A be a relations matrix for a minimal resolution (1) of $\mathscr{I}_Z$. We may arrange A such that $\alpha_{11} \geq \alpha_{12} \geq \cdots \geq \alpha_{1\nu}$; then $\deg f_1 = r$ and f_1 is the determinant of a $(\nu - 1) \times (\nu - 1)$ matrix whose nonzero entries are of degree at least one. Therefore $r = \deg f_1 \geq \nu(I(Z)) - 1$, i.e. $\nu(I(Z)) \leq r + 1$. This is a theorem of Dubreil ([4], Thm. I).

For points in uniform position, the upper bound of (1.8) can be improved:

PROPOSITION 1.9. *Let Z be a set of s points in uniform position in $\mathbf{P}^2$, such that $\binom{r+1}{2} + 1 \leq s \leq \binom{r+2}{2} - 2$ for some r. Then $\nu(I(Z)) \leq r$.*

Proof. Let $k = \binom{r+2}{2} - s$; so that $2 \le k \le r$. Suppose $\nu(I(Z)) = r + 1$, and let A be the relations matrix of a minimal resolution (1) of $\mathscr{I}_Z$.

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1,r+1} \\ \vdots & & \vdots \\ a_{r,1} & \cdots & a_{r,r+1} \end{pmatrix}$$

By (1.2), $0 \le \alpha_{ij} \le 2$, and $\alpha_{ij} = \deg a_{ij}$. We may assume $\alpha_{11} \ge \alpha_{12} \ge \cdots \ge \alpha_{1,r+1}$; then we have

$$\alpha_{i1} = \alpha_{i2} = \cdots = \alpha_{ik} = \alpha_{i,k+1} + 1 = \cdots = \alpha_{i,r+1} + 1. \tag{2}$$

We may also assume $\alpha_{11} \le \alpha_{21} \le \cdots \le \alpha_{r1}$.

Since $\deg f_{r+1} = r + 1$, we have

$$\begin{aligned} r + 1 &= \alpha_{11} + \alpha_{22} + \cdots \alpha_{rr} \\ &= \alpha_{11} + \cdots + \alpha_{kl} + (\alpha_{k+1,1} - 1) + \cdots + (\alpha_{r1} - 1) \\ &= \alpha_{11} + \cdots + \alpha_{r1} - r + k \end{aligned}$$

and thus

$$2r - k + 1 = \alpha_{11} + \cdots + \alpha_{r1}. \tag{3}$$

By (2), $\alpha_{i1} \ge 1$ for all i.

Suppose $\alpha_{k-1,1} \ge 2$. Then $2 \le \alpha_{k-1,1} \le \alpha_{k1} \le \cdots \le \alpha_{r1}$ and so $\alpha_{11} + \cdots + \alpha_{r1} \ge k - 2 + 2(r - k + 2) = 2r - k + 2$, contradicting (3). Therefore, $1 = \alpha_{k-1,1} = \alpha_{k-2,1} = \cdots = \alpha_{11}$.

By (2), $\alpha_{ij} = 0$ for $1 \le i \le k - 1$, $k + 1 \le j \le r + 1$. Since all entries of A lie in the maximal homogeneous ideal (X_0, X_1, X_2) of $k[X_0, X_1, X_2]$, $a_{ij} = 0$ for $1 \le i \le k - 1$, $k + 1 \le j \le r + 1$. A has the following form:

$$\left(\begin{array}{c|c|c} * & & \\ \vdots & A_1 & 0 \\ \hline \vdots & A_2 & A_3 \\ * & & \end{array}\right)\begin{array}{l} \left.\vphantom{\begin{array}{c}*\\ \vdots\end{array}}\right\}\ k-1 \\ \left.\vphantom{\begin{array}{c}\vdots\\ *\end{array}}\right\}\ r-k+1 \end{array}$$
$$\underbrace{\quad}_{1}\ \underbrace{\qquad}_{k-1}\ \underbrace{\qquad}_{r-k+1}$$

Case 1. If $s = \binom{r+1}{2} + 1$, then $k = r$, and the form of A is

$$\left(\begin{array}{c|c} & 0 \\ B & \vdots \\ & 0 \\ & a_{r,r+1} \end{array}\right)$$

where $\deg a_{r,r+1} = 1$. The complete intersection of the curves $\det B = 0$ and $a_{r,r+1} = 0$ lies in Z. Since $r \ge 2$, $\deg(\det B) \ge 3$, so Z contains three collinear points (counted with multiplicity), a contradiction to the uniform position assumption.

Case 2. If $\binom{r+1}{2} + 2 \leq s \leq \binom{r+2}{2} - 2$, then $f_1 = (\det A_1) \cdot (\det A_3)$, so f_1 is the composite of curves of degrees $k - 1 \geq 1$ and $\nu - k \geq 1$. This contradicts the following fact:

LEMMA 1.10 (*Geramita, Maroscia* [6], *Thm.* 3.4). *Let Z be a set of s points in uniform position in $\mathbf{P}^2$ where $\binom{r+1}{2} + 2 \leq s \leq \binom{r+2}{2} - 1$. Then every curve of degree r containing Z is irreducible.*

PROPOSITION 1.11. *Let Z be a set of s points in $\mathbf{P}^2$, and set*

$$N(s) = \begin{cases} r + 1 - s + \binom{r+1}{2} & \text{if } \binom{r+1}{2} \leq s \leq \dfrac{r(r+2)}{2} \\ r + 2 + s - \binom{r+2}{2} & \text{if } \dfrac{r(r+2)}{2} \leq s < \binom{r+2}{2} \end{cases}.$$

(a) *If Z lies in generic position, then $N(s) \leq \nu(I(Z)) \leq r + 1$.*
(b) *If Z lies in uniform position, then*

$$N(s) \leq \nu(I(Z)) \leq r \qquad \text{if } \binom{r+1}{2} < s < \binom{r+2}{2} - 1;$$

$$N(s) = \nu(I(Z)) = r + 1 \quad \text{if } s = \binom{r+1}{2} \text{ or } \binom{r+2}{2} - 1.$$

Proof. Follows from (1.7), (1.8) and (1.9).

COROLLARY 1.12. (*Geramita, Maroscia, Orecchia* [6], [7]). (a) *If Z lies in generic position and $s = \binom{r+1}{2}$ or $\binom{r+2}{2} - 1$, then $\nu(I(Z)) = r + 1$.* (b) *If Z lies in uniform position and $s = \binom{r+1}{2} + 1$ or $\binom{r+2}{2} - 2$, then $\nu(I(Z)) = r$.*

COROLLARY 1.13. *If Z lies in uniform position and $s = \binom{r+2}{2} - 3$, then $r - 1 \leq \nu(I(Z)) \leq r$.*

REMARK 1.13.1. It follows from (1.11) that if $s \leq 11$, s points in uniform position have exactly $N(s)$ ideal generators. In the next section we show that this statement fails for $s = 12$.

2. A counterexample. The following result is related to the classical Cayley-Bacharach Theorem.

LEMMA 2.1. *Let S, T be zero-dimensional subschemes of $\mathbf{P}^2$ linked by two curves C, D of degrees c and d, respectively, having no common components. Suppose $d \leq c + 2$. Then $h^0(\mathcal{I}_S(d - 3)) = h^1(\mathcal{I}_T(c))$.*

Proof. Suppose $\mathscr{I}_T$ has minimal resolution

$$0 \to \sum_{i=1}^{\nu-1} \mathscr{O}_{\mathbf{P}^2}(-t_i) \to \sum_{j=1}^{\nu} \mathscr{O}_{\mathbf{P}^2}(-r_j) \to \mathscr{I}_T \to 0.$$

Then linkage implies ([9], Prop. 2.5)

$$0 \to \sum_{j=1}^{\nu} \mathscr{O}_{\mathbf{P}^2}(r_j - c - d)$$
$$\to \sum_{i=1}^{\nu-1} \mathscr{O}_{\mathbf{P}^2}(t_i - c - d) \oplus \mathscr{O}_{\mathbf{P}^2}(-c) \oplus \mathscr{O}_{\mathbf{P}^2}(-d) \to \mathscr{I}_S \to 0$$

is a projective resolution for $\mathscr{I}_S$. By the long exact sequences of cohomology,

$$\begin{aligned} h^0(\mathscr{I}_S(d-3)) &= \sum_{i=1}^{\nu-1} h^0(\mathscr{O}_{\mathbf{P}^2}(t_i - c - 3)) - \sum_{j=1}^{\nu} h^0\big(\mathscr{O}_{\mathbf{P}^2}(r_j - c - 3)\big) \\ &= \sum_{i=1}^{\nu-1} h^2(\mathscr{O}_{\mathbf{P}^2}(c - t_i)) - \sum_{j=1}^{\nu} h^2\big(\mathscr{O}_{\mathbf{P}^2}(c - r_j)\big) \\ &= h^1(\mathscr{I}_T(c)). \end{aligned}$$

LEMMA 2.2. *Suppose $P_1, \ldots, P_{13}$ are points in $\mathbf{P}^2$ such that no three are collinear, and suppose $\{Q_1, Q_2, Q_3\}$ is linked to $\{P_1, \ldots, P_{13}\}$ via two quartic curves. Then every subset of $\{Q_1, Q_2, Q_3, P_1, \ldots, P_{10}\}$ imposes independent conditions on quartics.*

Proof. Set $T = \{Q_1, Q_2, Q_3, P_1, \ldots, P_{10}\}$ and $S = \{P_{11}, P_{12}, P_{13}\}$ in (2.1). Since P_{11}, P_{12}, P_{13} are not collinear, $h^1(\mathscr{I}_T(4)) = 0$, i.e. T imposes independent conditions on quartics. Since each point imposes at most one condition, it follows that every subset of T imposes independent conditions on quartics. □

According to (1.13) a set of 12 points in uniform position must have either 3 or 4 ideal generators. We know almost all have 3 ideal generators (1.7). If $\nu(I(Z)) = 3$, then the minimal resolution for $\mathscr{I}_Z$ must be

$$0 \to 2\mathscr{O}_{\mathbf{P}^2}(-6) \xrightarrow{A} 3\mathscr{O}_{\mathbf{P}^2}(-4) \xrightarrow{(f_1 f_2 f_3)} \mathscr{I}_Z \to 0$$

and A has form

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{pmatrix}$$

where $\deg c_{ij} = 2$.

Suppose there are two quartic curves with no common component containing Z. We may assume they are given by $f_1 = 0$ and $f_2 = 0$. By (1.3), Z is linked via f_1 and f_2 to the complete intersection of the two conics $c_{13} = 0$ and $c_{23} = 0$.

EXAMPLE 2.3. A set of 12 points in uniform position in $\mathbf{P}^2$ with $\nu(I(Z)) = 4$.

We will construct a set Z in uniform position that is linked via two quartics to four points, three of which are collinear. By the preceding argument, we conclude $\nu(I(Z)) = 4$.

The following criterion for uniform position will be used below.

LEMMA 2.4 (*Brun*, [**1**]). *A set of points Z in $\mathbf{P}^2$ is in uniform position if and only if*

(a) *at most* $\binom{r+2}{2} - 1$ *points of Z lie on a degree r curve, for each r, and*

(b) *for each $z \in Z$, there exists a degree d curve Y such that $Y \cap Z = Z \setminus \{z\}$, where d is the smallest integer such that* $\binom{d+2}{2} \geq s$.

Construction. Fix a nonsingular quartic C in $\mathbf{P}^2$ and three distinct collinear points Q_1, Q_2, Q_3 on C. We will show that the general quartic passing through Q_1, Q_2, Q_3, intersects C in $Q_1, Q_2, Q_3, P_1, \ldots, P_{13}$ such that any twelve of $P_1, \ldots, P_{13}$ lie in uniform position.

The family of quartics in $\mathbf{P}^2$ is parametrized by $\mathbf{P}^{14}$, and an 11-dimensional subfamily S passes through Q_1, Q_2, Q_3, since it is clear that none of these three points is a base point for quartics through the other two.

The quartics in S missing the point Q_4, the residual intersection of C and the line L connecting Q_1, Q_2, Q_3, form again an 11-dimensional family S'. Suppose $C' \in S'$ and $C \cap C' = \{Q_1, Q_2, Q_3, P_1, \ldots, P_{13}\}$, counting multiplicities, and that P_1, P_2, P_3 are collinear. None of the P_i lie on L. P_3 is not a base point for quartics through $\{Q_1, Q_2, Q_3, P_1, P_2\}$: for example, set $L_1 = \overline{P_1 Q_i}$, where we choose $Q_i \notin \overline{P_1 P_2}$, and L_2, L_3, L_4 lines containing P_2, Q_j, Q_k, respectively, where $\{i, j, k\} = \{1, 2, 3\}$. Then there exists a quartic $L_1 L_2 L_3 L_4$ containing $\{Q_1, Q_2, Q_3, P_1, P_2\}$ but not P_3. Similar arguments show that none of the six points $\{Q_1, Q_2, Q_3, P_1, P_2, P_3\}$ is a base point for quartics through the other five, so that the six points induce independent conditions on quartics.

Thus there are 8 dimensions of quartics through $\{P_1, P_2, P_3, Q_1, Q_2, Q_3\}$. The choice of a set of collinear points $\{P_1, P_2, P_3\}$ on $C \setminus L$ is 2-dimensional (since there is a finite choice (four) of three-point sets associated with each line in $\mathbf{P}^2$). Therefore there is a 10-dimensional set of quartics C' belonging to S' such that the remaining 13 points of the complete intersection $C \cap C'$ contain three collinear points. We conclude that the general quartic through Q_1, Q_2, Q_3 is such that the residual intersection with C does not contain three collinear points.

Showing the general residual intersection contains no six on a conic and no ten on a cubic requires similar arguments; we argue only the latter. Suppose C' is a general quartic such that $C \cap C' = \{Q_1, Q_2, Q_3, P_1, \dots, P_{13}\}$ where $\{P_1, \dots, P_{10}\}$ are situated on a cubic. Since no three of $P_1, \dots, P_{13}$ are collinear, (2.2) applies to show $\{Q_1, \dots, P_{10}\}$ imposes independent conditions on quartics, so that the family of quartics through $\{Q_1, \dots, P_{10}\}$ is one-dimensional. The family of 10-point sets on C lying on a cubic is at most 9-dimensional, therefore the family of quartics through Q_1, Q_2, Q_3 such that the residual 13 points of $C \cap C'$ contain 10 cocubic points is at most 10-dimensional. We conclude that the general quartic through Q_1, Q_2, Q_3 intersects C in 13 residual points that have no three collinear, no six on a conic, and no ten on a cubic.

Let C' be such a general quartic. We will show that any twelve of $P_1, \dots, P_{13}$ satisfy the criterion of (2.4). Condition (a) is verified; for (b) we reason as follows. Given any point z among a subset of twelve, choose a cubic containing nine of the rest and a line containing the other two. The point z cannot lie on this reducible quartic Y, therefore $Y \cap Z = Z \setminus \{z\}$. Such a set Z of twelve points lies in uniform position and is linked via C and C' to a set of four points, three of which are collinear, so $\nu(I(Z)) = 4$.

REMARK 2.4.1. It is clear, at least if $\operatorname{char} k = 0$, that by Bertini's theorem the general such C' intersects C transversally, i.e. in the terminology of [6], provides sets of 12 points in "transversal uniform position" and $\nu = 4$.

Appendix. We present a proof of (1.3). Let R be a commutative ring and

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1\nu} \\ \vdots & & \vdots \\ a_{\nu-1,1} & \cdots & a_{\nu-1,\nu} \end{pmatrix}$$

be a matrix whose entries lie in R. Set

$$f_i = (-1)^{i+1} \det \begin{pmatrix} (\nu-1) \times (\nu-1) \text{ matrix} \\ \text{obtained by deleting} \\ \text{column } i \end{pmatrix}$$

$$\Gamma_{ij}^{k} = (-1)^{i+j+k+1}\det\begin{pmatrix} (\nu-2)\times(\nu-2) \text{ matrix} \\ \text{obtained by deleting} \\ \text{columns } i, j \text{ row } k \end{pmatrix}.$$

LEMMA 1. *If* $i < j$, *then*

$$\left(\Gamma_{ij}^{1},\dots,\Gamma_{ij}^{\nu-1}\right)\cdot A = \Big(0,\dots,0,\underset{i}{f_j},0,\dots,0,\underset{j}{-f_i},0,\dots,0\Big).$$

LEMMA 2. *If* $1 \le i < j \le \nu$, $l \ne i$ *or* j, *then*

$$\Gamma_{ij}^{k} f_l = \Gamma_{il}^{k} f_j - \Gamma_{jl}^{k} f_i \quad \textit{for } k = 1,\dots,\nu-1.$$

Proof. For $l \ne i$ or j, multiply the equation in Lemma 1 on the right by the matrix

$$B_l = \begin{pmatrix} \Gamma_{1l}^{1} & \Gamma_{1l}^{2} & \cdots & \Gamma_{ll}^{\nu-1} \\ \vdots & \vdots & & \vdots \\ \Gamma_{l-1,l}^{1} & \Gamma_{l-1,l}^{2} & \cdots & \Gamma_{l-1,l}^{\nu-1} \\ 0 & 0 & \cdots & 0 \\ -\Gamma_{l+1,l}^{1} & -\Gamma_{l+1,l}^{2} & \cdots & -\Gamma_{l+1,l}^{\nu-1} \\ \vdots & \vdots & & \vdots \\ -\Gamma_{\nu l}^{1} & -\Gamma_{\nu l}^{2} & \cdots & -\Gamma_{\nu l}^{\nu-1} \end{pmatrix}.$$

LEMMA 3. *Let* $(R, \mathfrak{M})$ *be a regular local ring,* A *a* $(\nu-1)\times\nu$ *matrix with entries in* $\mathfrak{M}$. *If* $\{f_i, f_j\}$ *is a regular sequence in* $\mathfrak{M}$, *then* $(f_i, f_j)\colon (f_1, f_2,\dots,f_\nu) = (\Gamma_{ij}^{1},\dots,\Gamma_{ij}^{\nu-1})$.

Proof. By Lemma 2, $(\Gamma_{ij}^{1},\dots,\Gamma_{ij}^{\nu-1}) \subseteq (f_i, f_j)\colon (f_1,\dots,f_\nu)$. On the other hand, let $r \in (f_i, f_j)\colon (f_1,\dots,f_\nu)$. Then

$$\begin{pmatrix} f_1 \\ \vdots \\ f_\nu \end{pmatrix} r = C\begin{pmatrix} f_i \\ f_j \end{pmatrix},$$

where

$$C = \begin{pmatrix} c_{11} & c_{12} \\ \vdots & \vdots \\ c_{\nu 1} & c_{\nu 2} \end{pmatrix}, \qquad c_{ij} \in R,$$

and we may assume $c_{i1} = r$, $c_{i2} = 0$, $c_{j1} = 0$, $c_{j2} = r$. Multiply on the left by A:

$$0 = AC\begin{pmatrix} f_i \\ f_j \end{pmatrix}.$$

AC is a $(\nu - 1) \times 2$ matrix whose first column is divisible by f_j and whose second column is divisible by f_i, since $\{f_i, f_j\}$ form a regular sequence. So write

$$AC = \begin{pmatrix} d_1 f_j & -d_1 f_i \\ \vdots & \vdots \\ d_{\nu-1} f_j & -d_{\nu-1} f_i \end{pmatrix}$$

where $d_k \in R$ for all $k = 1, \ldots, \nu - 1$. Multiplying on the left by the matrix $(\Gamma_{ij}^1, \ldots, \Gamma_{ij}^{\nu-1})$ and using Lemma 1, we get

$$r = \sum_{k=1}^{\nu-1} d_k \Gamma_{ij}^k \in \left(\Gamma_{ij}^1, \ldots, \Gamma_{ij}^{\nu-1}\right).$$

Proposition (1.3) follows from

LEMMA 4. *Let $(R, \mathcal{M})$ be a regular local ring, I an ideal with a minimal resolution*

$$0 \to R^{\nu-1} \xrightarrow{A} R^{\nu} \xrightarrow{\begin{pmatrix} f_1 \\ \vdots \\ f_\nu \end{pmatrix}} I \to 0,$$

f_i the maximal minors of A. Suppose $\{f_i, f_j\}$ is a regular sequence in R, and let $J = (f_i, f_j) \colon (f_1, \ldots, f_\nu)$ be the ideal linked to I by $\{f_i, f_j\}$. Then J has a minimal resolution

$$0 \to R^{\nu-2} \xrightarrow{B} R^{\nu-1} \xrightarrow{\begin{pmatrix} \Gamma_{ij}^{\nu-1} \\ \vdots \\ \Gamma_{ij}^1 \end{pmatrix}} J \to 0$$

where B is the transpose of the matrix obtained by eliminating columns i and j from A.

Proof. Easy to check that

$$B\begin{pmatrix} \Gamma_{ij}^{\nu-1} \\ \vdots \\ \Gamma_{ij}^1 \end{pmatrix} = 0.$$

Therefore

$$0 \to R^{\nu-2} \xrightarrow{B} R^{\nu-1} \xrightarrow{\begin{pmatrix} \Gamma_{ij}^{\nu-1} \\ \vdots \\ \Gamma_{ij}^{1} \end{pmatrix}} J \to 0$$

is a complex. Rank $A = \nu - 1 \Rightarrow$ rank $B = \nu - 2$, so the complex is exact on the left. It is exact on the right by Lemma 3. The grade of J is 2 by ([**9**], Prop. 1.3), so (1.6) applies to prove exactness in the middle. Since the entries of B lie in $\mathscr{M}$, this exact sequence is a minimal resolution of J.

Note added in proof. An example with the properties of Example 2.3, constructed by different techniques, is included in a later version of [**6**].

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