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EXTENSION OF THE HARDY-LITTLEWOOD-FEfferman-STEIN INEQUALITY

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We will show inequalities concerning the functions of the form $f * t^{-n}\varphi(\cdot/t)(x)$ defined on R_+^{n+1} and give their applications to real Hardy spaces. These inequalities can be regarded as weak extensions of the Hardy-Littlewood-Fefferman-Stein inequality concerning harmonic functions.

1. Introduction. In C. Fefferman and E. M. Stein [6] (p. 172 Lemma 2), (see also Hardy and Littlewood [8]), they showed

THEOREM 1.A. *Let $u(x)$ be a complex-valued harmonic function defined on*

$$B = \left\{ x = (x_1, \dots, x_n) \in R^n : \sum_{j=1}^n x_j^2 < 1 \right\}.$$

Let $p > 0$. Then

$$|u(0)|^p \leq C \int_B |u(x)|^p dx,$$

where C is a constant depending only on p and n .

Consequently, if $u(x, t)$ is harmonic on $R_+^{n+1} = \{(x, t) : x \in R^n, t > 0\}$ and if $p > 0$, then we have

$$(1.1) \quad |u(0, 1)|^p \leq C \int_{|x|<1} dx \int_{1/2}^{3/2} |u(x, t)|^p dt.$$

This inequality has some interesting applications to the theory of real Hardy spaces. (See [6].)

In this paper we show analogous inequalities for functions of the form $f * t^{-n}\varphi(\cdot/t)(x)$ defined on R_+^{n+1} , where $f \in \bigcup_{1 \leq p \leq +\infty} L^p(R^n)$ is arbitrary and where $\varphi \in C(R^n) \cap \bigcap_{1 \leq p \leq +\infty} L^p(R^n)$ satisfies certain conditions. Our results have weaker forms than (1.1) but still they have some interesting applications to real Hardy spaces.

First we prepare several definitions.

Functions considered are complex-valued and measurable. Sets considered are measurable. D_t denotes $\partial/\partial t$. For a multi-index $\gamma = (\gamma_1, \dots, \gamma_n)$, where $\gamma_1, \dots, \gamma_n$ are nonnegative integers, D_x^γ and $l(\gamma)$ denote $\partial^{\gamma_1 + \dots + \gamma_n} / \partial x_1^{\gamma_1} \dots \partial x_n^{\gamma_n}$ and $\gamma_1 + \dots + \gamma_n$, respectively. For $f \in L^2(\mathbb{R}^n)$, $\mathcal{F}f$ denotes its Fourier transform. For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $|x|$ denotes $(\sum_{j=1}^n x_j^2)^{1/2}$. For $(x, t) \in \mathbb{R}_+^{n+1}$, $B(x, t)$ denotes $\{y \in \mathbb{R}^n: |x - y| < t\}$. $(\varphi)_t(x)$ denotes $t^{-n}\varphi(x/t)$. For a real number α , $[\alpha]$ denotes its integral part.

For $\kappa \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $\alpha > 0$ let

$$\|\kappa\|_{\Lambda_\alpha} = \sup_B \inf_{P: \deg P \leq \alpha} |B|^{-1-\alpha/n} \int_B |\kappa(x) - P(x)| dx,$$

where the supremum is taken over all balls B in \mathbb{R}^n , $|B|$ denotes the Lebesgue measure of B and where the infimum is taken over all polynomials $P(x)$ of degree $\leq \alpha$. Let

$$\Lambda_\alpha(\mathbb{R}^n) = \{\kappa \in L^1_{\text{loc}}(\mathbb{R}^n): \|\kappa\|_{\Lambda_\alpha} < +\infty\}.$$

Let

$$\mathcal{B}_\alpha(\mathbb{R}^n) = \{\kappa \in \Lambda_\alpha(\mathbb{R}^n): \text{supp } \kappa \subset B(0, 1), \|\kappa\|_{\Lambda_\alpha} \leq 1\},$$

$$\mathcal{B}'_\alpha(\mathbb{R}^n) = \left\{ \kappa \in \Lambda_\alpha(\mathbb{R}^n): \text{there exists a sequence of functions} \right. \\ \left. \{\kappa_j\}_{j=0}^\infty \subset \mathcal{B}_\alpha(\mathbb{R}^n) \text{ such that} \right.$$

$$(1.2) \quad \left. \kappa(x) = \sum_{j=0}^\infty 2^{-j}(\kappa_j)_{2^j}(x) \right\},$$

$$\mathcal{B}_\alpha^0(\mathbb{R}^n) = \left\{ \kappa \in \mathcal{B}_\alpha(\mathbb{R}^n): \int_{\mathbb{R}^n} \kappa(x) dx = 0 \right\},$$

$$\mathcal{B}'_\alpha{}^0(\mathbb{R}^n) = \left\{ \kappa \in \Lambda_\alpha(\mathbb{R}^n): \text{there exists a sequence of functions} \right. \\ \left. \{\kappa_j\}_{j=0}^\infty \subset \mathcal{B}_\alpha^0(\mathbb{R}^n) \text{ such that (1.2) holds} \right\}.$$

REMARK 1.1. Notice the following simple fact: if $0 < \beta < \alpha$ and if $\varphi \in \mathcal{B}_\alpha(\mathbb{R}^n)$ (or $\varphi \in \mathcal{B}'_\alpha(\mathbb{R}^n)$), then $c\varphi \in \mathcal{B}_\beta(\mathbb{R}^n)$ (or $c\varphi \in \mathcal{B}'_\beta(\mathbb{R}^n)$), where c is a positive constant depending only on α , β and n .

DEFINITION 1.1. For $\alpha > 0$ and $\varphi \in \Lambda_\alpha(\mathbb{R}^n)$ we say that φ satisfies the condition (I. α) if the following three conditions hold:

$$(I.\alpha.1) \quad \varphi \in \mathcal{B}'_\alpha(\mathbb{R}^n),$$

$$(I.\alpha.2) \quad \mathcal{F}\varphi(\xi) \text{ is } (n + [\alpha] + 2)\text{-times differentiable except } \xi = 0 \text{ and}$$

$$|D_\xi^\gamma \mathcal{F}\varphi(\xi)| \leq |\xi|^{1-l(\gamma)}, \quad \xi \neq 0,$$

for any γ with $1 \leq l(\gamma) \leq n + [\alpha] + 2$,

$$(I.\alpha.3) \quad \mathcal{F}\varphi(0) \neq 0.$$

DEFINITION 1.2. For $\alpha > 0$ and $\{\varphi_1, \varphi_2, \dots, \varphi_N\} \subset \Lambda_\alpha(R^n)$ we say that the set of functions $\{\varphi_1, \varphi_2, \dots, \varphi_N\}$ satisfies the condition (II. α) if the following three conditions hold:

(II. α .1) $\varphi_i \in \mathcal{B}_\alpha^{0'}(R^n), i = 1, 2, \dots, N,$

(II. α .2) $\mathcal{F}\varphi_i(\xi) (i = 1, 2, \dots, N)$ are $(n + [\alpha] + 2)$ -times differentiable except $\xi = 0$ and

$$|D_\xi^\gamma \mathcal{F}\varphi_i(\xi)| \leq |\xi|^{1-l(\gamma)}, \quad \xi \neq 0, i = 1, 2, \dots, N,$$

for any γ with $0 \leq l(\gamma) \leq n + [\alpha] + 2,$

(II. α .3) $\inf_{\xi \in R^n \setminus \{0\}} \sup_{t > 0} \sum_{i=1}^N |\mathcal{F}\varphi_i(t\xi)| > 0.$

For the sake of simplicity we put

(1.3) $\delta =$ the left-hand side of (II. α .3).

EXAMPLE 1.1. Let

$$P(x, t) = C_n t (|x|^2 + t^2)^{-(n+1)/2}$$

that is the Poisson kernel. Then $cP(x, 1)$ satisfies the condition (I. α) for any $\alpha > 0$ if $c (> 0)$ is small enough depending on α and $n.$

EXAMPLE 1.2. Let $\varphi \in \mathcal{S}(R^n)$ and $\int_{R^n} \varphi(x) dx \neq 0.$ Then $c\varphi$ satisfies the condition (I. α) for any $\alpha > 0$ if $c (> 0)$ is small enough depending only on α and $\varphi.$

EXAMPLE 1.3. Let $N = 1$ and $\varphi(x) = cD_i P(x, 1).$ Then $\{\varphi\}$ satisfies the condition (II. α) for any $\alpha > 0$ if $c (> 0)$ is small enough depending on α and $n.$

EXAMPLE 1.4. Let $N = n$ and

$$\varphi_i(x) = cD_{x_i} P(x, 1), \quad i = 1, 2, \dots, n.$$

Then the set of functions $\{\varphi_1, \varphi_2, \dots, \varphi_n\}$ satisfies the condition (II. α) for any $\alpha > 0$ if $c (> 0)$ is small enough depending on α and $n.$

EXAMPLE 1.5. Let $\varphi_1, \varphi_2, \dots, \varphi_N \in \mathcal{S}(R^n),$

$$\mathcal{F}\varphi_i(0) = 0, \quad i = 1, 2, \dots, N,$$

and let (II. α .3) hold. Then the set of functions $\{c\varphi_1, \dots, c\varphi_N\}$ satisfies the

condition (II.α) for any α > 0 if c (> 0) is small enough depending on α and {ϕ₁, . . . , ϕ_N}.

REMARK 1.2. The author learned the condition (II.α.3) from A. Calderón and Torchinsky [1], in which they investigated the area integrals defined from the kernels that satisfy the conditions of Example 1.5.

Our first result is the following.

THEOREM 1.1. *Let*

$$(1.4) \quad 0 < \beta < \alpha, \quad 0 < \varepsilon < 1 \quad \text{and} \quad 0 < q \leq 1.$$

Let $\varphi \in \Lambda_\alpha(R^n)$ satisfy the condition (I.α). Let $\kappa \in \mathcal{B}'_\beta(R^n)$ and $f \in \bigcup_{1 \leq p \leq +\infty} L^p(R^n)$. Then

$$(1.5) \quad \left| \int_{R^n} f(x)\kappa(x) dx \right| \leq C \left(\int \int_{R^{n+1}_+} |f * (\varphi)_t(y)|^q k_{\beta,\varepsilon}(y, t)^q t^{n(q-1)-1} dy dt \right)^{1/q},$$

where

$$(1.6) \quad k_{\beta,\varepsilon}(y, t) = t^\beta(1 + |y| + t)^{-n-\beta-1+\varepsilon}$$

and where C is a constant depending only on α, β, ε, q, $\mathcal{F}\varphi(0)$ and n.

If ϕ, α and β are as in Theorem 1.1, then it follows from Remark 1.1 that $c\varphi \in \mathcal{B}'_\beta(R^n)$ provided c (> 0) is small enough. Thus substituting $\kappa(x) = \varphi(-x)$ into (1.5) we get

$$|f * \varphi(0)| \leq C \left(\int \int_{R^{n+1}_+} |f * (\varphi)_t(y)|^q k_{\beta,\varepsilon}(y, t)^q t^{n(q-1)-1} dy dt \right)^{1/q}.$$

Applying this inequality to the case of Example 1.1, i.e. $(\varphi)_t(x) = P(x, t)$, and putting

$$u(x, t) = (f * P(\cdot, t))(x)$$

we get

$$\begin{aligned} |u(0, 1)|^p &\leq C \left(\int \int_{R^{n+1}_+} |u(y, t)|^q k_{\beta,\varepsilon}(y, t)^q t^{n(q-1)-1} dy dt \right)^{p/q} \\ &\leq C \int \int_{R^{n+1}_+} |u(y, t)|^p k_{\beta,\varepsilon}(y, t)^q t^{n(q-1)-1} dy dt \end{aligned}$$

for any $\beta > 0$, any $q \in (n/(n + \beta), 1]$, any $p \in [q, +\infty)$ and any $\varepsilon \in (0, 1)$, where C depends only on β, q, p, ε and n . (The second inequality follows from Hölder's inequality and from the fact that

$$\int \int k_{\beta, \varepsilon}(y, t)^q t^{n(q-1)-1} dy dt < +\infty$$

provided $q > n/(n + \beta)$.) This is the reason why we regard Theorem 1.1 as a weak extension of the inequality (1.1).

If we replace the condition (I. α) by the condition (II. α), then we have the following.

THEOREM 1.2. *Let (1.4) hold. Let $\{\varphi_1, \varphi_2, \dots, \varphi_N\} \subset \Lambda_\alpha(R^n)$ satisfy the condition (II. α). Let $\kappa \in \mathcal{B}_\beta^{0'}(R^n)$ and $f \in \bigcup_{1 \leq p \leq +\infty} L^p(R^n)$. Then*

$$(1.7) \quad \left| \int_{R^n} f(x)\kappa(x) dx \right| \leq C \sum_{i=1}^N \left(\int \int_{R^{n+1}} |f * (\varphi_i)_t(y)|^q k_{\beta, \varepsilon}(y, t)^q t^{n(q-1)-1} dy dt \right)^{1/q},$$

where $k_{\beta, \varepsilon}$ is defined by (1.6) and where C is a constant depending only on $\alpha, \delta, \beta, \varepsilon, q, N$ and n . (For the definition of δ recall (1.3).)

We prove Theorems 1.1 and 1.2 in §§4–6. In §§2 and 3 we give applications of these theorems to real Hardy spaces. In §2 using Theorem 1.1 we give another proof of the estimate of grand maximal functions in terms of radial maximal functions. In §3 using Theorems 1.1 and 1.2 and using the ideas suggested by Robert Fefferman we give another proof of the estimate of the Lusin S -functions in terms of the Littlewood-Paley g -functions. Since we do not need harmonicity at all in Theorems 1.1 and 1.2, we can develop these arguments in general setting.

NOTATION. For $q > 0, f \in L^1_{loc}(R^n)$ and $x \in R^n$ let

$$M_q(f)(x) = \sup_{B: B \ni x} \left(|B|^{-1} \int_B |f(y)|^q dy \right)^{1/q},$$

where the supremum is taken over all balls $B (\subset R^n)$ that contain x . $\check{f}(x)$ denotes $f(-x)$. For a measurable set $E, \chi_E(x)$ denotes the characteristic function of E . For $(x, s) \in R^{n+1}_+$ let

$$Q(x, s) = \{(y, t) \in R^{n+1}_+ : 0 < t \leq s, |x - y| \leq s\}$$

$$Q'(x, s) = \{(y, t) \in R^{n+1}_+ : s/2 \leq t \leq 2s, |x - y| \leq s/2\}$$

and

$$Q''(x, s) = \{(y, t) \in R_+^{n+1}: 3s/4 \leq t \leq 3s/2, |x - y| \leq s/6\}.$$

The letter C denotes various positive constants.

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2. Grand maximal functions from radial maximal functions. In this section we give an application of Theorem 1.1. Let $\alpha > 0$ and let φ satisfy the condition (I. α). We fix φ . For $f \in \cup_{1 \leq p \leq +\infty} L^p(R^n)$ we define a radial maximal function $f_\varphi^+(x)$ by

$$f_\varphi^+(x) = \sup_{t>0} |f * (\varphi)_t(x)|.$$

For $f \in \cup_{1 \leq p \leq +\infty} L^p(R^n)$ and $\beta > 0$ we define a grand maximal function $f_\beta^*(x)$ by

$$f_\beta^*(x) = \sup\{|f * (\kappa)_t(x)|: t > 0, \kappa \in \mathcal{B}_\beta(R^n)\}.$$

As an application of Theorem 1.1 we get

THEOREM 2.1. *Let $0 < \beta < \alpha$. Let φ satisfy the condition (I. α). Let $q > n/(n + \beta)$, $x \in R^n$, and $f \in \cup_{1 \leq p \leq +\infty} L^p(R^n)$. Then*

$$f_\beta^*(x) \leq CM_q(f_\varphi^+)(x),$$

where C is a constant depending only on $\alpha, \beta, q, \mathcal{F}\varphi(0)$ and n .

Proof. We may assume $x = 0$. Let $\kappa \in \mathcal{B}_\beta(R^n)$ and $\varepsilon \in (0, 1)$. By Theorem 1.1 we get

$$\begin{aligned} |f * \kappa(0)| &\leq C \left(\int \int_{R_+^{n+1}} f_\varphi^+(y)^q k_{\beta,\varepsilon}(y, t)^q t^{n(q-1)-1} dy dt \right)^{1/q} \\ &= C \left(\int_{R^n} f_\varphi^+(y)^q dy \int_0^{+\infty} k_{\beta,\varepsilon}(y, t)^q t^{n(q-1)-1} dt \right)^{1/q} \\ &\leq C \left(\int f_\varphi^+(y)^q (1 + |y|)^{-n-q(1-\varepsilon)} dy \right)^{1/q} \quad \text{by } q > \frac{n}{n + \beta} \\ &\leq CM_q(f_\varphi^+)(0). \end{aligned}$$

By the argument of dilation we get

$$|f * (\kappa)_t(0)| \leq CM_q(f_\varphi^+)(0)$$

for any $t > 0$. Since $\kappa \in \mathcal{B}_\beta(R^n)$ is arbitrary, we have

$$f_\beta^*(0) \leq CM_q(f_\varphi^+)(0).$$

□

As a consequence of Theorem 2.1, using the Hardy-Littlewood maximal theorem we get the following which was originally proved by C. Fefferman and E. Stein [6].

COROLLARY 2.A. *Let α, β and φ be as in Theorem 1.1. Let $p > n/(n + \beta)$ and $f \in \bigcup_{1 \leq p \leq +\infty} L^p(\mathbb{R}^n)$. Then*

$$\|f_\beta^*\|_{L^p} \leq C \|f_\varphi^+\|_{L^p},$$

where C is a constant depending only on $\alpha, \beta, p, \mathcal{F}\varphi(0)$ and n .

REMARK 2.1. Theorem 2.1 had been proved by Uchiyama [13] in a somewhat different method. (See also [12].) Theorem 2.1 for the case φ is the Poisson kernel had been proved by C. Fefferman and E. M. Stein [6] p. 170 by using the inequality (1.1). The above proof of Theorem 2.1 is a generalization of their argument.

3. S-functions from g-functions. The author owes the fundamental idea in this section to Robert Fefferman.

For $a > 0, r \in \mathbb{R}, x \in \mathbb{R}^n$ and for a continuous function $u(y, t)$ defined on \mathbb{R}_+^{n+1} let

$$(3.1) \quad S_{a,r}(u)(x)^2 = a^{-n} \int \int_{\Gamma(x,a)} |u(y, t)|^2 t^{2r-n-1} dy dt$$

and

$$(3.2) \quad S_{0,r}(u)(x)^2 = \int_0^{+\infty} |u(x, t)|^2 t^{2r-1} dt,$$

where

$$\Gamma(x, a) = \{(y, t) \in \mathbb{R}_+^{n+1}: |x - y| < at\}.$$

Note that if u is harmonic, then $S_{0,1}(D_t u)(x)$ and $S_{0,1}(D_{x_i} u)(x)$ are the Littlewood-Paley g -functions and $S_{1,1}(D_t u)(x)$ and $S_{1,1}(D_{x_i} u)(x)$ are the Lusin S -functions. Our concern is to get the estimates of $S_{a,r}(u)$ in terms of $S_{0,r}(u)$.

First we explain the idea suggested by R. Fefferman, which is also implicit in R. Fefferman and Stein [7], for the case when u is harmonic. The following is crucial.

THEOREM 3.A. *Let $u(y, t)$ be a complex-valued harmonic function defined on \mathbb{R}_+^{n+1} . Let $r \in \mathbb{R}, q > 0, a \geq 0$ and $x \in \mathbb{R}^n$. Then*

$$S_{a,r}(u)(x)^2 \leq C \int_0^{+\infty} M_q(u(\cdot, t))(x)^2 t^{2r-1} dt,$$

where C is a constant depending only on r, q, a and n .

Proof. We give a proof only for the case $a = 1$. We may assume $q \leq 2$. Applying the inequality (1.1) with dilation we get that if $(y, t) \in \Gamma(x, 1)$, then

$$\begin{aligned} |u(y, t)|^q &\leq Ct^{-n-1} \int_{B(y, t)} dz \int_{t/2}^{3t/2} |u(z, s)|^q ds \\ &\leq Ct^{-n-1} \int_{B(x, 2t)} dz \int_{t/2}^{3t/2} |u(z, s)|^q ds \\ &\leq C \int_{t/2}^{3t/2} M_q(u(\cdot, s))(x)^q \frac{ds}{s}. \end{aligned}$$

Thus

$$\begin{aligned} &\int \int_{\Gamma(x, 1)} |u(y, t)|^2 t^{2r-n-1} dy dt \\ &\leq \int_0^{+\infty} t^{2r-1} dt \left(C \int_{t/2}^{3t/2} M_q(u(\cdot, s))(x)^q \frac{ds}{s} \right)^{2/q} \\ &= \int_0^{+\infty} t^{2r-1} dt \left(C \int_{1/2}^{3/2} M_q(u(\cdot, st))(x)^q \frac{ds}{s} \right)^{2/q} \\ &\leq C \int_0^{+\infty} M_q(u(\cdot, t))(x)^2 t^{2r-1} dt \end{aligned}$$

by $2/q \geq 1$ and by Minkowski's inequality. □

COROLLARY 3.A. *Let $0 < p < +\infty$. Let u, r and a be as in Theorem 3.A. Then*

$$\|S_{a,r}(u)\|_{L^p} \leq C \|S_{0,r}(u)\|_{L^p},$$

where C is a constant depending only on r, p, a and n .

Proof. Take $q \in (0, \min(p, 2))$. By the continuous version of the vector maximal theorem of C. Fefferman and Stein [5] we get

$$\begin{aligned} (3.3) \quad &\left\{ \int_{R^n} \left(\int_0^{+\infty} M_q(u(\cdot, t))(x)^2 t^{2r-1} dt \right)^{p/2} dx \right\}^{1/p} \\ &= \left\{ \int_{R^n} \left(\int_0^{+\infty} M_1(|u(\cdot, t)|^q)(x)^{2/q} t^{2r-1} dt \right)^{(q/2)(p/q)} dx \right\}^{(q/p)/q} \\ &\leq C \left\{ \int_{R^n} \left(\int_0^{+\infty} |u(x, t)|^{q(2/q)} t^{2r-1} dt \right)^{(q/2)(p/q)} dx \right\}^{(q/p)/q} \\ &= C \|S_{0,r}(u)\|_{L^p}. \end{aligned}$$

Combining Theorem 3.A and the inequality (3.3) we get Corollary 3.A. □

The above clever idea to use the result of [5] is in R. Fefferman and Stein [7].

COROLLARY 3.B. *Let u , a and p be as in Corollary 3.A. Then*

$$\|S_{a,1}(D_t u)\|_{L^p} \leq C \|S_{0,1}(D_t u)\|_{L^p}$$

and

$$\|S_{a,1}(D_{x_i} u)\|_{L^p} \leq C \|S_{0,1}(D_{x_i} u)\|_{L^p}, \quad i = 1, \dots, n,$$

where C is a constant depending only on p , a and n .

Since $D_t u$ and $D_{x_i} u$ are harmonic, Corollary 3.B is a direct consequence of Corollary 3.A. Corollary 3.B means that the integrals of the p th powers of g -functions dominate those of S -functions.

The argument we have explained so far was suggested by R. Fefferman. In the following part of this section, replacing the inequality (1.1) in the above argument by Theorems 1.1 and 1.2, we extend the above results to the functions of the form $u(y, t) = f * (\varphi)_t(y)$, which are no longer harmonic. We define

$$S_{a,r}(f * (\kappa):)(x)$$

by the formulae (3.1) and (3.2) with

$$u(y, t) = f * (\kappa)_t(y).$$

(Two dots : mean two variables y and t .) Our results are the following.

THEOREM 3.1. *Let $0 < \beta < \alpha$ and $-1 < r < \beta$. Let φ satisfy the condition (I. α). Let $\kappa \in \mathcal{B}'_\beta(\mathbb{R}^n)$, $q > n/(n + \beta - r)$, $a \geq 0$, $f \in \bigcup_{1 \leq p \leq +\infty} L^p(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. Then*

$$(3.4) \quad S_{a,r}(f * (\kappa):)(x)^2 \leq C \int_0^{+\infty} M_q(f * (\varphi)_t(\cdot))(x)^2 t^{2r-1} dt,$$

where C is a constant depending only on α , $\mathcal{F}\varphi(0)$, β , r , q , a and n .

COROLLARY 3.1. *Let α , β , r , φ , κ , a and f be as in Theorem 3.1. Let $p \in (n/(n + \beta - r), +\infty)$. Then*

$$\|S_{a,r}(f * (\kappa):)\|_{L^p} \leq C \|S_{0,r}(f * (\varphi):)\|_{L^p},$$

where C is a constant depending only on α , $\mathcal{F}\varphi(0)$, β , r , p , a and n .

COROLLARY 3.2. *Let $1 < \beta < \alpha$. Let φ , κ and a be as in Theorem 3.1. Let $p \in (n/(n + \beta - 1), +\infty)$ and let $f \in \mathcal{S}(\mathbb{R}^n)$. Then*

$$\|S_{a,1}(D_{x_i}(f * (\kappa):))\|_{L^p} \leq C \|S_{0,1}(D_{x_i}(f * (\varphi):))\|_{L^p}, \quad i = 1, \dots, n,$$

where C is a constant depending only on α , $\mathcal{F}\varphi(0)$, β , p , a and n .

REMARK 3.1. By Remark 1.1 we can substitute $\kappa = \varphi$ in the above three results.

REMARK 3.2. If $r > -1$ and if $u(y, t)$ is a harmonic function defined by $f * P(\cdot, t)(y)$ with $f \in \cup_{1 \leq p \leq +\infty} L^p(R^n)$, then Theorem 3.A follows from Theorem 3.1 by substituting

$$\varphi(x) = \kappa(x) = P(x, 1)$$

and by taking α and β so that $\max(0, r, r - n + n/q) < \beta < \alpha$, where $P(x, t)$ is the Poisson kernel in Example 1.1. Similarly, if $r > -1$ and if $u(y, t) = f * P(\cdot, t)(y)$ with $f \in \cup_{1 \leq p \leq +\infty} L^p(R^n)$, then Corollary 3.A follows from Corollary 3.1. If $u(y, t) = f * P(\cdot, t)(y)$ with $f \in \mathcal{S}(R^n)$, then the latter half of Corollary 3.B follows from Corollary 3.2.

THEOREM 3.2. Let $0 < \beta < \alpha$ and $-1 < r < \beta$. Let $\{\varphi_1, \dots, \varphi_N\}$ satisfy the condition (II.α). Let $\kappa \in \mathcal{B}_\beta^{0'}(R^n)$, $q > n/(n + \beta - r)$, $a \geq 0$, $f \in \cup_{1 \leq p \leq +\infty} L^p(R^n)$ and $x \in R^n$. Then

$$(3.5) \quad S_{a,r}(f * (\kappa):)(x)^2 \leq C \sum_{i=1}^N \int_0^{+\infty} M_q(f * (\varphi_i)_t(\cdot))(x)^2 t^{2r-1} dt,$$

where C is a constant depending only on $\alpha, \delta, \beta, r, q, a, N$ and n . (For the definition of δ recall (1.3).)

COROLLARY 3.3. Let $\alpha, \beta, r, \{\varphi_1, \dots, \varphi_N\}, \kappa, a$ and f be as in Theorem 3.2. Let $p \in (n/(n + \beta - r), +\infty)$. Then

$$\|S_{a,r}(f * (\kappa):)\|_{L^p} \leq C \sum_{i=1}^N \|S_{0,r}(f * (\varphi_i):)\|_{L^p},$$

where C is a constant depending only on $\alpha, \delta, \beta, r, p, a, N$ and n .

COROLLARY 3.4. Let $\alpha > 0$. Let $\{\varphi_1, \dots, \varphi_N\}$ satisfy the condition (II.α). Let $f \in \cup_{1 \leq p < +\infty} L^p(R^n)$. Let $p \in (n/(n + \alpha), +\infty)$. Then

$$(3.6) \quad \|f\|_{H^p} \leq C \sum_{i=1}^N \|S_{0,0}(f * (\varphi_i):)\|_{L^p},$$

where C is a constant depending only on α, δ, p, N and n .

REMARK 3.3. In Corollary 3.4 $\|f\|_{H^p}$ means

$$\left(\int_{R^n} \sup_{t>0} |f * P(\cdot, t)(x)|^p dx \right)^{1/p}.$$

There are several characterizations of $H^p(R^n)$. See [6] and [2] for details.

REMARK 3.4. If $\{\varphi_1, \dots, \varphi_N\}$ is as in Example 1.3 or 1.4, if f is as in Corollary 3.4 and if $p \in (0, +\infty)$, then the inequality (3.6) had been shown by [6] p. 172 Remark.

REMARK 3.5. If $\{\varphi_1, \dots, \varphi_N\}$ is as in Example 1.5, $p \in (0, +\infty)$ and if $f \in H^p(\mathbb{R}^n)$, then the inequality

$$\|f\|_{H^p} \leq \sum_{i=1}^N \|S_{1,0}(f * (\varphi_i):)\|_{L^p}$$

had been shown by Calderón and Torchinsky [1] and the inequality (3.6) had been shown by [14].

REMARK 3.6. The converse inequality of (3.6) (with another constant $C > 0$) is known to hold for $p \in (n/(n + \alpha), +\infty)$. Let $\mathcal{H} = L^2_{dt/t}(0, +\infty)$. Let T be the operator that assigns $(\varphi_i)_t * f(x) \in L^2(\mathbb{R}^n, \mathcal{H})$ to $f \in L^2(\mathbb{R}^n)$. If $p \in (1, +\infty)$, then the argument of Hilbert space valued singular integral operators ([9] p. 83) gives $\|Tf\|_{L^p(\mathbb{R}^n, \mathcal{H})} \leq C\|f\|_{L^p(\mathbb{R}^n)}$ provided $f \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$. If $p \in (n/(n + \alpha), 1]$, then it is easy to show that $\|Tf\|_{L^p(\mathbb{R}^n, \mathcal{H})} \leq C\|f\|_{H^p(\mathbb{R}^n)}$ provided f is a p -atom. (For the definition of p -atoms see [2].) The converse inequality of (3.6) follows from the above two inequalities and Fatou’s lemma.

From now we prove the above results. For the proof of Theorem 3.1 we need the following.

LEMMA 3.1. *Let $0 < \beta < \alpha$, $0 < \varepsilon < 1$ and $n/(n + \beta + 1 - \varepsilon) < q \leq 1$. Let φ, κ and f be as in Theorem 3.1. Then*

$$(3.7) \quad \left| \int_{\mathbb{R}^n} f(x)\kappa(x) dx \right| \leq C \left(\int_0^{+\infty} M_q(f * (\varphi)_t(\cdot))(0)^q h_{\beta, q, \varepsilon}(t) \frac{dt}{t} \right)^{1/q},$$

where

$$(3.8) \quad h_{\beta, q, \varepsilon}(t) = t^{\beta q + nq - n} (1 + t)^{(-n - \beta - 1 + \varepsilon)q + n}$$

and where C is a constant depending only on $\alpha, \mathcal{F}\varphi(0), \beta, \varepsilon, q$ and n .

Proof. By Theorem 1.1 it is enough to show that the right-hand side of (1.5) is bounded by the right-hand side of (3.7).

Since $k_{\beta,\varepsilon}(y, t)$ in (1.6) is radial and decreasing as a function of y variable for each fixed $t > 0$, we get

$$\begin{aligned} & \int_{R^n} |f * (\varphi)_t(y)|^q k_{\beta,\varepsilon}(y, t)^q dy \\ & \leq M_q(f * (\varphi)_t(\cdot))(0)^q \int k_{\beta,\varepsilon}(y, t)^q dy \\ & \leq CM_q(\dots)(0)^q t^{\beta q} (1 + t)^{(-n-\beta-1+\varepsilon)q+n} \\ & = CM_q(\dots)(0)^q t^{-nq+n} h_{\beta,q,\varepsilon}(t). \end{aligned}$$

Hence the right-hand side of (1.5) is bounded by the right-hand side of (3.7). □

Proof of Theorem 3.1. We may assume $x = 0$ and $q \leq 1$. First we give a proof for the case $a = 0$. Applying Lemma 3.1 with $\varepsilon \in (0, \min(1, 1 + r))$ and with dilation we get

$$\begin{aligned} S_{0,r}(f * (\kappa):)(0)^2 &= \int_0^{+\infty} |f * (\kappa)_t(0)|^2 t^{2r-1} dt \\ &\leq C \int_0^{+\infty} t^{2r-1} dt \left(\int_0^{+\infty} M_q(f * (\varphi)_{st}(\cdot))(0)^q h_{\beta,q,\varepsilon}(s) \frac{ds}{s} \right)^{2/q} \\ &\leq C \left\{ \int_0^{+\infty} h_{\beta,q,\varepsilon}(s) \frac{ds}{s} \left(\int_0^{+\infty} M_q(f * (\varphi)_{st}(\cdot))(0)^2 t^{2r-1} dt \right)^{q/2} \right\}^{2/q} \\ & \hspace{15em} \text{by Minkowski's inequality} \\ &= C \left(\int_0^{+\infty} h_{\beta,q,\varepsilon}(s) s^{-rq-1} ds \right)^{2/q} \int_0^{+\infty} M_q(f * (\varphi)_t(\cdot))(0)^2 t^{2r-1} dt. \end{aligned}$$

Since

$$\int_0^{+\infty} h_{\beta,q,\varepsilon}(s) s^{-rq-1} ds < +\infty$$

by $q > n/(n + \beta - r)$ and by $\varepsilon < 1 + r$, we get (3.4) for the case $a = 0$ and $x = 0$.

Next we show the case $a > 0$. Put $\tau_z \kappa(y) = \kappa(y + z)$. Then

$$(3.9) \quad S_{a,r}(f * (\kappa):)(0)^2 = a^{-n} \int_{B(0, a)} S_{0,r}(f * (\tau_z \kappa):)(0)^2 dz.$$

If $z \in B(0, a)$ and if $c > 0$ is small enough depending on a, β and n , then $c\tau_z \kappa \in \mathcal{B}'_c(R^n)$. Thus by the result of the case $a = 0$ we get

$$(3.10) \quad S_{0,r}(f * (\tau_z \kappa):)(0)^2 \leq C \int_0^{+\infty} M_q(f * (\varphi)_t(\cdot))(0)^2 t^{2r-1} dt.$$

Substituting (3.10) into (3.9) we get (3.4) for the case $a > 0$ and $x = 0$. □

The way Corollary 3.1 follows from Theorem 3.1 is the same with the way Corollary 3.A followed from Theorem 3.A. Corollary 3.2 is obtained by applying Corollary 3.1 not to the function f but to the function $D_x f$.

For the proof of Theorem 3.2 we need the following Lemma 3.2, which can be proved in exactly the same way with Lemma 3.1 just by replacing Theorem 1.1 by Theorem 1.2. We omit the proof.

LEMMA 3.2. *Let $0 < \beta < \alpha$, $0 < \varepsilon < 1$ and $n/(n + \beta + 1 - \varepsilon) < q \leq 1$. Let $\{\varphi_1, \dots, \varphi_N\}$, κ and f be as in Theorem 3.2. Then*

$$\left| \int f(x)\kappa(x) dx \right| \leq C \sum_{i=1}^N \left(\int_0^{+\infty} M_q(f * (\varphi_i)_t(\cdot))(0)^q h_{\beta, q, \varepsilon}(t) \frac{dt}{t} \right)^{1/q},$$

where $h_{\beta, q, \varepsilon}$ is defined by (3.8) and where C is a constant depending only on $\alpha, \delta, \beta, \varepsilon, q, N$ and n .

The way Theorem 3.2 follows from Lemma 3.2 is the same with the way Theorem 3.1 followed from Lemma 3.1. The way Corollary 3.3 follows from Theorem 3.2 is the same with the way Corollary 3.A followed from Theorem 3.A.

Proof of Corollary 3.4. Put

$$\kappa(x) = D_t P(x, 1),$$

where $P(x, t)$ is the Poisson kernel. By the result of [6] we have

$$\|S_{a,0}(f * (\kappa))\|_{L^p} = \|S_{a,1}(D_t(f * P(\cdot, \cdot)))\|_{L^p} \geq c \|f\|_{H^p}.$$

(Recall Remark 3.4.) Thus, putting $r = 0$, taking $\beta \in (\max(0, n/p - n), \alpha)$ and applying Corollary 3.3 to the above κ we get (3.6). □

4. Proof of Theorem 1.2.

LEMMA 4.1. *Let $0 < \beta < \alpha$ and $0 < \varepsilon < 1$. Let $\{\varphi_1(x), \dots, \varphi_N(x)\}$ satisfy the condition (II.α). Let $\kappa \in \mathcal{B}_\beta^{0'}(R^n)$. Let $0 < a < C_{4.1}$. Let $\mathcal{E} \subset R_+^{n+1}$ be a measurable set that satisfies*

$$(4.1) \quad \int \int_{\mathcal{E} \cap Q''(x,s)} dy dt \leq as^{n+1}$$

for any $(x, s) \in R_+^{n+1}$. Then, there exist measurable functions $k_1(y, t), \dots, k_N(y, t)$ defined on R_+^{n+1} such that

$$(4.2) \quad |k_i(y, t)| \leq C_{4.2} k_{\beta, \varepsilon}(y, t), \quad i = 1, \dots, N,$$

for any $(y, t) \in R_+^{n+1}$ and that

$$(4.3) \quad \kappa(x) = \int \int_{\mathcal{E}^c} \sum_{i=1}^N (\varphi_i)_t(y - x) k_i(y, t) dy \frac{dt}{t}$$

for any $x \in R^n$, where $k_{\beta, \varepsilon}$ is defined by (1.6) and where $C_{4.1}$ and $C_{4.2}$ are positive constants depending only on $\alpha, \beta, \varepsilon, \delta, N$ and n . (For the definition of $Q''(x, s)$ and δ see §1.)

Since the proof of Lemma 4.1 is lengthy, we postpone it to §5.

Accepting Lemma 4.1 temporarily we prove Theorem 1.2. For $(x, s) \in R_+^{n+1}$ let

$$w_q(x, s) = \left\{ s^{-n-1} \int \int_{Q'(x, s)} \left(\sum_{i=1}^N |f * (\varphi_i)_t(y)| \right)^q dy dt \right\}^{1/q}.$$

Let $A > 1$ and let

$$(4.4) \quad \mathcal{E} = \left\{ (x, s) \in R_+^{n+1} : \left(\sum_{i=1}^N |f * (\varphi_i)_s(x)| \right)^q > Aw_q(x, s)^q \right\}.$$

Let $(y, t) \in Q''(x, s)$. Since $Q'(y, t) \supset Q''(x, s)$, we have

$$(4.5) \quad w_q(y, t)^q \geq \left(\frac{2}{3} \right)^{n+1} s^{-n-1} \int \int_{Q''(x, s)} \left(\sum_{i=1}^N |f * (\varphi_i)_u(z)| \right)^q dz du.$$

Therefore

$$\int \int_{\mathcal{E} \cap Q''(x, s)} dy dt \leq \int \int_{Q''(x, s)} \left(\sum |f * (\varphi_i)_t(y)| \right)^q A^{-1} w_q(y, t)^{-q} dy dt$$

by the definition of \mathcal{E}

$$\leq (3/2)^{n+1} s^{n+1} A^{-1} \quad \text{by (4.5).}$$

So, the set \mathcal{E} defined by (4.4) satisfies (4.1) with

$$(4.6) \quad a = (3/2)^{n+1} A^{-1}.$$

Let $A > 1$ be so large that (4.6) $< C_{4.1}$. Applying Lemma 4.1 with (4.6) and (4.4) gives $k_1(y, t), \dots, k_N(y, t)$ that satisfy (4.2) and (4.3). Thus

$$(4.7) \quad \left| \int_{R^n} f(x) \kappa(x) dx \right| \\ = \left| \int_{R^n} f(x) dx \int \int_{\mathcal{E}^c} \sum_{i=1}^N (\varphi_i)_t(y-x) k_i(y, t) dy \frac{dt}{t} \right| \quad \text{by (4.3)} \\ = \left| \int \int_{\mathcal{E}^c} \sum_i f * (\varphi_i)_t(y) k_i(y, t) dy \frac{dt}{t} \right| \quad \text{by } f \in \bigcup_{1 \leq p \leq +\infty} L^p(R^n) \\ \leq \int \int_{R_+^{n+1}} A^{1/q} w_q(y, t) Ck_{\beta, \varepsilon}(y, t) dy \frac{dt}{t}$$

by (4.2) and by the definition of \mathcal{E} .

Put $\omega = n^{-1/2}$. For $(y, t) \in R_+^{n+1}$ let

$$S(y, t) = \{(j_1, \dots, j_n, i) \in Z^{n+1}: t/2 \leq 2^i \leq 2t, |y - \omega 2^i j| \leq 2^{i+1}\},$$

where Z is the set of all integers and where $j = (j_1, \dots, j_n) \in Z^n$. Note that

$$Q'(y, t) \subset \bigcup_{(j, i) \in S(y, t)} Q'(\omega 2^i j, 2^i)$$

and that if $(j, i) \in S(y, t)$, then

$$ck_{\beta, \epsilon}(y, t) \leq k_{\beta, \epsilon}(\omega 2^i j, 2^i) \leq Ck_{\beta, \epsilon}(y, t).$$

So

$$(4.8) \quad w_q(y, t)k_{\beta, \epsilon}(y, t) \leq C \sum_{(j, i) \in S(y, t)} w_q(\omega 2^i j, 2^i)k_{\beta, \epsilon}(\omega 2^i j, 2^i).$$

Note that

$$(4.9) \quad \int \int_{\{(y, t) \in R_+^{n+1}: S(y, t) \ni (\omega 2^i j, 2^i)\}} dy \frac{dt}{t} \leq C2^{in}$$

for any $(j, i) \in Z^{n+1}$ and that

$$(4.10) \quad \left\| \sum_{(j, i) \in Z^{n+1}} \chi_{Q'(\omega 2^i j, 2^i)} \right\|_{L^\infty} \leq C.$$

Therefore,

$$(4.7) \leq CA^{1/q} \int \int_{R_+^{n+1}} \sum_{(j, i) \in S(y, t)} w_q(\omega 2^i j, 2^i)k_{\beta, \epsilon}(\omega 2^i j, 2^i) dy \frac{dt}{t} \tag{by (4.8)}$$

$$\leq CA^{1/q} \sum_{(j, i) \in Z^{n+1}} w_q(\omega 2^i j, 2^i)k_{\beta, \epsilon}(\omega 2^i j, 2^i)2^{in} \quad \text{by } q \leq 1$$

$$\leq CA^{1/q} \left(\sum_{(j, i) \in Z^{n+1}} w_q(\dots)^q k_{\beta, \epsilon}(\dots)^q 2^{inq} \right)^{1/q} \quad \text{by (4.9)}$$

$$\leq CA^{1/q} \left(\int \int_{R_+^{n+1}} \left(\sum_{i=1}^N |f * (\varphi_i)_t(y)| \right)^q k_{\beta, \epsilon}(y, t)^q t^{nq} t^{-n-1} dy dt \right)^{1/q}$$

by (4.10)

which implies Theorem 1.2.

5. Proof of Lemma 4.1.

LEMMA 5.1. Let $a > 0$ and let $\mathcal{E} \subset R_+^{n+1}$ be a measurable set that satisfies (4.1) for any $(x, s) \in R_+^{n+1}$. Let $b \geq 1$ and let $K(y, t)$ be a positive measurable function defined on R_+^{n+1} such that

$$(5.1) \quad K(y, t) \leq bK(x, s)$$

for any $(x, s) \in R_+^{n+1}$ and any $(y, t) \in Q'(x, s)$. Let $t_0 > 0$. Let

$$T = Q(0, t_0) \quad \text{or} \quad R_+^{n+1} \setminus Q(0, t_0) \quad \text{or} \quad Q(0, 2t_0) \setminus Q(0, t_0).$$

Then

$$\int \int_{\mathcal{E} \cap T} K(y, t) \, dy \, dt \leq Cba \int \int_T K(y, t) \, dy \, dt,$$

where C is a constant depending only on n .

Proof. Let $\{(y_i, t_i)\}_{i=1}^\infty \subset R_+^{n+1}$ be maximal with respect to the properties

$$(y_i, t_i) \in T \quad \text{and} \quad |y_i - y_j| + |t_i - t_j| \geq 0.01(t_i + t_j) \quad \text{if } i \neq j.$$

Then geometric observation gives

$$(5.2) \quad T \subset \bigcup_{i=1}^\infty Q''(y_i, t_i),$$

$$(5.3) \quad \int \int_{Q''(y_i, t_i)} dy \, dt \leq C \int \int_{T \cap Q''(y_i, t_i)} dy \, dt$$

and

$$(5.4) \quad \left\| \sum_{i=1}^\infty \chi_{Q''(y_i, t_i)} \right\|_{L^\infty} \leq C,$$

where the constant C in (5.3)–(5.4) depends only on n . Then

$$\begin{aligned} & \int \int_{\mathcal{E} \cap T} K(y, t) \, dy \, dt \\ & \leq \sum_{i=1}^\infty \int \int_{\mathcal{E} \cap T \cap Q''(y_i, t_i)} K(y, t) \, dy \, dt \quad \text{by (5.2)} \\ & \leq Cb \sum \int \int_{\mathcal{E} \cap T \cap Q''(y_i, t_i)} dy \, dt \inf_{(x, s) \in Q''(y_i, t_i)} K(x, s) \quad \text{by (5.1)} \\ & \leq Cba \sum \int \int_{T \cap Q''(y_i, t_i)} K(y, t) \, dy \, dt \quad \text{by (4.1) and (5.3)} \\ & \leq Cba \int \int_T K(y, t) \, dy \, dt \quad \text{by (5.4),} \end{aligned}$$

which implies the desired result. \square

We can extend Lemma 5.1 to more general T , but we do not need it in the sequel.

LEMMA 5.2. *Let $0 < \beta < \alpha$ and $0 < \epsilon' < \epsilon < 1$. Let $a > 0$ and let $\mathcal{E} \subset \mathbb{R}_+^{n+1}$ be a measurable set that satisfies (4.1) for any $(x, s) \in \mathbb{R}_+^{n+1}$. Let $(y, t) \in \mathbb{R}_+^{n+1}$. Then*

$$(5.5) \quad \int \int_{\mathcal{E}} k_{\beta, \epsilon}(x, s) s^{-n} k_{\alpha, \epsilon'}\left(\frac{y-x}{s}, \frac{t}{s}\right) dx \frac{ds}{s} \leq C a k_{\beta, \epsilon}(y, t),$$

where $k_{\beta, \epsilon}$ is defined by (1.6) and where C is a constant depending only on $\alpha, \beta, \epsilon, \epsilon'$ and n .

Proof. Put

$u(y, t) =$ the left-hand side of (5.5)

$$= t^\alpha \int \int_{\mathcal{E}} s^{\beta-\epsilon'} (1 + |x| + s)^{-n-\beta-1+\epsilon} (|y-x| + t + s)^{-n-\alpha-1+\epsilon'} dx ds.$$

If $(y, t) \in Q(0, 1)$, then

$$\begin{aligned} u(y, t) &\leq t^\alpha \int \int_{\mathcal{E} \cap Q(0, 2)} s^{\beta-\epsilon'} (|y-x| + t + s)^{-n-\alpha-1+\epsilon'} dx ds \\ &\quad + C t^\alpha \int \int_{\mathcal{E} \cap Q(0, 2)^c} s^{\beta-\epsilon'} (|x| + s)^{-2n-2-\alpha-\beta+\epsilon+\epsilon'} dx ds \\ &\leq C a t^\alpha \int \int_{Q(0, 2)} s^{\beta-\epsilon'} (|y-x| + t + s)^{-n-\alpha-1+\epsilon'} dx ds \\ &\quad + C a t^\alpha \int \int_{\mathbb{R}_+^{n+1} \setminus Q(0, 2)} s^{\beta-\epsilon'} (|x| + s)^{-2n-2-\alpha-\beta+\epsilon+\epsilon'} dx ds \end{aligned}$$

by Lemma 5.1

$$\begin{aligned} &\leq C a t^\alpha \int_0^2 s^{\beta-\epsilon'} (t + s)^{-\alpha-1+\epsilon'} ds \\ &\quad + C a t^\alpha \int_0^{+\infty} s^{\beta-\epsilon'} (1 + s)^{-n-2-\alpha-\beta+\epsilon+\epsilon'} ds \\ &\leq C a t^\beta \quad \text{by } \beta < \alpha. \end{aligned}$$

If $(y, t) \notin Q(0, 1)$ and if $t > |y|$, then

$$\begin{aligned}
u(y, t) &\leq t^\alpha \int \int_{\mathcal{E} \cap Q(0, 2t)} s^{\beta-\varepsilon'} (1 + |x| + s)^{-n-\beta-1+\varepsilon} t^{-n-\alpha-1+\varepsilon'} dx ds \\
&\quad + Ct^\alpha \int \int_{\mathcal{E} \cap Q(0, 2t)^c} s^{\beta-\varepsilon'} (|x| + s)^{-2n-2-\alpha-\beta+\varepsilon+\varepsilon'} dx ds \\
&\leq Cat^{-n-1+\varepsilon'} \int_0^{2t} s^{\beta-\varepsilon'} (1 + s)^{-\beta-1+\varepsilon} ds \\
&\quad + Cat^\alpha \int_0^{+\infty} s^{\beta-\varepsilon'} (t + s)^{-n-2-\alpha-\beta+\varepsilon+\varepsilon'} ds \quad \text{by Lemma 5.1} \\
&\leq Cat^{-n-1+\varepsilon} \quad \text{by } \varepsilon' < \varepsilon.
\end{aligned}$$

If $(y, t) \notin Q(0, 1)$ and if $t \leq |y|$, then

$$\begin{aligned}
u(y, t) &\leq Ct^\alpha \int \int_{\mathcal{E} \cap Q(0, |y|/2)} s^{\beta-\varepsilon'} (1 + |x| + s)^{-n-\beta-1+\varepsilon} |y|^{-n-\alpha-1+\varepsilon'} dx ds \\
&\quad + Ct^\alpha \int \int_{\mathcal{E} \cap (Q(0, 2|y|) \setminus Q(0, |y|/2))} s^{\beta-\varepsilon'} |y|^{-n-\beta-1+\varepsilon} \\
&\quad \quad \quad \times (|y - x| + t + s)^{-n-\alpha-1+\varepsilon'} dx ds \\
&\quad + Ct^\alpha \int \int_{\mathcal{E} \cap Q(0, 2|y|)^c} s^{\beta-\varepsilon'} (|x| + s)^{-2n-2-\alpha-\beta+\varepsilon+\varepsilon'} dx ds \\
&\leq Cat^\alpha |y|^{-n-\alpha-1+\varepsilon'} \int_0^{|y|/2} s^{\beta-\varepsilon'} (1 + s)^{-\beta-1+\varepsilon} ds \\
&\quad + Cat^\alpha |y|^{-n-\beta-1+\varepsilon} \int_0^{2|y|} s^{\beta-\varepsilon'} (t + s)^{-\alpha-1+\varepsilon'} ds \\
&\quad + Cat^\alpha \int_0^{+\infty} s^{\beta-\varepsilon'} (|y| + s)^{-n-2-\alpha-\beta+\varepsilon+\varepsilon'} ds \quad \text{by Lemma 5.1} \\
&\leq Cat^\beta |y|^{-n-\beta-1+\varepsilon} \quad \text{by } \beta < \alpha \text{ and } \varepsilon' < \varepsilon.
\end{aligned}$$

Thus in all cases we get $u(y, t) \leq Cak_{\beta, \varepsilon}(y, t)$. \square

LEMMA 5.3. *Let $\alpha > 0$. Let $\{\varphi_1, \dots, \varphi_N\}$ satisfy the condition (II.α). Then there exist $\psi_1, \dots, \psi_N \in L^2(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ such that*

$$(5.6) \quad \int_0^{+\infty} \sum_{i=1}^N \mathcal{F}\varphi_i(t\xi) \mathcal{F}\psi_i(t\xi) \frac{dt}{t} = 1$$

for any $\xi \neq 0$,

$$(5.7) \quad \text{supp } \mathcal{F}\psi_i \not\equiv 0, \quad i = 1, \dots, N,$$

and that

$$(5.8) \quad |D_x^\gamma \psi_i(x)| \leq C(1 + |x|)^{-n-[\alpha]-2}, \quad i = 1, \dots, N,$$

for any multi-index γ with $l(\gamma) \leq 1$, where C is a constant depending only on α, δ, N and n .

Proof. Let $\epsilon \in (0, 1)$. Let $\theta \in C^\infty(0, +\infty)$ be a nonnegative function such that $\theta(t) = 1$ if $t \in (\epsilon, 1/\epsilon)$ and that $\text{supp } \theta \subset (\epsilon/2, 2/\epsilon)$. By (II.α.1) and (II.α.2) we have $|\mathcal{F}\varphi_i(\xi)| \leq \min(|\xi|, C|\xi|^{-\alpha})$ and $|D_\xi^\gamma \mathcal{F}\varphi_i(\xi)| \leq 1$ if $l(\gamma) = 1$. Hence if $\epsilon > 0$ is small enough depending on α, δ, N and n , then

$$\inf_{|\xi|=1} \int_0^{+\infty} \sum_{i=1}^N |\mathcal{F}\varphi_i(t\xi)|^2 \theta(t) \frac{dt}{t} > c_{\alpha, \delta, N, n} > 0.$$

For this small ϵ take $\psi_1, \dots, \psi_N \in L^2(\mathbb{R}^n)$ so that

$$\mathcal{F}\psi_i(\xi) = \overline{\mathcal{F}\varphi_i(\xi)} \theta(|\xi|) \left\{ \int_0^{+\infty} \sum_{j=1}^N \left| \mathcal{F}\varphi_j \left(\frac{t\xi}{|\xi|} \right) \right|^2 \theta(t) \frac{dt}{t} \right\}^{-1},$$

$i = 1, \dots, N.$

(This is due to [1].) Then (5.6)–(5.7) are easy. (5.8) follows from (II.α.2). \square

REMARK 5.1. Similarly we can show that if φ satisfies (I.α), then there exists $\psi \in L^2(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ satisfying

$$(5.6)' \quad \int_0^{+\infty} \mathcal{F}\varphi(t\xi) \mathcal{F}\psi(t\xi) \frac{dt}{t} = 1 \quad \text{for any } \xi \neq 0,$$

$$(5.7)' \quad \text{supp } \mathcal{F}\psi \neq \emptyset$$

and

$$(5.8)' \quad |D_x^\gamma \psi(x)| \leq C(1 + |x|)^{-n-[\alpha]-2} \quad \text{for any } \gamma \text{ with } l(\gamma) \leq 1,$$

where C is a constant depending only on $\alpha, \mathcal{F}\varphi(0)$ and n .

LEMMA 5.4. Let $\alpha > 0$. Let $\psi \in L^2(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ satisfy (5.7)' and

$$(5.8)'' \quad |D_x^\gamma \psi(x)| \leq (1 + |x|)^{-n-[\alpha]-2} \quad \text{for any } \gamma \text{ with } l(\gamma) \leq 1.$$

Then, there exists a sequence of functions $\{v_i\}_{i=0}^\infty \subset L^\infty(\mathbb{R}^n)$ such that

$$(5.9) \quad \psi(x) = \sum_{i=0}^\infty 2^{-i([\alpha]+2)} (v_i)_2(x),$$

$$(5.10) \quad \text{supp } v_i \subset B(0, 1),$$

$$(5.11) \quad \|v_i\|_{L^\infty} \leq C$$

and

$$(5.12) \quad \int v_i(x)x^\gamma dx = 0$$

for any multi-index γ with $l(\gamma) \leq [\alpha] + 1$, where C is a constant depending only on α and n .

Proof. Let $h \in \mathcal{S}(R^n)$ be a nonnegative function such that

$$\text{supp } h \subset B(0, 1) \setminus B(0, 1/4)$$

and that

$$\sum_{j=1}^{\infty} h(2^{-j}x) = 1 \quad \text{if } |x| > 1.$$

Let $\{\pi_j(x)\}_{j=1}^L$ be an orthonormal basis for the Hilbert space of polynomials of degree $\leq [\alpha] + 1$ with norm

$$\|P\| = \left(\int |P(x)|^2 h(x) dx \right)^{1/2}.$$

Put

$$\begin{aligned} \psi(x) &= \left(1 - \sum_{i=1}^{\infty} h(2^{-i}x) \right) \psi(x) + \sum_{i=1}^{\infty} h(2^{-i}x) \psi(x) \\ &= \theta_0(x) + \sum_{i=1}^{\infty} \theta_i(x) \end{aligned}$$

and

$$\zeta_i(x) = 2^{-in} \sum_{j=1}^L \int \sum_{k=i+1}^{\infty} \theta_k(y) \pi_j(2^{-i}y) dy \pi_j(2^{-i}x) h(2^{-i}x).$$

Note that by (5.8)'' and by $\deg \pi_j \leq [\alpha] + 1$ the above integrands are integrable and that

$$(5.13) \quad \|\zeta_i\|_{L^\infty} \leq C 2^{-i(n+[\alpha]+2)}.$$

Put

$$\begin{aligned} \psi &= \sum_{i=0}^{\infty} \theta_i = (\theta_0 + \zeta_0) + \sum_{i=1}^{\infty} (\theta_i - \zeta_{i-1} + \zeta_i) \\ &= v_0 + \sum_{i=1}^{\infty} 2^{-i([\alpha]+2)} (v_i)_2. \end{aligned}$$

Then (5.10) is clear. (5.11) follows from (5.8)'' and (5.13). Since $\{\sum_{k=i+1}^{\infty} \theta_k - \xi_i\}_{i=0}^{\infty}$ are orthogonal to all the polynomials of degree $\leq [\alpha] + 1$, (5.12) for $i \geq 1$ is easy. Since $\int \psi(x)x^\gamma = 0$ for any γ with $l(\gamma) \leq [\alpha] + 1$ by (5.7)' and (5.8)'', (5.12) holds for the case $i = 0$, too. \square

REMARK 5.2. In a similar way with Lemma 5.4 we can show that if $\beta > 0$, $\kappa \in \mathcal{B}'_\beta(R^n)$ and if $\int \kappa(x) dx = 0$, then $c\kappa \in \mathcal{B}^0_\beta(R^n)$ with $c > 0$ small enough depending only on β and n .

LEMMA 5.5. Let $0 < \nu \leq [\alpha] + 1$ and let $0 < \varepsilon < 1$. Let $\psi \in L^2(R^n) \cap C^\infty(R^n)$ satisfy (5.7)' and (5.8)''. Let $\kappa \in \mathcal{B}'_\nu(R^n)$. Then

$$(5.14) \quad |\kappa * (\psi)_t(y)| \leq Ck_{\nu,\varepsilon}(y, t),$$

where $k_{\nu,\varepsilon}$ is defined by (1.6) and where the constant C depends only on α, ν, ε and n .

Proof. First we assume

$$(5.15) \quad \kappa \in \mathcal{B}^0_\nu(R^n).$$

Lemma 5.4 gives $\{v_i(x)\}_{i=0}^{\infty}$ that satisfy (5.9)–(5.12). Thus

$$(5.16) \quad \begin{aligned} |\kappa * (\psi)_t(y)| &\leq \sum_{i=0}^{\infty} 2^{-i([\alpha]+2)} \left| \int \kappa(y-x)(v_i)_{2^i t}(x) dx \right| \\ &\leq \sum_{i=0}^{\infty} 2^{-i([\alpha]+2)} \inf_{\deg P \leq \nu} \int |\kappa(y-x) - P(x)| |(v_i)_{2^i t}(x)| dx \\ &\leq \sum 2^{-i([\alpha]+2)} (2^i t)^\nu \leq Ct^\nu. \end{aligned} \tag{5.12}$$

If $(y, t) \notin Q(0, 2)$, then

$$(5.17) \quad \begin{aligned} |\kappa * (\psi)_t(y)| &= \left| \int \{(\psi)_t(y-x) - (\psi)_t(y)\} \kappa(x) dx \right| \\ &\leq C \sup_{x \in B(0,1)} |(\psi)_t(y-x) - (\psi)_t(y)| \\ &\leq Ct^{-n-1} (1 + |y|/t)^{-n-[\alpha]-2} \text{ by (5.8)''} \end{aligned} \tag{by } \int \kappa = 0$$

Since $[\alpha] + 1 \geq \nu$, (5.16) and (5.17) imply

$$(5.18) \quad |\kappa * (\psi)_t(y)| \leq Ck_{\nu,0}(y, t).$$

Next we remove the restriction (5.15). Let $\kappa \in \mathcal{B}'_\nu(R^n)$. By its definition κ can be decomposed into the sum $\sum_{j=0}^\infty 2^{-j}(\kappa_j)_{2^j}$ with $\kappa_j \in \mathcal{B}'_\nu(R^n)$. Applying (5.18) with dilation to each κ_j gives

$$(5.19) \quad \begin{aligned} |2^{-j}(\kappa_j)_{2^j} * (\psi)_t(y)| &\leq C2^{-j(n+1)}k_{\nu,0}(y/2^j, t/2^j) \\ &= Ct^\nu(2^j + |y| + t)^{-n-\nu-1}. \end{aligned}$$

Summing up (5.19) with respect to j gives (5.14). □

LEMMA 5.6. *Let $0 < \beta < \alpha$ and $0 < \varepsilon' < \varepsilon < 1$. Let $\{\varphi_1, \dots, \varphi_N\}$ satisfy (II.α). Let $\psi_1, \dots, \psi_N \in L^2(R^n) \cap C^\infty(R^n)$ satisfy (5.6)–(5.8). Let $\kappa \in \mathcal{B}'_\beta(R^n)$. Let $a > 0$ and let $\mathcal{E} \subset R_+^{n+1}$ be a measurable set that satisfies (4.1) for any $(x, s) \in R_+^{n+1}$. Let*

$$(5.20) \quad k_{i,j}(y, t) = \check{\varphi}_i * (\check{\psi}_j)_t(y),$$

$$(5.21) \quad k_i^1(y, t) = \kappa * (\check{\psi}_i)_t(y)$$

and

$$(5.22) \quad k_i^m(y, t) = \int \int_{\mathcal{E}} \sum_{j=1}^N k_j^{m-1}(x, s) s^{-n} k_{j,i} \left(\frac{y-x}{s}, \frac{t}{s} \right) dx \frac{ds}{s},$$

where $i, j \in \{1, \dots, N\}$, $m \in \{2, 3, 4, \dots\}$ and $(y, t) \in R_+^{n+1}$. Then

$$(5.23) \quad |k_{i,j}(y, t)| \leq Ck_{\alpha,\varepsilon'}(y, t),$$

$$(5.24) \quad \varphi_i(-x) = \int \int_{R_+^{n+1}} \sum_{j=1}^N (\varphi_j)_s(z-x) k_{i,j}(z, s) dz \frac{ds}{s},$$

$$(5.25) \quad |k_i^m(y, t)| \leq C(Ca)^{m-1} k_{\beta,\varepsilon}(y, t)$$

and

$$(5.26) \quad \begin{aligned} \kappa(x) &= \int \int_{\mathcal{E}^c} \sum_{i=1}^N (\varphi_i)_t(y-x) \sum_{l=1}^m k_i^l(y, t) dy \frac{dt}{t} \\ &\quad + \int \int_{\mathcal{E}} \sum_{i=1}^N (\varphi_i)_t(y-x) k_i^m(y, t) dy \frac{dt}{t} \end{aligned}$$

for any $i, j \in \{1, \dots, N\}$, any $m \in \{1, 2, 3, \dots\}$, any $x \in R^n$ and any $(y, t) \in R_+^{n+1}$, where $k_{\alpha,\varepsilon'}$ and $k_{\beta,\varepsilon}$ are defined by (1.6) and where C is a constant depending only on $\alpha, \beta, \varepsilon, \varepsilon', \delta, N$ and n .

(5.23) follows directly from Lemma 5.5 (with $\nu = \alpha$). (5.24) follows from (5.6).

Proof of (5.25). We prove this by induction with respect to m . The case $m = 1$ is clear from Lemma 5.5 (with $\nu = \beta$). Suppose that (5.25) holds for some m . Then

$$\begin{aligned} |k_i^{m+1}(y, t)| &\leq \int \int_{\mathcal{E}} \sum_{j=1}^N |k_j^m(x, s)| s^{-n} \left| k_{j,i} \left(\frac{y-x}{s}, \frac{t}{s} \right) \right| dx \frac{ds}{s} \\ &\leq \int \int_{\mathcal{E}} C(Ca)^{m-1} k_{\beta,\varepsilon}(x, s) s^{-n} k_{\alpha,\varepsilon} \left(\frac{y-x}{s}, \frac{t}{s} \right) dx \frac{ds}{s} \\ &\quad \text{by (5.23) and by the hypothesis of induction} \\ &\leq (Ca)^m k_{\beta,\varepsilon}(y, t) \quad \text{by Lemma 5.2.} \end{aligned}$$

Thus the induction is completed. □

Proof of (5.26). We prove this by induction with respect to m . The case $m = 1$ is clear from (5.6). Suppose that (5.26) holds for some m . Note that applying translation and dilation to (5.24) gives

$$(\varphi_i)_t(y-x) = \int \int_{R_+^{n+1}} \sum_{j=1}^N (\varphi_j)_s(z-x) t^{-n} k_{i,j} \left(\frac{z-y}{t}, \frac{s}{t} \right) dz \frac{ds}{s}.$$

Thus, the second term on the right-hand side of (5.26)

$$\begin{aligned} &= \int \int_{\mathcal{E}} \sum_{i=1}^N (\varphi_i)_t(y-x) k_i^m(y, t) dy \frac{dt}{t} \\ &= \int \int_{\mathcal{E}} \sum_{i=1}^N k_i^m(y, t) dy \frac{dt}{t} \\ &\quad \times \int \int_{R_+^{n+1}} \sum_{j=1}^N (\varphi_j)_s(z-x) t^{-n} k_{i,j} \left(\frac{z-y}{t}, \frac{s}{t} \right) dz \frac{ds}{s} \\ &= \int \int_{R_+^{n+1}} \sum_{j=1}^N (\varphi_j)_s(z-x) dz \frac{ds}{s} \\ &\quad \times \int \int_{\mathcal{E}} \sum_{i=1}^N k_i^m(y, t) t^{-n} k_{i,j} \left(\frac{z-y}{t}, \frac{s}{t} \right) dy \frac{dt}{t} \\ &= \int \int_{R_+^{n+1}} \sum_{j=1}^N (\varphi_j)_s(z-x) k_j^{m+1}(z, s) dz \frac{ds}{s}. \end{aligned}$$

Substituting this into (5.26) gives

$$\begin{aligned} \kappa(x) &= \int \int_{\mathcal{E}^c} \sum_{i=1}^N (\varphi_i)_t(y-x) \sum_{l=1}^m k_i^l(y,t) dy \frac{dt}{t} \\ &\quad + \int \int_{R_+^{n+1}} \sum_{i=1}^N (\varphi_i)_t(y-x) k_i^{m+1}(y,t) dy \frac{dt}{t} \\ &= \int \int_{\mathcal{E}^c} \sum_{i=1}^N (\varphi_i)_t(y-x) \sum_{l=1}^{m+1} k_i^l(y,t) dy \frac{dt}{t} \\ &\quad + \int \int_{\mathcal{E}} \sum_{i=1}^N (\varphi_i)_t(y-x) k_i^{m+1}(y,t) dy \frac{dt}{t}. \end{aligned}$$

Thus the induction is completed. □

Proof of Lemma 4.1. The assumptions in Lemma 4.1 give $\alpha, \beta, \varepsilon, \{\varphi_1, \dots, \varphi_N\}, \kappa, a$ and \mathcal{E} . Lemma 5.3 gives $\{\psi_1, \dots, \psi_N\}$. Take $\varepsilon' \in (0, \varepsilon)$. Then, applying Lemma 5.6 to these gives us $\{k_i^m(y, t)\}_{m=1,2,3,\dots; i=1,\dots,N}$ that satisfy (5.25)–(5.26). If $C_{4.1}$ in Lemma 4.1 is small enough, then by (5.25) $\sum_{m=1}^\infty k_i^m(y, t)$ converges everywhere. Put

$$k_i(y, t) = \sum_{m=1}^\infty k_i^m(y, t).$$

Then (4.2) follows from (5.25). Since the second term on the right-hand side of (5.26) goes to zero by (5.25), (4.3) follows from (5.26).

6. Proof of Theorem 1.1. The proof of Theorem 1.1 is very similar to the proof of Theorem 1.2. All we need is the following Lemma 6.1, which corresponds to Lemma 4.1. The way Theorem 1.1 follows from Lemma 6.1 is exactly the same with the way Theorem 1.2 followed from Lemma 4.1. We do not repeat this argument.

LEMMA 6.1. *Let $0 < \beta < \alpha$ and $0 < \varepsilon < 1$. Let φ satisfy the condition (I.α). Let $\kappa \in \mathcal{B}'_\beta(R^n)$. Let $0 < a < C_{6.1}$. Let $\mathcal{E} \subset R_+^{n+1}$ be a measurable set that satisfies (4.1) for any $(x, s) \in R_+^{n+1}$. Then there exists a measurable function $k(y, t)$ defined on R_+^{n+1} such that*

$$(6.1) \quad |k(y, t)| \leq C_{6.2} k_{\beta, \varepsilon}(y, t)$$

for any $(y, t) \in R_+^{n+1}$ and that

$$(6.2) \quad \kappa(x) = \int \int_{\mathcal{E}^c} (\varphi)_t(y-x) k(y, t) dy \frac{dt}{t}$$

for any $x \in R^n$, where $k_{\beta, \varepsilon}$ is defined by (1.6) and where $C_{6.1}$ and $C_{6.2}$ are positive constants depending only on α , $\mathcal{F}\varphi(0)$, β , ε and n .

Proof. By Remark 5.1 we get ψ that satisfies (5.6)'–(5.8)'. Take $\varepsilon' \in (0, \varepsilon)$.

Put

$$L = \int \int_{Q'(0,1)} dy \frac{dt}{t},$$

$$\theta(x) = L^{-1} \int \int_{Q'(0,1)} (\varphi)_t(y-x) dy \frac{dt}{t}$$

and

$$k(y, t) = (\check{\varphi} - \theta) * (\check{\psi})_t(y) + L^{-1} \chi_{Q'(0,1)}(y, t).$$

Since

$$\int \check{\varphi}(x) - \theta(x) dx = 0$$

and since $c(\check{\varphi} - \theta) \in \mathcal{B}'_\alpha(R^n)$ with $c > 0$ small enough, Remark 5.2 implies that $c(\check{\varphi} - \theta) \in \mathcal{B}^{0'}_\alpha(R^n)$ if $c > 0$ is small enough. Applying Lemma 5.5 (with $\nu = \alpha$) gives

$$|(\check{\varphi} - \theta) * (\check{\psi})_t(y)| \leq Ck_{\alpha, \varepsilon'}(y, t).$$

So,

$$(6.3) \quad |k(y, t)| \leq Ck_{\alpha, \varepsilon'}(y, t).$$

By (5.6)' and by the definition of θ we get

$$(6.4) \quad \int \int_{R^{n+1}_+} (\varphi)_t(y-x) k(y, t) dy \frac{dt}{t}$$

$$= (\check{\varphi}(x) - \theta(x)) + \theta(x) = \varphi(-x).$$

Similarly put

$$\eta(x) = \mathcal{F}\kappa(0) \mathcal{F}\varphi(0)^{-1} L^{-1} \int \int_{Q'(0,1)} (\varphi)_t(y-x) dy \frac{dt}{t}$$

and

$$k^1(y, t) = (\kappa - \eta) * (\check{\psi})_t(y) + \mathcal{F}\kappa(0) \mathcal{F}\varphi(0)^{-1} L^{-1} \chi_{Q'(0,1)}(y, t).$$

Since

$$\int \kappa(x) - \eta(x) dx = 0$$

and since $c(\kappa - \eta) \in \mathcal{B}'_\beta(R^n)$ with $c > 0$ small enough (recall Remark 1.1), Remark 5.2 implies that $c(\kappa - \eta) \in \mathcal{B}^{0'}_\beta(R^n)$ if $c > 0$ is small enough. Applying Lemma 5.5 (with $\nu = \beta$) gives

$$|(\kappa - \eta) * (\check{\psi})_t(y)| \leq Ck_{\beta,\varepsilon}(y, t).$$

Thus

$$(6.5) \quad |k^1(y, t)| \leq Ck_{\beta,\varepsilon}(y, t).$$

By (5.6)' and by the definition of η we get

$$(6.6) \quad \int \int_{\mathbb{R}^{n+1}_+} (\varphi)_t(y - x)k^1(y, t) dy \frac{dt}{t} \\ = (\kappa(x) - \eta(x)) + \eta(x) = \kappa(x).$$

(6.3) and (6.4) correspond to (5.23) and (5.24). (6.5) and (6.6) correspond to (5.25) and (5.26) of the case $m = 1$. Then, the rest is the same with the preceding section.

Inductively we define

$$k^m(y, t) = \int \int_{\mathcal{G}} k^{m-1}(x, s)s^{-n}k\left(\frac{y-x}{s}, \frac{t}{s}\right) dx \frac{ds}{s}$$

for $m = 2, 3, 4, \dots$, which corresponds to (5.22). Then by the same argument with the preceding section we get

$$(6.7) \quad |k^m(y, t)| \leq C(Ca)^{m-1}k_{\beta,\varepsilon}(y, t)$$

and

$$(6.8) \quad \kappa(x) = \int \int_{\mathcal{G}^c} (\varphi)_t(y - x) \sum_{l=1}^m k^l(y, t) dy \frac{dt}{t} \\ + \int \int_{\mathcal{G}} (\varphi)_t(y - x)k^m(y, t) dy \frac{dt}{t}$$

which correspond to (5.25)–(5.26). Put

$$k(y, t) = \sum_{m=1}^\infty k^m(y, t).$$

If $C_{6.1}$ is small enough, then by (6.7) this converges everywhere and satisfies (6.1). Letting $m \rightarrow \infty$ in (6.8) gives (6.2). □

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