

# Pacific Journal of Mathematics

**ON BANACH SPACES HAVING A RADON-NIKODÝM DUAL**

C. DEBIÈVE

## ON BANACH SPACES HAVING A RADON-NIKODYM DUAL

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The purpose of this paper is to prove a new characterisation of Banach spaces having a Radon-Nikodym dual, namely that if  $E$  is a Banach space, then  $E'$  has the Radon-Nikodym property if and only if there exists an equivalent norm on  $E$  such that for each  $E$ -valued measure  $m$  of bounded variation, there exists an  $E'$ -valued function  $f$  with norm 1  $|m|$ -a.e. such that  $|m| = f.m.$

**1. Introduction.** In [1], we have proved that if  $E$  is a Banach space,  $m$  an  $E$ -valued measure defined on a  $\sigma$ -algebra  $\mathcal{A}$  of subsets of a set  $T$ , with bounded variation  $|m|$ , and if  $\varepsilon$  is any positive number, then there exists an  $E'$ -valued strongly measurable function  $f$  defined on the set  $T$ , such that  $\|f\| < 1 + \varepsilon$  and  $|m|(A) = \int_A f dm$  for each  $A$  in  $\mathcal{A}$ .

A very natural question which arises is the following: Does there always exist an  $E'$ -valued strongly measurable function with norm 1 such that  $|m|(A) = \int_A f dm$  for each  $A$  in  $\mathcal{A}$ ? Following the example given in [1], this seems to be possible.

Finally, an answer to that question was provided by F. Delbaen who proved the following unpublished theorem: If  $E$  is a Banach space, the following are equivalent:

(a)  $E'$  has the Radon-Nikodym property

(b) For each equivalent norm on  $E$ , for each  $E$ -valued measure  $m$  of bounded variation defined on a  $\sigma$ -algebra  $\mathcal{A}$  of subsets of a set  $T$ , there exists a  $|m|$ -strongly measurable function  $f$  from  $T$  to  $E'$  such that  $\|f\| = 1$   $|m|$ -a.e. and  $|m|(A) = \int_A f dm$  for each  $A$  in  $\mathcal{A}$ .

The purpose of this paper is to provide a positive answer to the following question: Is it possible to weaken assertion (b) by requiring the existence of an equivalent norm on the space having the property instead of assuming it for each equivalent norm on  $E$ .

**2. Proof of the theorem.** Before proving our theorem let us recall the Mazur density theorem and prove two lemmas.

**THEOREM** (*Mazur density theorem* [5] p. 171). *If  $E$  is a separable Banach space, then for each equivalent norm on  $E$ , the set of smooth points of the unit sphere of  $E$  is dense in the unit sphere.*

**LEMMA 1.** *Let  $E$  be a Banach space such that  $E'$  is not separable,  $B$  a dense subset of  $S(E) = \{x | x \in E, \|x\| = 1\}$  and  $\varepsilon > 0$ . If we denote by  $\Omega$  the first uncountable ordinal and by  $S$  the set  $\{i | i < \Omega\}$ , then for each  $i$  in  $S$ , there exists  $x_i$  in  $B$  and  $x'_i$  in  $S(E')$ , the unit sphere of  $E'$  such that  $x'_i(x_i) = 1$  and  $\|x'_i - x'_j\| > 1 - \varepsilon$  if  $i \neq j$ .*

*Proof.* Let  $i$  in  $S$  and suppose that the families  $(x_j)$  and  $(x'_j)$  are chosen for  $j < i$ .

As  $E'$  is not separable,  $\bigcap_{j < i} \text{Ker } x'_j \neq \{0\}$ .

Let  $x \in S(E) \cap \bigcap_{j < i} \text{Ker } x'_j$  and choose  $x_i$  in  $B$  such that  $\|x - x_i\| < \varepsilon$ .

Now, if we choose  $x'_i$  in  $S(E')$  such that  $x'_i(x_i) = 1$  it is easy to see that we are done.

$\|x'_i - x'_j\| > 1 - \varepsilon$  follows from the fact that if  $j < i$ ,  $(x'_i - x'_j)(x_i) > 1 - \varepsilon$ .

**LEMMA 2.** *For the same set  $S$  as in Lemma 2, there exists a positive scalar measure  $\mu$  on the  $\sigma$ -algebra  $\mathcal{P}(S)$  of the subsets of  $S$  such that  $\mu(S) = 1$  and  $\mu(A) = 0$  if  $A$  is countable.*

*Proof.* Let  $i$  in  $S$  and define  $\mu_i$  as the evaluation measure at the point  $i$ . As the set of measures on the  $\sigma$ -algebra of the subsets of  $S$  is the dual of the space of continuous bounded functions on  $S$  for a locally convex topology, the family of measures has a cluster point which is a measure satisfying our requirement.

We are now ready for the proof of the following

**THEOREM.** *For any Banach space  $E$ , the following are equivalent:*

(1)  *$E'$  has the Radon-Nikodym property.*

(2) *For each equivalent norm on  $E$ , for each  $E$ -valued measure  $m$  of bounded variation defined on a  $\sigma$ -algebra  $\mathcal{A}$  of subsets of a set  $T$ , there exists a function  $f$  from  $T$  into  $E'$   $|m|$ -strongly measurable such that  $\|f(t)\| = 1$   $|m|$ -a.e. and  $|m|(A) = \int_A f \, dm$  for each  $A$  in  $\mathcal{A}$ .*

(3) *There exists an equivalent norm on  $E$  such that for each  $E$ -valued measure  $m$  of bounded variation defined on a  $\sigma$ -algebra  $\mathcal{A}$  of subsets of a set  $T$ , there exists a function  $f$  from  $T$  into  $E'$   $|m|$ -strongly measurable such that  $\|f(t)\| = 1$   $|m|$ -a.e. and  $|m|(A) = \int_A f \, dm$  for each  $A$  in  $\mathcal{A}$ .*

*Proof.* (1)  $\Rightarrow$  (2) It follows from the theorem we proved in [1] that for each integer  $n$ , there exists a function  $f_n$  from  $T$  into  $E'$  such that  $f_n$  is  $|m|$ -strongly measurable,  $1 \leq \|f_n(t)\| < 1 + 1/n$  and  $|m|(A) = \int_A f_n dm$  for each  $A$  in  $\mathcal{A}$ .

Let  $G$  be the Banach subspace of  $E'$  generated by  $\bigcup_{n=1}^{\infty} f_n(T)$ .

As  $G$  is separable and  $E'$  has Radon-Nikodym property, there exists a Banach space  $F$  such that  $F'$  is separable and  $G \subseteq F'$  ([3]). Let  $f$  be a pointwise  $\sigma(F', F)$ -cluster point of the sequence  $(f_n)$ .  $f$  is  $G$ -valued, thus  $E'$ -valued.

It is clear that  $\|f\| \leq 1$  and that  $f$  is  $\sigma(F', F)$ -measurable. As  $|m|(A) = \int_A f dm$  for each  $A$  in  $\mathcal{A}$ , if we prove that  $f$  is strongly measurable, the norm of  $f$  will be greater than 1 and our assertion will be proved.

Let  $m_0$  from  $\mathcal{A}$  into  $F'$  defined by  $m_0(A)(y) = \int_A \langle f, y \rangle d|m|$ .

It is clear that  $m_0$  is a measure with finite variation and that  $|m_0| = |m|$ .

As  $F$  has the Radon-Nikodym property, there exists a measurable function  $g$  from  $T$  into  $F'$  such that  $m_0(A) = \int_A g d|m|$  for each  $A$  in  $\mathcal{A}$ .

It follows that if  $y \in F$ ,  $m_0(A)(y) = \int_A \langle g, y \rangle d|m|$  which shows that  $\langle g, y \rangle = \langle f, y \rangle, |m|$ -a.e. for each  $y$  in  $F$ .

As  $F$  is separable, it follows that  $f = g$   $|m|$ -a.e. and that  $f$  is strongly measurable which proves the first assertion.

As (2)  $\Rightarrow$  (3) is obvious, it remains to show that

(3)  $\Rightarrow$  (1) It is easy to prove that if property (3) is satisfied for  $E$  it is also satisfied for each Banach subspace of  $E$ . Now as we have to prove that each separable subspace of  $E$  has a separable dual, we only have to prove that if a separable Banach space satisfies (3), it has a separable dual.

Let us suppose that there exists a separable Banach space  $E$  satisfying property (3) and such that  $E'$  is not separable. Let  $B$  be the set of smooth points of the unit sphere  $S(E)$  of  $E$  which is dense in  $S(E)$  by Mazur density theorem,  $\varepsilon = 1/4$  and apply Lemma 1.

We define the function  $f$  from  $S$  to  $E$  by  $f(i) = x_i$ . If  $\mathcal{A}$  is defined as the set of inverse images by  $f$  of the open subsets of  $S(E)$ , the function  $f$  is strongly measurable. Let us choose on  $\mathcal{A}$  a positive scalar measure  $\mu$  such that  $\mu(S) = 1$  and  $\mu(A) = 0$  if  $A$  is countable. Such a  $\mu$  exists by Lemma 2. Now we define  $m$  from  $\mathcal{A}$  to  $E$  by  $m(A) = \int_A f d\mu$ .

$m$  is clearly a measure of bounded variation and  $|m| = \mu$ . So there exists a function  $g$  from  $S$  into  $E'$  which is  $\mu$ -strongly measurable,  $\|g\| = 1$   $\mu$ -a.e. and  $\mu(A) = \int_A g dm$  for each  $A$  in  $\mathcal{A}$ .

It follows that  $\mu(A) = \int_A \langle f, g \rangle d\mu$  for each  $A$  in  $\mathcal{A}$  and that  $\langle f, g \rangle = 1$   $\mu$ -a.e.

So there exists a  $\mu$ -negligible subset  $N$  of  $S$  such that  $g(i)(f(i)) = 1$  if  $i \notin N$  and  $g(S \setminus N)$  is separable. If  $i \notin N$ ,  $g(i)(f(i)) = g(i)(x_i) = 1$ .

As  $x_i$  is a smooth point and  $\|g(i)\| = 1$ ,  $g(i) = x_i$ ,

It follows that  $\|g(i) - g(j)\| \geq 1 - \varepsilon = 3/4$  for  $i \neq j$  in  $S \setminus N$  which shows that  $g(S \setminus N)$  is discrete.

As it is separable, it has to be countable. So  $S \setminus N$  has to be countable which is impossible.

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