Pacific Journal of Mathematics

SOME ROGERS-RAMANUJAN TYPE PARTITION THEOREMS

M. V. Subba Rao

Vol. 120, No. 2 October 1985

PACIFIC JOURNAL OF MATHEMATICS
Vol. 120, No 2, 1985

SOME ROGERS-RAMANUJAN TYPE PARTITION THEOREMS

M. V. SUBBARAO

Analogous to the celebrated Rogers-Ramanujan partition theorems, we obtain four partition theorems wherein the minimal difference for 'about the first half' of the parts of a partition (arranged in non-increasing order of magnitude) is 2. For example, we prove that the number of partitions of n, such that the minimal difference of the 'first half of the summands' (that is, first $[(t+1)/2]$ summands in a partition into t summands) of any partition is 2, equals the number of partitions n into summands congruent to $\pm 1, \pm 2, \pm 5, \pm 6, \pm 8, \pm 9 \pmod{20}$.

1. Introduction. Throughout this paper, $|x| < 1$ and we use the notation:

$$(x)_n = (1-x)(1-x^2)\cdots(1-x^n), \qquad n = 1,2,\ldots;$$

$$\phi(a,x) = (1-a)(1-ax)(1-ax^2)\cdots \quad \text{to } \infty.$$

$$\phi(x) = \phi(x,x).$$

$\pi_t(n)$ denotes a partition of n into t parts arranged in non-increasing order of magnitude, say,

$$(1.1) \qquad \pi_t(n) = n_1 + n_2 + \cdots + n_t; \qquad n_1 \geq n_2 \geq \cdots \geq n_t.$$

For convenience, we shall refer to n_1, $n_2, \ldots$ as the first, second,... part in $\pi_t(n)$. The differences $n_1 - n_2$, $n_2 - n_3, \ldots$ are referred to as the first, second,... differences of the parts of $\pi_t(n)$. The "first half of the parts in $\pi_t(n)$" are defined to be the parts $n_1, n_2, \ldots, n_{[(t+1)/2]}$, where $[x]$ denotes the greatest integer function. Thus these parts are

$$n_1, n_2, \ldots, n_{t/2} \quad \text{if } t \text{ is even,}$$

and

$$n_1, n_2, \ldots, n_{(t+1)/2} \quad \text{if } t \text{ is odd.}$$

These are also described as the parts in the first half of the partition. We shall also have occasion to speak frequently about the minimal differences of the first half of the parts in $\pi_t(n)$. This will be the minimum of the differences

$$n_1 - n_2, n_2 - n_3, \ldots, n_{[(t-1)/2]} - n_{[(t+1)/2]},$$

the last of these being "the last difference in the first half of the summands of $\pi_t(n)$".

The celebrated Rogers-Ramanujan identities have a well-known partition-theoretic interpretation first noticed by MacMahon and can be stated thus ([**1**], Theorems 364, 365):

(1.2) The number of partitions of n into parts with minimal difference 2 equals the number of partitions of n into parts which are $\equiv \pm 1 \pmod 5$.

(1.3) The number of partitions of n with minimal part 2 and minimal difference 2 equals the number of partitions of n into parts which are $\equiv \pm 2 \pmod 5$.

In this paper, we obtain analogous theorems for partitions wherein the minimal difference of the parts in the *first half* of the partition is 2 (with one possible exception in the cases of some theorems).

For establishing our theorems, we utilize some identities of Slater [**3**]. Four of these identities were earlier used by Hirschhorn [**2**] to establish entirely different combinatorial results. However the partition theorems that we obtain here from these four identities are much more analogous in structure to the Rogers-Ramanujan partitions than those obtained by Hirschhorn.

2. The partition theorems.

2.1. THEOREM. *The number of partitions of n such that the parts in the first half of each partition have minimal difference* 2 *is equal to the number of partitions of n into parts which are congruent to* $\pm 1, \pm 2, \pm 5, \pm 6, \pm 8, \pm 9 \pmod{20}$.

2.2. THEOREM. *The number of partitions of n such that the parts in the first half of each partition have minimal difference* 2*—with the possible exception of the last difference which is at least* 1*—is equal to the number of partitions of n into parts which are congruent to* $\pm 1, \pm 3, \pm 4, \pm 5, \pm 7, \pm 9 \pmod{20}$.

2.3. THEOREM. *The number of partitions of n into parts not less than* 2 *and such that in any partition into t parts, the first* $[t/2]$ *parts have minimal size* $[t/2]+1$ *or* $[t/2]+3$ *according as t is even or odd and minimal difference* 2, *equals the number of partitions of n into parts which are congruent to* $\pm 2, \pm 3, \pm 4, \pm 5 \pmod{16}$. *Equivalently, the number of partitions of n with* $n = a_1 + \cdots + a_{2s-1}$, *where* $a_1 - a_2 \geq 2, \ldots, a_{s-2} - a_{s-1} \geq 2$, $a_{s-1} \geq s+2$, $a_s \geq a_{s+1} \geq \cdots \geq a_{2s-1} \geq 2$, *or with* $n = a_1 + \cdots + a_{2s}$ *with* $a_1 - a_2 \geq 2, \ldots, a_{s-1} - a_s \geq 2$, $a_s \geq s+1$, $a_{s+1} \geq a_{s+2} \geq \cdots \geq a_{2s} \geq 2$ = *number of partitions of n with parts congruent to* $\pm 2, \pm 3, \pm 4, \pm 5 \pmod{16}$.

2.4. THEOREM. *The number of partitions of n into parts not less than* 2 *and such that for the first* $[t/2]$ *parts of a partition of n into t parts, the minimum difference is* 2 *and minimum part* $[(t+1)/2]+1$, *and further, if t is odd, the middle part* (*i.e. the* $(t+1)/2$*-th*) *is at least* $(t+1)/2$, *equals the numbers of partitions of n into parts congruent to* $\pm 1, \pm 4, \pm 6, \pm 7$ (mod 16).

3. Proofs of the theorems.

3.1. *Proof of Theorem* 2.1. Consider a partition $\pi_t(n)$, satisfying the conditions on the differences of parts stated in the first part of the theorem. Let

$$\pi_t(n) = n_1 + n_2 + \cdots + n_t,$$

the parts being in non-increasing order of magnitude.

Case 1. t even $= 2s$. Then we have

$$1 \le n_{2s} \le n_{2s-1} \le n_{2s-2} \le \cdots \le n_{s+1} \le n_s;$$

further

$$n_{s-1} \ge n_s + 2, \quad n_{s-2} \ge n_{s-1} + 2, \quad n_2 \ge n_3 + 2, \quad n_1 \ge n_2 + 3.$$

Hence

$$n_s \ge 1, \quad n_{s-1} \ge 3, \quad n_{s-2} \ge 5, \ldots, n_1 \ge 2s - 1.$$

It follows that

$$n_1 + n_2 + \cdots + n_s \ge ((2s-1) + (2s-3) + \cdots + 1) = s^2$$

and

$$n_{s+1} + n_{s+2} + \cdots + n_{2s} \ge 1 + 1 + \cdots + 1 = s.$$

Thus

$$n - (s^2 + s) = (n_1 - (2s-1)) + (n_2 - (2s-3)) + \cdots + (n_s - 1) \\ + (n_{s+1} - 1) + (n_{s+2} - 1) + \cdots + (n_{2s} - 1),$$

which represents a partition of $n - (s^2 + s)$ into at most $2s$ parts. Thus the partitions of the type $\pi_{2s}(n)$ with the stated restrictions on differences of parts are generated by $x^{s^2+s}/(x)_{2s}$ $(s = 1, 2, \ldots)$.

Case 2. t odd $= 2s - 1$. A typical partition $\pi_{2s-1}(n)$ is of the form

$$n = n_1 + n_2 + \cdots + n_{2s-1},$$

with

$$n_1 \geq n_2 + 2, \qquad n_2 \geq n_3 + 2,$$
$$\cdots$$
$$n_{s-1} \geq n_s + 2, \qquad n_s \geq n_{s+1}, \qquad n_{s+1} \geq n_{s+2},$$
$$\cdots$$
$$n_{2s-2} \geq n_{2s-1}, \qquad n_{2s-1} \geq 1.$$

Hence we have that each of

$$n_{s+1}, n_{s+2}, \ldots, n_{2s-1} \text{ is } \geq 1,$$

while

$$n_s \geq 1, \quad n_{s-1} \geq 3, \quad n_{s-2} \geq 5, \ldots, n_2 \geq 2s-3, \quad n_1 \geq 2s-1.$$

Hence

$$\begin{aligned} n - (s^2 + s - 1) &= (n_1 + \cdots + n_{2s-1}) - ((2s-1) + (2s-3) + \cdots + 1) - s \\ &= (n_1 - (2s-1)) + (n_2 - (2s-3)) + \cdots + (n_s - 1) \\ &\quad + (n_{s+1} - 1) + \cdots + (n_{2s-1} - 1), \end{aligned}$$

and this represents a partition of $n - (s^2 + s - 1)$ into at most $2s - 1$ parts. Thus the partitions of the type $\pi_{2s-1}(n)$ with the stated restrictions on the parts are generated by $x^{s^2+s-+1}/(x)_{2s-1}$ $(s = 1, 2, \ldots)$. Noting that

$$1 + \sum_{s=1}^{\infty} \left\{ \frac{x^{s^2+s-1}}{(x)_{2s-1}} + \frac{x^{s^2+s}}{(x)_{2s}} \right\} = \sum_{s=0}^{\infty} \frac{x^{s^2+s}}{(x)_{2s+1}},$$

we get Theorem 2.1 in view of the following identity of L. J. Slater ([**3**], 94):

$$\begin{aligned} \sum_{r=0}^{\infty} \frac{x^{r^2+r}}{(x)_{2r+1}} = \{ &\phi(x, x^{20}) \phi(x^2, x^{20}) \cdot \phi(x^5, x^{20}) \cdot \phi(x^6, x^{20}) \\ &\cdot \phi(x^8, x^{20}) \cdot \phi(x^9, x^{20}) \cdot \phi(x^{11}, x^{20}) \cdot \phi(x^{12}, x^{20}) \\ &\cdot \phi(x^{14}, x^{20}) \cdot \phi(x^{15}, x^{20}) \cdot \phi(x^{18}, x^{20}) \cdot \phi(x^{19}, x^{20}) \}^{-1}. \end{aligned}$$

3.2. *Indication of the proof of the other theorems.* As suggested by the referee, we omit proofs of the other theorems, since they are analogous to that of Theorem 2.1. For Theorems 2.2, 2.3, 2.4, we utilize, respectively, the identities 79, 98, 39, 83 and 38, 86 of Slater [**3**]. We only observe that if

$$a_1 - a_2 \geq 2, \ldots, a_{s-2} - a_{s-1} \geq 2, \qquad a_{s-1} \geq s + 2,$$
$$a_s \geq a_{s+1} \geq \cdots \geq a_{2s-1} \geq 2,$$

then

$$a_1 + \cdots + a_{2s-1} \geq 2s^2;$$

while if

$$a_1 - a_2 \geq 2, \ldots, a_{s-1} - a_s \geq 2, \qquad a_s \geq s + 1,$$
$$a_{s+1} \geq a_{s+2} \geq \cdots \geq a_{2s} \geq 2,$$

then

$$a_1 + \cdots + a_{2s} \geq 2s^2 + 2s.$$

The interested reader can no doubt supply the details, or obtain them from the author.

The author thanks the referee for his helpful comments.

References

[1] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Clarendon Press, Oxford (4th Edition).

[2] M. N. Hirschhorn, *Some partition theorems of the Rogers-Ramanujan type*, J. Combinatorial Analysis, **A27** (1979), 33–37.

[3] L. J. Slater, *Further identities of the Rogers-Ramanujan type*, Proc. London Math. Soc., **54** (1951–52), 147–167.

Received October 1, 1981 and in revised form August 23, 1984. This research is partly supported by a Canadian National Research Council Grant A-3103.

UNIVERSITY OF ALBERTA
EDMONTON, ALBERTA, CANADA
T6G 2G1

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph must be capable of being used separately as a synopsis of the entire paper. In particular it should contain no bibliographic references. Please propose a heading for the odd numbered pages of less than 35 characters. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. 39. Supply name and address of author to whom proofs should be sent. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California 90024.

There are page-charges associated with articles appearing in the Pacific Journal of Mathematics. These charges are expected to be paid by the author's University, Government Agency or Company. If the author or authors do not have access to such Institutional support these charges are waived. Single authors will receive 50 free reprints; joint authors will receive a total of 100 free reprints. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: $190.00 a year (5 Vols., 10 issues). Special rate: $66.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Old back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

The Pacific Journal of Mathematics at P.O. Box 969, Carmel Valley, CA 93924 (ISSN 0030-8730) publishes 5 volumes per year. Application to mail at Second-class postage rates is pending at Carmel Valley, California, and additional mailing offices. Postmaster: Send address changes to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Pacific Journal of Mathematics

Vol. 120, No. 2 October, 1985

Philip Marshall Anselone and Mike Treuden, Regular operator approximation theory 257

Giuseppe Baccella, Semiprime ℵ-QF3 rings 269

Earl Robert Berkson and Thomas Alastair Gillespie, The generalized M. Riesz theorem and transference 279

Joachim Boidol, A Galois-correspondence for general locally compact groups 289

Joseph Eugene D'Atri, Josef Dorfmeister and Yan Da Zhao, The isotropy representation for homogeneous Siegel domains 295

C. Debiève, On Banach spaces having a Radon-Nikodým dual 327

Michael Aaron Freedman, Existence of strong solutions to singular nonlinear evolution equations 331

Francisco Jose Freniche, Grothendieck locally convex spaces of continuous vector valued functions 345

Hans-Peter Künzi and Peter Fletcher, Extension properties induced by complete quasi-uniformities 357

Takaŝi Kusano, Charles Andrew Swanson and Hiroyuki Usami, Pairs of positive solutions of quasilinear elliptic equations in exterior domains . . . 385

Angel Rafael Larotonda and Ignacio Zalduendo, Spectral sets as Banach manifolds 401

J. Martínez-Maurica and C. Pérez García, A new approach to the Kreĭn-Milman theorem 417

Christian Pommerenke, On the boundary continuity of conformal maps 423

M. V. Subba Rao, Some Rogers-Ramanujan type partition theorems 431

Stephen Edwin Wilson, Bicontactual regular maps 437

Jaap C. S. P. van der Woude, Characterizations of (H)PI extensions 453

Kichoon Yang, Deformation of submanifolds of real projective space 469

Subhashis Nag, Errata: "On the holomorphy of maps from a complex to a real manifold" 493