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**$\omega$ -ELONGATIONS AND CRAWLEY'S PROBLEM**

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$\omega$ -elongations of  $Z(p)$  by separable  $p$ -primary groups are studied. Assuming  $(V = L)$ , direct sums of cyclic groups are characterized using  $\omega$ -elongations. Also assuming  $(V = L)$  much information is obtained about  $\omega$ -elongations of  $Z(p)$  by groups which are not direct sums of cyclic groups. Finally it is shown that it is consistent that there is an uncountable group  $B$  with a countable basic subgroup such that there is a unique  $\omega$ -elongation of  $Z(p)$  of  $B$ .

In this paper all groups are  $p$ -primary Abelian groups. Suppose  $B$  is a separable group; i.e.  $p^\omega B = 0$ . Here  $p^n B = \{p^n x: x \in B\}$  and  $p^\omega B = \bigcap p^n B$ . A group  $H$  is said to be an  $\omega$ -elongation of  $A$  by  $B$  if  $A \simeq p^\omega H$  and  $B \simeq H/p^\omega H$ . In this paper we will study whether or not the  $\Sigma$ -cyclic groups (i.e. direct sum of cyclic groups) can be characterized by their elongations of  $Z(p)$ . Our main theorem is:

**THEOREM 2.2.** *Assume  $(V = L)$ . A group  $G$  is  $\Sigma$ -cyclic iff for every  $\omega$ -elongation  $H$  of  $Z(p)$  by  $G$  there is a homomorphism  $f$  from  $H$  to  $P$  such that  $f(p^\omega H) \neq 0$ . Here  $P$  is the Prüfer group generated by  $\{x_n: n < \omega\}$  subject to the relations  $px_0 = 0$  and  $p^{n+1}x_{n+1} = x_0$ . If we assume  $\text{MA} + \neg\text{CH}$  this criterion fails to characterize the  $\Sigma$ -cyclic groups.*

Following Megibben [M] we call a group  $B$  a *Crawley group*, if all elongations of  $Z(p)$  by  $B$  are isomorphic (as groups). Crawley asked if all Crawley groups were  $\Sigma$ -cyclic. Megibben [M] showed, assuming  $\text{MA} + \neg\text{Ch}$ , there is a Crawley group which is not  $\Sigma$ -cyclic. Further he showed, assuming  $(V = L)$ , any Crawley group of cardinality  $\aleph_1$  must be  $\Sigma$ -cyclic. Megibben's proof is somewhat indirect. In particular, the proof using  $(V = L)$  involves valuated vector spaces. On aesthetic grounds it seems worthwhile to give a strictly group theoretic proof. In fact we get additional information on the structure of  $\omega$ -elongations of  $Z(p)$  by groups of cardinality  $\aleph_1$  in  $L$ . The following result speaks of "rigid" systems of  $\omega$ -elongations.

**THEOREM 1.3.** *Assume  $(V = L)$ . Suppose  $B$  is a separable group of cardinality  $\aleph_1$  and  $B$  is not  $\Sigma$ -cyclic. There are  $2^{\aleph_1}$   $\omega$ -elongations  $\{H_\alpha: \alpha < 2^{\aleph_1}\}$  of  $Z(p)$  by  $B$  such that: for  $\alpha \neq \beta$  if  $f: H_\alpha \rightarrow H_\beta$  and  $t \in p^\omega H_\alpha$  then  $f(t) = 0$ .*

Also in §1 we give a proof of Megibben’s result that if  $MA + \neg CH$  holds then any  $\omega_1$ -separable group of cardinality  $\aleph_1$  is a Crawley group. This proof is a run up for the more difficult result in §3:

**THEOREM 3.2.** *It is consistent that there is a Crawley group of cardinality  $\aleph_1$  which is not  $\aleph_1$ -separable.*

Finally in §2, we use our methods to obtain additional information on Crawley groups in  $L$ . Perhaps the most interesting result is Corollary 2.3 which asserts if  $V = L$  then any strongly Crawley group is  $\Sigma$ -cyclic. Here  $G$  is *strongly* Crawley if for all  $\Sigma$ -cyclic groups  $H$ ,  $G \oplus H$  is a Crawley group.

Eklof and Huber [EH] have studied  $\omega$ -filtered vector spaces of cardinality  $\aleph_1$ . Eklof has found  $\omega$ -filtered vector space versions of many of our theorems. We wish to thank him for his helpful comments on the first draft of this paper.

**1. Crawley groups.** We begin with a group theoretic lemma.

1.1. **LEMMA.** *Suppose  $Z(p) \twoheadrightarrow H \xrightarrow{\pi} B$  is an  $\omega$ -elongation. Further assume  $B \subseteq B'$ ,  $B'$  is separable,  $|B'/B| = \aleph_0$  and  $B'/B$  is not  $\Sigma$ -cyclic. Let  $t$  generate  $p^\omega H$ . There are  $\omega$ -elongations  $Z(p) \twoheadrightarrow H_1 \xrightarrow{\pi_1} B'$  and  $Z(p) \twoheadrightarrow H_2 \xrightarrow{\pi_2} B'$  so that:  $\pi \subseteq \pi_1, \pi_2$ , and for all  $f: H \rightarrow G$ , if  $f(t) \neq 0$  and  $p^\omega G \cong Z(p)$  then  $f$  extends to at most one of  $H_1$  and  $H_2$ .*

*Proof.* Let  $Z(p) \twoheadrightarrow H_1 \xrightarrow{\pi_1} B'$  be any  $\omega$ -elongation such that  $\pi \subseteq \pi_1$ . Since  $|B'/B| = \aleph_0$ ,  $B'/B$  is not separable. Hence there are  $\{x_n: n < \omega\} \subseteq B' \setminus B$  and  $\{b_n: n < \omega\} \subseteq B$  so that  $px_0 = b_0$  and  $p^{n+1}x_{n+1} = x_0 + b_{n+1}$ . Choose preimages  $x_n^1, b_n^1$  ( $n < \omega$ ) of the  $x_n$ ’s and  $b_n$ ’s so that  $px_0^1 = b_0^1$  and  $p^{n+1}x_{n+1}^1 = x_0^1 + b_{n+1}^1$ . There is another  $\omega$ -elongation  $Z(p) \twoheadrightarrow H_2 \xrightarrow{\pi_2} B'$  such that:  $\pi \subseteq \pi_2$ ; and the  $x_n$ ’s have pre-images  $x_n^2$  where  $px_0^2 = b_0^1 + t$  and  $p^{n+1}x_{n+1}^2 = x_0^2 + b_{n+1}^1$ . (The existence of this second elongation can be established by direct computation. An elegant proof of a more general result can be found in [EHM].)

Suppose  $f: H \rightarrow G$  and  $G$  as above and  $f$  extends to both  $H_1$  and  $H_2$ . Then in  $G$  there are  $\{y_n: n < \omega\}$  and  $\{y'_n: n < \omega\}$  such that:  $p^{n+1}y_{n+1} = y_0 + f(b_{n+1}^1)$ ;  $p^{n+1}y'_{n+1} = y'_0 + f(b_{n+1}^1)$ ;  $py_0 = f(b_0^1)$ ; and  $py_0^1 = f(b_0^1) + f(t)$ . Subtracting we have  $p(y'_0 - y_0) = f(t)$  and  $p^{n+1}(y'_{n+1} - y_{n+1}) = (y'_0 - y_0)$ . So  $(y'_0 - y_0) \in p^\omega G$ , but  $p(y_0 - y_0) \neq 0$ . This is a contradiction.

1.2. THEOREM. ([M]). Assume  $(V = L)$ . Every Crawley group of cardinality  $\aleph_1$  is  $\Sigma$ -cyclic.

*Proof.* Assume  $B$  is a non- $\Sigma$ -cyclic Crawley group and  $|B| = \omega_1$ . Let  $Z(p) \twoheadrightarrow H \twoheadrightarrow B$  be any  $\omega$ -elongation of  $Z(p)$  by  $B$ . Let  $\{B_\alpha: \alpha < \omega_1\}$  be an  $\omega_1$ -filtration of  $B$ ; i.e. for all  $\alpha$ ,  $|B_\alpha| = \omega$ ; if  $\lambda$  is a limit ordinal,  $B_\lambda = \bigcup_{\alpha < \lambda} B_\alpha$ ;  $B = \bigcup B_\alpha$ ; and for all  $\alpha$ ,  $B_{\alpha+1}/B_\alpha$  is  $\Sigma$ -cyclic iff for all  $\tau > \alpha$   $B_\tau/B_\alpha$  is  $\Sigma$ -cyclic. We can assume  $B_0$  has infinite final rank (i.e. for all  $n$ ,  $p^n B_0 \neq 0$ ). Since  $B$  is not  $\Sigma$ -cyclic,  $W = \{\alpha: B_{\alpha+1}/B_\alpha \text{ is not } \Sigma\text{-cyclic}\}$  is stationary. Applying  $\diamond(W)$  choose  $\{f_\alpha: \alpha < \omega_1\}$  such that:  $f_\alpha: Z(p) \times B_\alpha \rightarrow H$ ; and for all  $f: Z(p) \times B \rightarrow H$  there is  $\alpha \in W$  so that  $f \upharpoonright Z(p) \times B_\alpha = f_\alpha$ . (All the set-theoretic notation and results used are standard. See [E] for example.)

We construct an increasing sequence of  $\omega$ -elongations  $Z(p) \twoheadrightarrow G_\alpha \twoheadrightarrow B_\alpha$  by induction on  $\alpha$ .  $G_\alpha$  will always be a group structure on  $Z(p) \times B_\alpha$  and  $\pi_\alpha$  is projection on the second coordinate. Let  $Z(p) \twoheadrightarrow G_0 \twoheadrightarrow B_0$  be any  $\omega$ -elongation. Let  $t$  be a generator of  $p^\omega G_0$ . The key case occurs when  $\alpha \in W$ ,  $f_\alpha: G_\alpha \rightarrow H$  is a homomorphism, and  $f_\alpha(t) \neq 0$ . Then by lemma 1.1 we can choose  $G_{\alpha+1}$  so that  $f_\alpha$  does not extend to a homomorphism from  $G_{\alpha+1}$  to  $H$ . At other successor steps  $G_{\alpha+1}$  can be chosen arbitrarily. At limit ordinals, we take unions. Let  $G = \bigcup_{\alpha < \omega_1} G_\alpha$ .

Suppose  $f: G \rightarrow H$  is a homomorphism so that  $f(t) \neq 0$ . (An isomorphism would be such an  $f$ .) Then there is some  $\alpha \in W$  so that  $f \upharpoonright G_\alpha = f_\alpha$ . But then  $f \upharpoonright G_{\alpha+1}$  is an extension of  $f_\alpha$  contradicting the choice of  $G_{\alpha+1}$ .

REMARKS. Lemma 1.1 is just what is needed to apply the "weak diamond" principle. So the hypothesis " $(V = L)$ " can be weakened to "weak diamond holds for every stationary subset of  $\omega_1$ ". Further the proof can be used to produce  $\aleph_2$  pairwise non-isomorphic  $\omega$ -elongations of  $Z(p)$  by  $B$ . All the above is known (cf. [EH]). In fact Eklof and Huber ([EH]) show CH and weak diamond for all stationary sets suffices to produce  $2^{\aleph_1}$  pairwise non-isomorphic  $\omega$ -elongations of  $Z(p)$  by  $B$ .

The following result on “rigid” systems of elongations is new.

1.3. THEOREM. *Assume  $(V = L)$ . Suppose  $B$  is a separable group of cardinality  $\aleph_1$  and  $B$  is not  $\Sigma$ -cyclic. There are  $2^{\aleph_1}$   $\omega$ -elongations  $\{H_\alpha: \alpha < 2^{\aleph_1}\}$  of  $Z(p)$  by  $B$  such that: for  $\alpha \neq \beta$  if  $f: H_\alpha \rightarrow H_\beta$  and  $t \in p^\omega H_\alpha$  then  $f(t) = 0$ .*

*Proof.* If we wanted  $\aleph_1$   $\omega$ -elongations the argument would be somewhat simpler. Let  $\{B_\alpha: \alpha < \omega_1\}$  be an  $\omega_1$ -filtration of  $B$  as is Theorem 1.2. We will define inductively a strictly monotone continuous function  $g: \omega_1 \rightarrow \omega_1$  and for each  $\eta \in {}^\alpha 2$  a group structure  $H_\eta$  on  $Z(p) \times B_{g(\alpha)}$  so that  $Z(p) \twoheadrightarrow H_\eta \twoheadrightarrow B_{g(\alpha)}$  is an  $\omega$ -elongation. Here  ${}^\alpha 2$  denotes the set of functions from  $\alpha$  to 2 and  $\pi_{g(\alpha)}$  is projection on the second coordinate. The construction will be done so that if for  $\eta \in {}^{\omega_1} 2$  we let  $H_\eta = \bigcup H_{\eta \upharpoonright \alpha}$  then  $\{H_\eta: \eta \in {}^{\omega_1} 2\}$  is the desired set of  $\omega$ -elongations.

Let  $W = \{\alpha: B_{\alpha+1}/B_\alpha \text{ is not } \Sigma\text{-cyclic}\}$ . Apply  $\diamond(W)$  to get  $\{(\eta_\alpha, \tau_\alpha, f_\alpha): \alpha \in W\}$  so that: for all  $\alpha, \eta_\alpha, \tau_\alpha \in {}^\alpha 2$ ; for all  $\alpha, f_\alpha: Z(p) \times B_\alpha \rightarrow Z(p) \times B_\alpha$ ; and for all  $\eta, \tau \in {}^{\omega_1} 2, f: Z(p) \times B \rightarrow Z(p) \times B, \{\alpha: \eta \upharpoonright \alpha = \eta_\alpha, \tau \upharpoonright \alpha = \tau_\alpha \text{ and } f \upharpoonright Z(p) \times B_\alpha = f_\alpha\}$  is stationary. Let  $g(0) = 0$  and  $H_{\langle \cdot \rangle}$  be any group structure on  $Z(p) \times B_0$  so that  $Z(p) \twoheadrightarrow H_{\langle \cdot \rangle} \twoheadrightarrow B_0$  is an  $\omega$ -elongation. The key case occurs when  $g(\alpha) = \alpha, f_\alpha: H_{\eta_\alpha} \rightarrow H_{\tau_\alpha}$  is a homomorphism, and for some  $t \in p^\omega H_{\eta_\alpha}, f_\alpha(t) \neq 0$ . Let  $H^1, H^2$  be  $\omega$ -elongations of  $Z(p)$  by  $B_{\alpha+1}$  extending  $H_{\eta_\alpha}$  as in Lemma 1.1. There are two subcases.

*Subcase (i).* Assume for all  $\beta > \alpha$  and  $H$  an  $\omega$ -elongation of  $Z(p)$  by  $B_\beta$  extending  $H_{\tau_\alpha}$  there is no extension of  $f_\alpha$  to a homomorphism from  $H^1$  to  $H$ . In this case let  $g(\alpha + 1) = \alpha + 1$ . Let  $H_{\eta_\alpha \frown \langle 0 \rangle} = H_{\eta_\alpha \frown \langle 1 \rangle} = H^1$ . For all  $\tau \neq \eta_\alpha$  and  $i = 0, 1, H_{\tau \frown \langle i \rangle}$  can be chosen arbitrarily.

*Subcase (ii).* For some  $\beta > \alpha$ , there is  $H$  an  $\omega$ -elongation of  $Z(p)$  by  $B_\beta$  extending  $H_{\tau_\alpha}$  and an extension of  $f_\alpha$  to a homomorphism from  $H^1$  to  $H$ . In this case let  $g(\alpha + 1) = \beta$  and  $H_{\tau_\alpha \frown \langle 0 \rangle} = H_{\tau_\alpha \frown \langle 1 \rangle} = H$ . Let  $H_{\eta \frown \langle 0 \rangle} = H_{\eta \frown \langle 1 \rangle}$  be any  $\omega$ -elongation of  $Z(p)$  by  $B_\beta$  extending  $H^2$ . For all other  $\tau$  and  $i = 0, 1, H_{\tau \frown \langle i \rangle}$  can be chosen arbitrarily.

The point here is that if  $H$  and  $G$  are  $\omega$ -elongations of  $Z(p)$  by  $B$  extending  $H_{\eta \frown \langle i \rangle}$  and  $H_{\tau \frown \langle j \rangle}$  respectively ( $i, j < 2$ ), then  $f_\alpha$  does not extend to a homomorphism from  $H$  to  $G$ .

In the other successor case let  $g(\alpha + 1) = g(\alpha) + 1$  and  $H_{\tau \frown \langle i \rangle}$  be any allowable  $\omega$ -elongation ( $\tau \in {}^\omega 2$  and  $i = 0, 1$ ). If  $\lambda$  is a limit ordinal we take unions; i.e.  $g(\lambda) = \sup_{\alpha < \lambda} g(\alpha)$  and for  $\eta \in {}^\lambda 2$   $H_\eta = \bigcup_{\alpha < \lambda} H_{\eta \upharpoonright \alpha}$ . The verification that  $\{H_\eta; \eta \in {}^{\omega_1} 2\}$  is the desired set is routine.

In the proof of Theorem 1.3, the hypothesis can be weakened from “ $(V = L)$ ” to “ $\diamond(W)$  holds for all stationary subsets  $W$  of  $\omega_1$ ”. The latter statement is consistent with  $2^{\aleph_1} > \aleph_2$ . Eklof has shown the hypothesis can be weakened further to “CH and weak diamond holds for every stationary subset of  $\omega_1$ ”. (His proof follows the proofs of Theorems 7.2 and 7.9 of [EH].)

We now turn to the proof that it is consistent that there are non- $\Sigma$ -cyclic Crawley groups. Recall a group is  $\omega_1$ -separable if it is separable and every countable subset is contained in a countable direct summand. There are  $\omega_1$ -separable groups of cardinality  $\omega_1$  which are not  $\Sigma$ -cyclic ([F] Thm. 75). The basic result we'll use is:

1.4. THEOREM. ([M]) *Assume MA +  $\neg$ CH. If  $B$  is an  $\omega_1$ -separable group of cardinality  $\aleph_1$  and  $A$  is a countable group, then  $\text{Pext}(B, A) = 0$ . I.e. any pure exact sequence  $0 \rightarrow A \rightarrow C \rightarrow B \rightarrow 0$  splits.*

Suppose  $B$  is an  $\omega_1$ -separable group. A countable subgroup  $A$  is a maximal direct summand of  $B$ , if  $A$  is a direct summand of  $B$ , any Ulm invariant of  $B/A$  is either 0 or  $\geq \omega$  and whenever an Ulm invariant of  $B$  is infinite so is the same invariant of  $a$ . Since we can calculate the Ulm invariants of  $A$ ,  $A$  is determined up to isomorphism.

1.5. LEMMA. *Suppose  $B$  is an  $\omega_1$ -separable group and  $A_1, A_2$  are maximal countable direct summands. Then  $B/A_1 \cong B/A_2$ .*

*Proof.* Choose  $K$  a countable direct summand of  $B$  such that  $A_1, A_2 \subseteq K$  and  $K/A_1 \cong K/A_2$ . (Such a  $K$  exists by the maximality of  $A_1$  and  $A_2$ .) There is an automorphism of  $K$  which induces an isomorphism from  $A_1$  to  $A_2$ .

1.6. THEOREM. ([M]) *Assume MA +  $\neg$ CH. There is a Crawley group which is not  $\Sigma$ -cyclic.*

*Proof.* Let  $B$  be an  $\omega_1$ -separable group of cardinality  $\aleph_1$  which is not  $\Sigma$ -cyclic. Suppose  $Z(p) \twoheadrightarrow H \twoheadrightarrow B$  is an  $\omega$ -elongation. Choose  $A$  a maximal countable direct summand of  $B$  such that  $Z(p) \twoheadrightarrow H' \twoheadrightarrow A$  is an

$\omega$ -elongation. Here  $H' = \pi^{-1}(A)$ . So  $H' \twoheadrightarrow H \twoheadrightarrow B/A$  is a pure exact sequence. Since  $B/A$  is  $\omega_1$ -separable,  $H \simeq H' \oplus B/A$ . As both  $H'$  and  $B/A$  are determined up to isomorphism, so is  $H$ .

**2. Characterizing  $\Sigma$ -cyclic groups in  $L$ .** Let  $P$  denote the Prüfer group; i.e. the group generated by  $\{x_n: n < \omega\}$  subject to the relations  $px_0 = 0$  and  $p^{n+1}x_{n+1} = x_0$ .

**2.1. LEMMA.** *Assume  $(V = L)$ . Suppose  $B \subseteq \frac{B'}{\pi}$ ,  $B'$  is separable and  $B'/B$  is not  $\Sigma$ -cyclic. Further suppose  $Z(p) \twoheadrightarrow H \twoheadrightarrow B$  is an  $\omega$ -elongation,  $f: H \rightarrow P$  is a homomorphism and  $f(p^\omega H) \neq 0$ . Then there is an  $\omega$ -elongation  $Z(p) \twoheadrightarrow H' \twoheadrightarrow B'$  extending  $Z(p) \twoheadrightarrow H \twoheadrightarrow B$  so that  $f$  does not extend to  $H'$ .*

*Proof.* Suppose  $\kappa = |B'/B|$  is the minimum cardinality of a counterexample. By Lemma 1.1  $\kappa > \aleph_0$ . Since the existence of an extension of  $f$  is an hereditary property, if  $C \subseteq B'$  and  $|C| < \kappa$  then  $C + B/B$  is  $\Sigma$ -cyclic. By [S1],  $\kappa$  must be regular. Choose an increasing continuous chain  $\{C_\alpha: \alpha < \kappa\}$  of subgroups of  $B'$  so that: for all  $\alpha$ ,  $|C_\alpha| < \kappa$ ; and  $\{B + C_\alpha/B: \alpha < \kappa\}$  is a  $\kappa$ -filtration of  $B'/B$ . We can assume  $C_0 = 0$ . Since  $B'/B$  is not  $\Sigma$ -cyclic,  $W = \{\alpha: C_{\alpha+1} + B/C_\alpha + B \text{ is not } \Sigma\text{-cyclic}\}$  is stationary in  $\kappa$ . Applying  $\diamond(W)$  we obtain  $\{g_\alpha: \alpha \in W\}$  so that:  $g_\alpha: C_\alpha \rightarrow P$  and for all  $g: C \rightarrow P$  there is  $\alpha \in W$  so that  $g \upharpoonright C_\alpha = g_\alpha$ .

We can assume  $H$  is a group structure on  $Z(p) \times B$  and  $\pi$  is projection on the second coordinate. We now define  $H_\alpha$  by induction on  $\alpha$  to be a group structure on  $Z(p) \times (B + C_\alpha)$  extending  $H$  so that  $Z(p) \twoheadrightarrow H_\alpha \twoheadrightarrow B + C_\alpha$  is an  $\omega$ -elongation and  $\pi_\alpha$  is projection on the second coordinate. We identify  $C_\alpha$  as a set with  $\{0\} \times C_\alpha$ . The key case occurs when  $Z(p) \twoheadrightarrow H_\alpha \twoheadrightarrow B + C_\alpha$  has been defined,  $\alpha \in W$  and  $f \cup g_\alpha$  induces a homomorphism from  $H_\alpha$  to  $P$ .

Then use the inductive hypothesis to choose  $Z(p) \twoheadrightarrow H_{\alpha+1} \twoheadrightarrow B + C_{\alpha+1}$  so that  $f \cup g_\alpha$  does not extend to a homomorphism from  $H_{\alpha+1}$  to  $P$ . The proof can be finished as usual.

$A$ , by now standard, argument using the  $\diamond$  principle yields the following theorem.

**2.2. THEOREM.** *Assume  $(V = L)$ . A group  $G$  is  $\Sigma$ -cyclic iff for every  $\omega$ -elongation  $H$  of  $Z(p)$  by  $G$  there is a homomorphism  $f$  from  $H$  to  $P$  such that  $f(p^\omega H) \neq 0$ .*

REMARK. The proof of Theorem 1.6 shows this characterization fails if we assume  $\text{MA} + \neg\text{CH}$ . The next result can be thought of as characterizing the *strongly Crawley* groups: i.e. those groups which remain Crawley when a  $\Sigma$ -cyclic group is added to them.

In Megibben's proof Crawley groups are attacked via Richman's criterion which links elongations of  $Z(p)$  by  $G$  with codimension 1 dense subgroups of  $G[p]$ .

2.3. COROLLARY. *Assume  $(V = L)$ . Suppose  $B$  is a group of unbounded exponent.  $B$  is  $\Sigma$ -cyclic iff for all  $\Sigma$ -cyclic groups  $A$ ,  $B \oplus A$  is a Crawley group.*

*Proof.* Let  $A$  be  $\bigoplus_{n < \omega} Z(p^n)$ . Then there is an  $\omega$ -elongation of  $Z(p)$  by  $B \oplus A$  isomorphic to  $P \oplus B$ . If  $B \oplus A$  is Crawley, then any  $\omega$ -elongation  $H$  of  $Z(p)$  by  $B \oplus A$  has a homomorphism  $f$  to  $P$  so that  $f(p^\omega H) \neq 0$ .

In Lemma 2.1,  $P$  can be replaced by any group,  $G$ , of cardinality  $\leq \aleph_1$  such that  $p^\omega G \simeq Z(p)$ . This remark allows us to obtain additional information on Crawley groups in  $L$ . (The proofs are similar to the foregoing ones.)

2.4. THEOREM. *Assume  $(V = L)$ . Suppose  $B$  has a direct summand  $C$  of unbounded exponent and  $|C| \leq \aleph_1$ . Then  $B$  is a Crawley group iff  $B$  is  $\Sigma$ -cyclic.*

A group,  $A$ , is said to be *weakly  $\kappa$ -separable* if every subset of  $A$  of cardinality  $< \kappa$  is contained in a pure subgroup  $B$  such that:  $B$  is  $\Sigma$ -cyclic; and for all  $C \supseteq B$  if  $|C/B| < \kappa$ , then  $C/B$  is  $\Sigma$ -cyclic.

2.5. THEOREM. *Assume  $(V = L)$ . If  $B$  is weakly  $\kappa$ -separable of cardinality  $\kappa$ , then  $B$  is a Crawley group iff  $B$  is  $\Sigma$ -cyclic.*

*Proof.* By [S1] we can assume  $\kappa$  is regular and by [M] (or Theorem 1.2) that  $\kappa > \aleph_1$ . Let  $\{B_\alpha: \alpha < \kappa\}$  be a  $\kappa$ -filtration of  $B$  by pure subgroups so that  $B_0$  is countable, has unbounded exponent and  $B_\alpha/B_0$  is  $\Sigma$ -cyclic for all  $\alpha < \kappa$ . We can further assume  $B_{\alpha+1}$  is a direct summand of  $B_\nu$  for all  $\nu < \alpha$ .

Let  $Z(p) \twoheadrightarrow H \twoheadrightarrow B$  be an  $\omega$ -elongation extending an  $\omega$ -elongation  $Z \twoheadrightarrow H_0 \twoheadrightarrow B_0$ . Let  $H_\alpha$  be the preimage of  $B_\alpha$ .



We now define a group structure  $H'$  on  $Z(p) \times B$  so that  $Z(p) \twoheadrightarrow H' \xrightarrow{\pi} B$  is an  $\omega$ -elongation where  $\pi$  is projection on the second coordinate. Let  $W = \{ \alpha: B_{\alpha+1}/B_\alpha \text{ is not } \Sigma\text{-cyclic} \}$ . Choose  $\{ g_\alpha: \alpha \in W \}$  so that: for all  $\alpha$ ,  $g_\alpha: Z(p) \times B_\alpha \rightarrow H_\alpha$  and for all  $g: Z(P) \times B \rightarrow H$  there is  $\alpha \in W$  with  $g \upharpoonright Z(p) \times B = g_\alpha$ . Choose  $H'_0$  so that  $Z(p) \twoheadrightarrow H'_0 \twoheadrightarrow B_0$  is an  $\omega$ -elongation. The key case occurs when  $H'_\alpha$  has been defined,  $\alpha \in W$  and  $g_\alpha: H'_\alpha \rightarrow H_\alpha$  is an isomorphism. Since  $H_{\alpha+1}/H_0 \cong B_{\alpha+1}/B_0$ , we can choose  $\rho$  a projection of  $H_{\alpha+1}$  onto  $H_0$ . By Lemma 2.1, we can choose  $H'_{\alpha+1}$  so that  $\rho \circ g_\alpha$  does not extend to a homomorphism from  $H'_{\alpha+1}$  to  $H_0$ . Let  $H' = \cup H'_\alpha$ .

Suppose  $g: H' \rightarrow H$  is an isomorphism. Choose  $\alpha$  so that  $g \upharpoonright H'_\alpha = g_\alpha$ , an isomorphism. Choose  $\nu > \alpha$  so that  $g(H'_{\alpha+1}) \subseteq H_\nu$ . Since  $B_{\alpha+1}$  is a direct summand of  $B_\nu$ ,  $H_\nu \cong H_{\alpha+1} \oplus B_\nu/B_\alpha$ . So we can take  $\rho': H_\nu \rightarrow H_0$  extending  $\rho$ . Then  $\rho' \circ g$  contradicts the choice of  $H'_{\alpha+1}$ .

**3. Must Crawley groups be  $\omega_1$ -separable?** So far the examples given of Crawley groups are all  $\omega_1$ -separable. In this section we will show that it is consistent that there is a Crawley group of cardinality  $\omega_1$  which is not  $\omega_1$ -separable. It would be more interesting to produce, in some model of set theory, a Crawley group with no unbounded  $\Sigma$ -cyclic direct summand. This problem seems quite hard. The proof of our result is a mixture of the techniques in [S2] for producing a Whitehead group of class I with the proof of Theorem 1.6.

To start we will introduce the set theoretic machinery. A *tree* of height  $\leq \omega$  is a partially ordered set such that any element has finitely many predecessors and these predecessors are totally ordered. Suppose  $T$  is a tree of height  $\omega$  (i.e. there are subchains of  $T$  of every finite length). A *branch* of  $T$  is a maximal chain of cardinality  $\aleph_0$ . Suppose  $d_0: \omega \rightarrow \omega$ ,  $d_1: \omega \rightarrow \omega + 1$  and  $A \subseteq \omega$ . A function  $f: A \rightarrow \omega$  is said to be  $(d_0, d_1)$ -admissible if for all  $n$   $f(n) < d_1(n)$  and there is some  $k$  so that for all  $n \geq k$ ,  $f(n) < d_0(n)$ .

Suppose  $T$  is a tree of height  $\omega$  where underlying set is  $\omega$  and that  $\mathcal{S}$  is a set of branches. Let  $d_0$  and  $d_1$  be as above.  $(T, \mathcal{S})$  satisfies  $(d_0, d_1)$ -uniformization if whenever  $\{ F_A: A \in \mathcal{S} \}$  is such that for all  $A$ ,  $F_A$  is a countable collection of  $(d_0, d_1)$ -admissible functions from  $A$  to  $\omega$ , and for all finite  $X \subseteq A$ ,  $f: X \rightarrow \omega$  if  $f(x) < d_1(x)$  then  $f$  has an extension in  $F_A$ ; then there is  $f: \omega \rightarrow \omega$  so that for all  $A$ ,  $f \upharpoonright A \in F_A$ . Note if  $(T, \mathcal{S})$  satisfies  $(d_0, d_1)$ -uniformization and for all  $n$   $d'_l(n) < d_l(n)$  ( $l = 0, 1$ ), then  $(T, \mathcal{S})$  satisfies  $(d'_0, d'_1)$ -uniformization. A modification of

the proof of Theorem 4.2 of [S2] yields:

3.1. THEOREM. Fix  $d_0; \omega \rightarrow \omega$  and  $d_1; \alpha \rightarrow \omega + 1$ . It is consistent that there is a tree  $T$  of height  $\omega$  whose underlying set is  $\omega$  and a collection  $\mathcal{S} = \{A_i; i \in \omega_1\}$  of pairwise distinct branches such that  $(T, \mathcal{S})$  satisfies  $(d_0, d_1)$ -uniformization.

REMARK. Our proof of Theorem 3.2 could actually be done using the more usual uniformization property in Theorem 4.2 of [S2]. We state this stronger notion of uniformization in the hope it may prove useful.

Let  $d_0$  and  $d_1$  be defined by  $d_0(n) = d_1(n) = p$ . Assume  $T$  and  $\{A_i; i < \omega_1\}$  satisfy  $(d_0, d_1)$ -uniformization where  $T$  is a tree of height  $\omega$  whose underlying set is  $\omega$  and  $\{A_i; i < \omega_1\}$  is a collection of pairwise distinct branches. For each  $i$ , let  $\langle a'_n; n < \omega \rangle$  enumerate  $A_i$  in increasing order (in the sense of  $T$ ). Note: if  $a'_m = a'_n$  then  $m = n$ . Let  $B = \{a'_n; i < \omega_1, n < \omega\}$ . Of course  $B \subseteq \omega$ . For clarity of notation we will distinguish between elements of  $\omega$ , denoted by  $a'_n$ , and integers.

Let  $G$  be the group freely generated by  $B$  and  $\{x'_n; i < \omega_1, n < \omega\}$  subject to the relations:  $p^{n+1}a'_n = 0$ , for  $i < \omega_1$  and  $n < \omega$ ;  $px'_0 = 0$ , for  $i < \omega_1$ ; and for  $i < \omega_1$  and  $n < \omega$ ,  $px'_{n+1} = x'_n + a'_n$ . Of course  $x'_n$  can be thought of as the formal sum  $\sum_{m \geq n} -p^{m-n}a'_m$ .  $G$  is contained in the torsion completion of a countable  $\Sigma$ -cyclic group. So  $G$  is separable but not  $\omega_1$ -separable.

3.2. THEOREM. It is consistent that there is a Crawley group  $G$  of cardinality  $\aleph_1$  which is not  $\omega_1$ -separable. In fact  $G$  has a countable basic subgroup.

*Proof.* Let  $G$  be as above. Suppose  $Z(p) \twoheadrightarrow H_0 \xrightarrow{\pi_0} G$  and  $Z(p) \twoheadrightarrow H_1 \xrightarrow{\pi_1} G$  are two  $\omega$ -elongations. Our first goal is to find subgroups  $G_1$  and  $G_2$ , so that:  $G = G_1 \oplus G_2$ ;  $G_1$  is countable; and if we let  $H'_0 = \pi_0^{-1}(G_1)$  and  $H'_1 = \pi_1^{-1}(G_1)$  then  $Z(p) \twoheadrightarrow H'_0 \xrightarrow{\pi_0} G_1$  and  $Z(p) \twoheadrightarrow H'_1 \xrightarrow{\pi_1} G_1$  are  $\omega$ -elongations. We will then show the pure exact sequences  $H'_0 \twoheadrightarrow H_0 \twoheadrightarrow G_2$  and  $H'_1 \twoheadrightarrow H_1 \twoheadrightarrow G_2$  split. Clearly this suffices.

To begin we wish to choose  $C$  such that: for some  $i, j < \omega_1$   $C \subseteq B \cup \{x'_n, x'_j; n < \omega\}$ ; for all  $n > 0$  and  $l = 0, 1$  there is  $n \leq m$   $c \in C$  and  $c^* \in H_l$  so that  $\pi_l(c^*) = c$ ,  $p^m c^* \in p^\omega H_l \setminus \{0\}$ ; and for all but finitely many  $i$ ,  $A_i \cap C$  is finite. Note: if  $\pi_l(c^*) = c$  and  $p^m c^* \in p^\omega H_l$ , then  $p^m c = 0$ . Also for all  $c^{**}$  if  $\pi_l(c^{**}) = c$  then  $p^m c^{**} = p^m c^*$ .

Fix  $l = 0, 1$ . There are two cases. Suppose for all  $n > 0$  there is  $n \leq m$ ,  $b \in B$ ,  $b^* \in H_l$  so that  $\pi_l(b^*) = b$  and  $p^m b^* \in p^\omega H_l \setminus \{0\}$ . In this case, let  $D_l = \{b \in B: \text{for some } m \text{ and } b^* \in H_l, \pi_l(b^*) = b \text{ and } p^m b^* \in p^\omega H_l \setminus \{0\}\}$ . If for some  $i < \omega_1$ ,  $D_l \cap A_i$  is infinite, let  $C_l = A_i$ . Otherwise let  $C_l = D_l$ .

In the second case pick  $n$  so that for all  $b \in B$ ,  $m \geq n$  and  $b^* \in H_l$  if  $\pi_l(b^*) = b$  and  $p^m b = 0$  then  $p^m b^* = 0$ . Choose  $i$  and  $x_n^{i*}$  so that  $\pi_l(x_n^{i*}) = x_n^i$  and  $p^{n+1} x_n^{i*} \neq 0$ . Such an  $i$  exists since  $p^n G[p]$  is generated by  $\{p^m b: b \in B, n \leq m \text{ and } p^{m+1} b = 0\} \cup \{p^n x_n^i: i < \omega_1\}$ . Let  $a_k^{i*}$  be  $\pi_l$ -preimages of  $a_k^i$  ( $n \leq k < m$ ). Then

$$p^{m+1} x_m^{i*} = p^{n+1} (x_n^{i*} + \sum p^{k-n} a_k^{i*}) = p^{n+1} x_n^{i*}.$$

(The last equality follows from the choice of  $n$ .) In this case let  $C_l = \{x_n^i: n < \omega\}$ .

If we let  $C = C_0 \cup c_1$ , then  $C$  is as desired. We can modify  $C$  so that, if  $x_n^i \in C$  then  $a_n^i \in C$ .

For convenience we can assume if  $a_n^i \in C$  then  $a_m^i \in C$  for all  $m \leq n$ . Let  $G_1$  be the subgroup of  $G$  generated by  $C$  together with  $\{x_n^i: n < \omega \text{ and } C \cap A_i \text{ is infinite}\}$ . Let  $G_2$  be the group generated by  $B \setminus C$  together with  $\{x_n^i: a_n^i \notin C, i < \omega_1\}$ . It is clear that  $G = G_1 \oplus G_2$ ,  $G_1$  is countable and  $Z(p) \twoheadrightarrow H_l' \twoheadrightarrow G_1$  is an  $\omega$ -elongation.

We now wish to show for  $l = 0, 1$ ,  $H_l' \twoheadrightarrow H_l \twoheadrightarrow G_2$  splits. Since the proof is the same for each  $l$  we will drop the subscript  $l$ . Reindexing we can assume  $\langle A_i: i < \omega_1 \rangle$  enumerates the branches such that  $A_i \cap C$  is finite. (The exceptional branches have been dropped.) For each  $i$ , let  $n(i)$  denote the least  $n$  such that  $a_n^i \notin C$ . To find a splitting we must find preimages  $\underline{x}_m^i, \underline{a}_m^i$  for  $x_m^i$  and  $a_m^i$  ( $i < \omega_1$  and  $n(i) \leq m < \omega$ ) such that:  $p^{m+1} \underline{a}_m^i = p^{m+1} \underline{x}_m^i = 0$ ; and for all  $m \geq n(i)$ ,  $p \underline{x}_{m+1}^i = \underline{x}_m^i + \underline{a}_m^i$ . (Of course,  $a_m^i = a_m^j$ , then  $\underline{a}_m^i = \underline{a}_m^j$ .) First for  $i < \omega_1$  and  $n(i) \leq m$  choose  $\mathbf{x}_m^i, \mathbf{a}_m^i$  so that  $\pi(\mathbf{x}_m^i) = x_m^i$  and  $\pi(\mathbf{a}_m^i) = a_m^i$ . Let  $t$  be a fixed element of  $p^\omega H' \setminus \{0\}$ . Since for all  $i, n(i) \leq m$ ,  $p \mathbf{x}_{m+1}^i = \mathbf{x}_m^i + \mathbf{a}_m^i + kt$  and  $pt = 0$ , by modifying our choice of the  $\mathbf{x}_m^i$ , we assume  $p \mathbf{x}_{m+1}^i = \mathbf{x}_m^i + \mathbf{a}_m^i$ . For  $m < \omega$  choose  $b_m \in H'$  so that  $p^{m+1} b_m = t$ .

*Claim.* Fix  $i < \omega_1$ . For  $m \geq n(i)$  choose  $0 \leq k_m^i < p$  so that  $p^{m+1} \mathbf{x}_m^i + p^{m+1} k_m^i b_m = 0$ .

(1) Then  $p^{m+1} (\mathbf{a}_m^i - k_m^i b_m - p k_{m+1}^i b_{m+1}) = 0$ .

(2) Suppose  $\underline{a}_m^i$  ( $m \geq n(i)$ ), preimages of the  $a_m^i$ , are given such that for all  $m$ ,  $p^{m+1} \underline{a}_m^i = 0$ ; and for some  $n$  if  $m \geq n$  then  $\underline{a}_m^i = \mathbf{a}_m^i - k_m^i b_m + p k_{m+1}^i b_{m+1}$ . Then there exists  $\underline{x}_m^i$  ( $m \geq n(i)$ ) preimages of the  $x_m^i$  so that: for all  $m$ : (i)  $p^{m+1} \underline{x}_m^i = 0$  and (ii)  $p \underline{x}_{m+1}^i = \underline{x}_m^i + \underline{a}_m^i$ .

*Proof (of claim).* (1) Since  $\mathbf{a}'_m = p\mathbf{x}'_{m+1} - \mathbf{x}'_m$ ,

$$p^{m+1}(\mathbf{a}'_m - k_m^i b_m + pk_{m+1}^i b_{m+1}) \\ = p^{m+2}(\mathbf{x}'_{m+1} + k_{m+1}^i b_{m+1}) - p^{m+1}(\mathbf{x}'_m + k_m^i b_m) = 0.$$

(2) Let  $\underline{x}'_m = \mathbf{x}'_m + k^i k_m b_m$ , for  $m \geq n$ . For  $n(i) \leq m < n$  there is only one choice of  $\underline{x}'_m$  so that equation (ii) will be satisfied for all  $m$ . Just as in (1) we can verify equation (i) holds.

We can now apply our choice of  $T$  and  $\{A_i; i < \omega_1\}$ . For  $i < \omega_1$ , let  $f \in F_i$  if (i) for  $m \geq n(i)$ ,  $f(a'_m) = (r, s)$  where  $0 \leq r, s < p$  and  $p^{m+1}(\mathbf{a}'_m - rb_m + psb_{m+1}) = 0$  and (ii) for all but finitely many  $m$ ,  $r = k'_m$  and  $s = k'_{m+1}$ . By the choice of  $t$  and  $\{A_i; i < \omega_1\}$  there is  $f: \{a'_m; i < \omega_1, m \geq n(i)\} \rightarrow \omega \times \omega$  such that for all  $i$   $f \upharpoonright A_i \in F_i$ . For  $i < \omega_1$  and  $m \geq n(i)$ , let  $\underline{a}'_m = \mathbf{a}'_m - rb_m + psb_{m+1}$  where  $f(a'_m) = (r, s)$ . By claim (2) we can choose  $\underline{x}'_m$  as desired.

REMARK. Warfield [W] has shown: if CH holds then any Crawley group with a countable basic subgroup is  $\Sigma$ -cyclic. This result can also be proved using weak diamond. See the remarks after Theorem 1.2 or Corollary 7.8 of [EH].

If the following question has a positive answer, then Theorem 2.2 shows: if  $(V = L)$  then every Crawley group is  $\Sigma$ -cyclic. A positive answer would also show our construction in §3 is in some sense best possible.

*Question.* Suppose  $G$  is a Crawley group and  $H$  is the  $\omega$ -elongation of  $Z(p)$  by  $G$ . Do there exist  $H_1$  and  $H_2$  so that:  $H = H_1 \oplus H_2$ ;  $H_1$  is countable and  $H_2$  is separable?

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