Pacific Journal of Mathematics

THE SOLUTION TO CRAWLEY'S PROBLEM

Alan Harvey Mekler and Saharon Shelah

Vol. 121, No. 1 November 1986

PACIFIC JOURNAL OF MATHEMATICS
Vol. 121, No. 1, 1986

THE SOLUTION TO CRAWLEY'S PROBLEM

ALAN H. MEKLER AND SAHARON SHELAH

In this supplement to our paper, ω-elongations and Crawley's problem, we show that if $(V = L)$ every Crawley group is a direct sum of cyclic groups.

A simple argument completes the work begun in [**MS**] and allows us to show if that if $(V = L)$ then every Crawley group (no matter what its cardinality) is a direct sum of cyclic groups. (See [**MS**] for definitions and conventions.) We first note that in Theorem 2.2 of [**MS**] we prove something stronger. Namely

THEOREM 1. *Assume $(V = L)$ and let G be a group of cardinality at most $\aleph_1$ such that $p^\omega G \simeq Z(p)$. A separable group A is a direct sum of cyclic groups iff for every ω-elongation H of $Z(p)$ by A there is a homomorphism f from H to G such that $f(p^\omega H) \neq 0$.*

LEMMA 2. *Suppose A is a separable group of length ω. There is an ω-elongation H of $Z(p)$ by A and a group G such that $p^\omega G \simeq Z(p)$, $|G| \leq 2^{\aleph_0}$, and there is a homomorphism f from H to G with $f(p^\omega H) \neq 0$.*

Proof. Choose $B \subseteq A$ a basic subgroup and write $B = B_0 \oplus B_1$ where B_0 is countable with elements of arbitrarily large order. Let A^* be the closure of B_1 (i.e. A^* is the maximal subgroup of A so that A^*/B_1 is divisible). The subgroup $A^* + B_0 = A^* \oplus B_0$. Choose H_0 an ω-elongation of $Z(p)$ by B_0. Let $H_1 = A^* \oplus H_0$. Finally choose H an ω-elongation of $Z(p)$ by A containing H_1. Since $H \supseteq A^*$, we can let $G = H/A^*$. Let t generate $p^\omega H$. We have the sequence $\langle t + A^* \rangle \rightarrowtail G \twoheadrightarrow A/A^*$. Since $t \notin A^*$, to complete the proof we only need to note that A/A^* is separable and $|A/A^*| \leq 2^{\aleph_0}$. Both of these claims are easy to establish by first choosing an independent set of generators of B_0 and then identifying A/A^* with a group of formal sums of multiples of these generators.

THEOREM 3. *Assume $(V = L)$. Every Crawley group is a direct sum of cyclic groups.*

Proof. Suppose A is a separable group which is not the direct sum of cyclic groups. By Lemma 2 we can choose an ω-elongation H of $Z(p)$ by A and a group G such that $p^\omega G \simeq Z(p)$; $|G| \leq 2^{\aleph_0}(= \aleph_1)$; and there is a homomorphism f: $H \to G$ with $f(p^\omega H) \neq 0$. But by Theorem 2.2 there is an ω-elongation H' of $Z(p)$ by A such that there is no homomorphism g: $H' \to G$ with $g(p^\omega H') \neq 0$. Hence A is not a Crawley group.

References

[MS] A. Mekler, and S. Shelah, *ω-elongations and Crawley's problem*, Pacific J. Math., **121** (1986), 121–132.

Received June 19, 1985. Research by the first author was supported by NSERC Grant A8948. Research by the second author was supported by NSERC Grant A3040.

SIMON FRASER UNIVERSITY
BURNABY, B.C., CANADA V5A 1S6

AND

THE HEBREW UNIVERSITY
JERUSALEM, ISRAEL

Pacific Journal of Mathematics

Vol. 121, No. 1 November, 1986