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# MINIMAL NON-PERMUTATIVE PSEUDOVARIETIES OF SEMIGROUPS. II

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**This paper is the continuation of a previous paper in which all minimal non-$\mathscr{P}$-permutative pseudovarieties of $\mathscr{C}$ were determined where $\mathscr{P}$-permutativity was one of several conditions implying permutativity and $\mathscr{C}$ was the class of either (finite) groups, monoids or semigroups. In this work, the most general case of this type is treated, namely when $\mathscr{P}$-permutativity is permutativity and $\mathscr{C}$ is the class of all finite semigroups.**

The notations and conventions adopted in this paper were introduced in [1].

**1. Introduction.** This paper is the continuation of [1] and uses the notation and conventions introduced there. In [1] we determined all minimal non-$\mathscr{P}$-permutative pseudovarieties of $\mathscr{C}$ where $\mathscr{P}$-permutativity was one of several conditions implying permutativity and $\mathscr{C}$ was the class of either (finite) groups, monoids or semigroups. Here, we treat the most general case of this type, namely when $\mathscr{P}$-permutativity is permutativity and $\mathscr{C}$ is the class of all finite semigroups.

**2. Some non-permutative semigroups.** In this section, we introduce some semigroups which will play an important role in the sequel. We also indicate a finite basis of identities for each of them.

Recall that any completely simple semigroup is isomorphic to a Rees $I \times \Lambda$ matrix semigroup $\mathscr{M}(G;\ I, \Lambda;\ P)$ over a group $G$ with sandwich matrix $P = (p_{\lambda i})$ (see Clifford and Preston [3]). For a prime number $p$, we let

$$K_p = \mathscr{M}\left(\mathbf{Z}_p;\ 2, 2;\ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right).$$

Rasin [5] has given a complete description of the lattice of varieties of completely simple semigroups over abelian groups. The appropriate universal algebraic type to deal with arbitrary completely simple semigroups involves not only one binary operation (product), but also one unary operation (inversion within the group containing a given element). However, in the context of finite semigroups, the need for the unary operation disappears, since the (group) inverse of an element is then one of its (positive) powers.

A regular semigroup $S$ is said to be *orthodox* if the set $E(S)$ of its idempotents is a subsemigroup of $S$. $S$ is said to be an *orthogroup* if it is an orthodox union of groups (cf. Clifford [2]).

From the results of Rasin [5], one easily deduces that if $S$ is a finite non-orthodox completely simple semigroup over an abelian group, then $K_p \in \underline{V}(S)$ for some prime $p$. Moreover, the $\underline{V}(K_p)$ ($p$ prime) are minimal non-permutative pseudovarieties of semigroups; they are defined by the following identities:

$$\underline{V}(K_p) = [\![x^{p+1} = x, (xyx)^p = x^p, xyxzx = xzxyx]\!]$$

and $K_p$ is a generator of minimum size of this pseudovariety.

Let

$$Y = \langle e, s, f;\ e^2 = e,\ f^2 = f,\ es = s = sf,\ ef = fe = 0\rangle.$$

Then $Y = \{e, s, f, 0\}$ and, from Edmunds [4], we obtain

$$\underline{V}(Y) = [\![x^2 = x^3,\ xyx = x^2y^2 = y^2x^2]\!];$$

we also observe that $Y$ is a generator of minimum size of $\underline{V}(Y)$.

The last special semigroup we need to consider is the following:

$$Q = \langle e, s, t;\ e^2 = e,\ es = s,\ t = te,\ se = et = ts = 0\rangle.$$

We note that $Q = \{e, s, t, st, 0\}$.

A characterization of the identities which fail in $Q$ will be crucial later on. To describe it, we require some further terminology. We let $X = \{x_1, x_2, \ldots, y_1, y_2, \ldots, x, y, z, t, \ldots\}$ be a countable set of variables; let $X^+$ be the free semigroup on $x$ and let $X^* = X^+ \cup \{1\}$, where 1 denotes the empty word. For a word $w \in X^+$, $c(w)$ denotes the set of all variables which occur in $w$, and $|w|_x$ denotes the number of occurrences of the variable $x$ in $w$. Two identities are said to be *equivalent* if they hold in exactly the same semigroups. $S \models u = v$ means that the identity $u = v$ holds in the semigroup $S$. $\Sigma \vdash u = v$ means that $u = v$ is a consequence of the set $\Sigma$ of identities.

LEMMA 2.1. *An identity $u = v$ fails in $Q$ if and only if it satisfies (up to equivalence) one of the following conditions*:

(i) $c(u) \neq c(v)$;

(ii) *there exists $x \in X$ such that $|u|_x = 1$ and $|v|_x > 1$;*

(iii) *there exist $y, z \in X$ such that $u = u_1yzu_2$ with $|u|_y = |u|_z = 1$ and some $u_1, u_2 \in X^*$ while, either $v = v_1zv_2yv_3$ for some $v_1, v_2, v_3 \in X^*$, or $v = v_1yv_2'zv_3$ for some $v_1, v_3 \in X^*$ and some $v_2' \in X^+$;*

(iv) *there exists* $y \in X$ *such that* $|u|_y = 1$ *and every variable which occurs to the right* (*left*) *of* $y$ *in* $u$ *occurs only once in* $u$, *while in* $v$ *there is some variable which occurs repeatedly and at least once to the right* (*left*) *of* $y$.

*Proof.* If (i) holds, then no nontrivial semilattice satisfies $u = v$; since the subsemigroup $\{e, 0\}$ of $Q$ is a semilattice, we obtain $Q \not\models u = v$.

If (ii) holds, substitute $st$ for $x$, $e$ for every other variable in $u = v$. This yields the value $st$ and $u$ but 0 for $v$. Hence, $Q \not\models u = v$.

If case (iii) holds, we substitute $s$ for $y$, $t$ for $z$ and $e$ for every other variable in $u = v$ to obtain the value $st$ for $u$ and 0 for $v$. Hence, $Q \not\models u = v$.

Suppose condition (iv) holds, say with the "right" option. Let $y$ be the rightmost variable satisfying (iv). By the above, we may assume that (i)-(iii) fail. Then, $u = u'y$ for some $u' \in X^+$. We substitute $s$ for $y$ and $e$ for every other variable in $u = v$ to obtain the values $s = es$ for $u$ and $se = 0$ for $v$. Hence $Q \not\models u = v$.

The above establishes half of the lemma. For the converse, let $\neg$ denote negation and assume that, up to equivalence, $u = v$ satisfies the conjunction $(C)$ of conditions $\neg$(i)–$\neg$(iv). We show that then $Q \models u = v$. Let

$$\Sigma = \{x^2 = x^3, xy^2x = x^2y^2 = y^2x^2, x^2yz^2 = z^2yx^2, xyx = x^2yx^2, xyxzx = xzxyx\}.$$

Simple direct calculations show that $Q \models \Sigma$. Thus, it suffices to show $\Sigma \vdash u = v$.

We note that, since each element of $\Sigma$ satisfies $(C)$, if the identities $u = v$ and $u' = v'$ are equivalent in the presence of $\Sigma$, then $u' = v'$ satisfies $(C)$ if and only if $u = v$ does. Moreover, using $\Sigma$, any word $w$ can be reduced to one of the form

$$w' = w_0x_1^2w_1x_2^2w_2 \cdots x_{n-1}^2w_{n-1}x_n^2x_{n+1}^2 \cdots x_m^2w_n$$

where $n, m \geq 0$, $x_i$ $(i = 1, \ldots, m)$ are not necessarily distinct variables, $c(w_i) \cap c(w_j) = \varnothing$ for $i \neq j$, $x_i \notin c(w_j)$ $(i = 1, \ldots, m;\ j = 0, \ldots, n)$, $|w_i|_x \leq 1$ for every $x \in X$, $w_i \in X^+$ $(i = 1, \ldots, n-1)$, and $w_0, w_n \in X^*$; up to the relative order of the $x_i^2$, the relative order of the $w_i$ $(i = 1, \ldots, n-1)$, and the positive number of their occurrences, this form is unique.

Let $u'$, $v'$ be words in the previous canonical form obtained respectively from $u$, $v$ using $\Sigma$, say

$$u' = u_0x_1^2u_1x_2^2u_2 \cdots x_{n-1}^2u_{n-1}x_n^2x_{n+1}^2 \cdots x_m^2u_n$$

and

$$v' = v_0 y_1^2 v_1 y_2^2 v_2 \cdots y_{k-1}^2 v_{k-1} y_k^2 y_{k+1}^2 \cdots y_l^2 v_k.$$

By a previous remark, $u' = v'$ satisfies $(C)$. By $\neg$(i) and $\neg$(ii), we may assume that $m = l$ and $x_i = y_i$ $(i = 1, \ldots, m)$. By $\neg$(ii) and $\neg$(iii), the variables in $c(u_0 u_1 \cdots u_n) = c(v_0 v_1 \cdots v_k)$ appear in the same order in $u'$ as in $v'$, and appear in adjacent positions on one side of $u' = v'$ if and only if they do so on the other side. Finally, by $\neg$(iv), $u_0 = 1$ if and only if $v_0 = 1$, and $u_n = 1$ if and only if $v_k = 1$. Hence $k = n$ and $u_i = v_i$, $(i = 0, 1, \ldots, n)$, up to a rearrangement of the $u_i$ $(i = 1, \ldots, n-1)$. Thus, the words $u'$ and $v'$ coincide. Therefore $\Sigma \vdash u = v$, as claimed.

As a corollary to the proof of Lemma 2.1, we have the following.

PROPOSITION 2.2.

$$\underline{V}(Q) = [\![x^2 = x^3, xy^2x = x^2y^2 = y^2x^2, x^2yz^2 = z^2yx^2, xyx = x^2yx^2,$$
$$xyxzx = xzxyx]\!].$$

REMARK 2.3. Examining the list of non-permutative varieties generated by semigroups of order four given by Edmunds [**4**], and noting that a semigroup of order less than four is either permutative or idempotent, one can show that $Q$ is a generator of minimum size of $\underline{V}(Q)$.

**3. Regular case.** Recall from [**1**] the semigroups $G_p$, $H_{p,q'}$, $N^1$, $B(1,2)^1$ and $B(2,1)^1$. For a semigroup $S$, $\rho(S)$ denotes the set of regular elements of $S$.

THEOREM 3.1. *Let $\underline{V}$ be a pseudovariety of semigroups such that*

(1) $G_p, H_{p,q} \notin \underline{V}$ $(p, q$ *distinct primes*$)$

(2) $N^1, B(1,2)^1, B(2,1)^1 \notin \underline{V}$

(3) $K_p \notin \underline{V}$ $(p$ *prime*$)$

(4) $Y \notin \underline{V}$

*and let $S \in \underline{V}$. Then $\rho(S)$ is a permutative orthogroup.*

We prove Theorem 3.1 in a number of steps. Henceforth, $\underline{V}$ and $S$ are as in Theorem 3.1. First of all, because of (1) and (2), it follows from Theorem 5.3 [**1**] that all monoids in $\underline{V}$ are commutative.

LEMMA 3.2. *$E = E(S)$ is a subsemigroup of $S$.*

*Proof.* Let $e, f \in E$ and suppose that $ef \notin E$. We let $S'$ denote the subsemigroup of $S$ generated by $\{e, f\}$.

Suppose $ef$, $fe$ are not $\mathscr{J}$-equivalent in $S'$. If $fe \in E$, then $fe = fefe <_{\mathscr{J}} ef$. So, we may assume $ef \not\leqslant_{\mathscr{J}} fe$. Let $I = \{u \in S' : ef \not\leqslant_{\mathscr{J}} u\}$, an ideal of $S'$. Then $S'/I$ consists of the four elements $e$, $f$, $ef$ and $0 = fe$ (i.e., $S'/I$ is isomorphic to the semigroup $D$ of example 3.11 [**1**]). However, it is easy to check that $S'/I \times S'/I$ has a subsemigroup isomorphic to $Y$, whence $Y \in \underline{V}$, contradicting (4).

Hence, we have $ef \mathscr{J} fe$ in $S'$. Since $ef \mathscr{R} fe$ implies $ef = fef$ and so $ef \in E$, we deduce that $ef$, $fe$ are not $\mathscr{R}$-equivalent. Hence, in the $\mathscr{D}$-class of $ef$ in $S'$, $\mathscr{R} \neq \mathscr{H}$. Let $ef \mathscr{R} efu$ but $ef \not\mathscr{H} efu$. Since $e$, $f$ are idempotents, it follows that $ef = (ef)^k$ for some $k > 1$. Hence, $G = H_{ef}$ is a subgroup of $S'$. Moreover, from $ef = (ef)^k$ we obtain $fefe = (fe)^{k+1}$, and so $H_{fe}$ is also a subgroup of $S'$. In fact, $R = J_{ef}$ is a completely simple subsemigroup of $S'$. Since $(ef)^{k-1}(fe)^{k-1} = (efe)^{2k-3} \notin E$, $R$ is not orthodox. By the results of Rasin quoted in §2, it follows that $K_p \in \underline{V}(R) \subseteq \underline{V}$ for some prime $p$, contradicting (3). This completes the proof of the lemma.

Let $BA_2$ denote the Brandt semigroup $\mathscr{M}^0(\{1\}; 2, 2; \Delta)$ (cf. Clifford and Preston [**3**]). One can easily check that $Y$ is isomorphic to a subsemigroup of $BA_2 \times BA_2$, and that if a semigroup $U$ has a regular $\mathscr{D}$-class which is not a subsemigroup of $U$, then $BA_2 \in \underline{V}(U)$.

LEMMA 3.3. *$\rho(S)$ is a subsemigroup of $S$ and a union of abelian groups.*

*Proof.* By the remarks preceding the lemma, every regular $\mathscr{D}$-class of $S$ is a completely simple subsemigroup of $S$. Hence, $\rho(S)$ is a union of abelian groups.

Let $a, b \in \rho(S)$. Let $e \in E \cap H_a$, $f \in E \cap H_b$. By Lemma 3.2, $ef \in E$. Further, if $a'a = e$ and $bb' = f$, then $a'(ab)b' = ef$ and $ab = aefb$. Hence, $ab \mathscr{J} ef$ and so $ab \in \rho(S)$, as desired.

LEMMA 3.4. (*Yamada* [**6**], *p.* 375.) *A band is normal if and only if it satisfies the identity $xyxzx = xzxyx$, i.e., if and only if all its submonoids are commutative.*

COROLLARY 3.5. *$E$ is a normal band.*

LEMMA 3.6. *If $e, f \in E$ and $s \in \rho(S)$, then $esef = esf$.*

*Proof.* Let $g \in E \cap H_{es}$. Then

$$\begin{aligned} esef &= esggef = esgegf \quad \text{by normality of } E \\ &= esggf \quad \text{since } g \mathscr{R} es \\ &= esf. \end{aligned}$$

LEMMA 3.7. *If $e \in E$ and $s, t \in \rho(S)$, then $este = esete$.*

*Proof.* Let $g \in E \cap H_{es}$, $h \in E \cap H_{te}$. Then

$$\begin{aligned} este &= esgeghte \quad \text{since } eg = g \\ &= esggehte \quad \text{by normality} \\ &= esete. \end{aligned}$$

To complete the proof of Theorem 3.1, we only need to quote Theorem 3.9(iv) [**1**].

REMARK 3.8. The reader should consult Yamada [**6**] for further results on regular permutative semigroups.

**4. Main result.** We are now ready to establish our main theorem.

THEOREM 4.1. *A pseudovariety of semigroups is permutative if and only if it does not contain any of the following semigroups*:

$$G_p,\ H_{p,q},\ N^1,\ B(1,2)^1,\ B(2,1)^1,\ K_p,\ Y,\ Q \tag{5}$$

($p, q$ *distinct primes*).

*Moreover, the pseudovarieties generated by each of these semigroups are minimal non-permutative and admit the corresponding semigroup in* (5) *as a generator of minimum size.*

To prove Theorem 4.1, all we really need to show is that if $\underline{V}$ is a pseudovariety which does not contain any of the semigroups in (5), then $\underline{V}$ is permutative. We establish this in several steps. By Theorem 5.3 [**1**], all monoids in $\underline{V}$ are commutative. Henceforth, $S$ denotes a given element of $\underline{V}$ and we may assume that $\underline{V}(S) = \underline{V}$. We suppose $S$ is not permutative. Also, let $E = E(S)$.

LEMMA 4.2. *If $S \not\models x^\omega yzx^\omega = x^\omega yx^\omega zx^\omega$, then $S \models x^\omega yx^\omega = (x^\omega yx^\omega)^{k+1}$ for some $k > 0$.*

*Proof.* Since $Q \notin \underline{V}$, there exists an identity $u = v$ such that $S \models u = v$ while $Q \not\models u = v$. Hence, $u = v$ satisfies one of the conditions (i)-(iv) of Lemma 2.1.

If (i) holds, then $\underline{V}$ does not contain any non-trivial semilattices. Since $\rho(S)$ is a union of groups subsemigroup of $S$ and $S$ is finite, it follows that $S$ is a nilpotent extension of a completely simple orthodox union of abelian groups. In particular, $S$ is permutative, contradicting our initial assumption.

If (ii) holds, say the variable $y$ satisfies (ii), then substitute $x^\omega$ for every variable in $u = v$ other than $y$ and pre- and post-multiply by $x^\omega$ to obtain a pseudo-identity of the form $x^\omega yx^\omega = x^\omega y^{k_1} x^\omega y^{k_2} \cdots x^\omega y^{k_r} x^\omega$ with $\Sigma_{i=1}^r k_i > 1$ which holds in $S$. Then, substituting $x^\omega yx^\omega$ for $y$, we obtain

$$S \models x^\omega yx^\omega = (x^\omega yx^\omega)^{k+1} \quad \text{for some } k > 0.$$

Suppose that (iii) holds. Substitute $x^\omega$ for every variable in $u = v$ other than $y$, $z$ to obtain a pseudo-identity $x^\omega yzx^\omega = x^\omega zyx^\omega$ or $x^\omega yzx^\omega = x^\omega yx^\omega zx^\omega$ which holds in $S$, contradicting the hypothesis of the lemma.

Finally, suppose that (iv) holds, say $y$ is the rightmost variable in $u$ satisfying the "right" option of (iv) and suppose that (i), (ii) and (iii) fail. Upon substitution of $x^\omega$ for every variable in $u$ other than $y$, we obtain $S \models x^\omega y = x^\omega yx^\omega$, again in contradiction with the assumption of the lemma.

LEMMA 4.3. *If $e \in E$ and $s, t \in S$, then $este = esete$.*

*Proof.* Suppose that $este \neq esete$. Then, by Lemma 4.2, we have

$$S \models x^\omega yx^\omega = (x^\omega yx^\omega)^{k+1} \quad \text{for some } k \geq 2. \tag{6}$$

We claim that this leads to a contradiction. We may assume that $\{e, s, t\}$ generates $S$ and that $s = es$ and $t = te$.

By (6), every element of $S$ which lies in a submonoid of $S$ also lies in a subgroup of $S$. In particular, $se = (se)^{k+1} = s^{k+1}e$, and so $se \mathscr{R} s^2$. Similarly, $t^2 \mathscr{L} et$.

Since $st = (st)^{k+1} = ststst \cdots st$, we have $st \mathscr{J} (ts)^2$ and so $tst \mathscr{L} st \mathscr{R} sts$. From Theorem 3.1, we deduce that $sts^2 t \mathscr{J} st$, whence $s^2 t \mathscr{J} st \mathscr{J} stse$, and so $st \mathscr{J} stse \cdot tst$, so that $st \leq_{\mathscr{J}} set$.

On the other hand,

$$\begin{aligned} set &= (set)^{k+1} && \text{by (6)} \\ &= set^k es^k et && \text{since } eSe \text{ is commutative} \\ &= ses^k t^k et && \text{by Theorem 3.1, since } se, s^k, t^k, et \in \rho(S) \\ &= stt^{k-1} s^k et = set^k s^{k-1} st && \text{since } eSe \text{ is commutative.} \end{aligned}$$

Hence, $set \mathscr{H} st$, so that, by (6), $(st)^k = (set)^k$. Thus,

$$\begin{aligned} este = st = st(st)^k &= st(set)^k \\ &= este(se)^{k-1}(et)^{k-1}esete = e(se)^{k-1}st(et)^{k-1}set = es^k t^k sete \\ &\qquad \text{since } eSe \text{ is commutative} \\ &= es^k sett^k e \quad \text{by Theorem 3.1, since } s^k,\ t^k,\ set \in \rho(s) \\ &= es^{k+1}et^{k+1}e = (ese)^{k+1}(ete)^{k+1} = esete \quad \text{by (6).} \end{aligned}$$

This contradicts the initial assumption in this proof, and thus the lemma is established.

LEMMA 4.4. *If $e, f \in E$ and $s \in S$, then $esf = esef$.*

*Proof.* Suppose that $esf \neq esef$. We will then reach a contradiction. Here, we may assume that $S$ is generated by $\{e, f, s\}$ and that $s = es$.

Using Lemma 4.3, we have $sef = esefef = esfef = sfef$. Since $sef \neq sf$, we get $sf \neq sfef$ so that we may replace $s$ by $sf$ and still have $e(sf)f \neq e(sf)ef$. Hence, we may assume that $s = sf$. By Theorem 3.1, we may also assume that $s$ is not regular, and so $s <_{\mathscr{J}} e, f$.

Let $I = \{x \in S: x \not\leq_{\mathscr{J}} x\}$. Note that $I$ is an ideal of $S$.

Suppose $s \leq_{\mathscr{J}} ef$. Then, there exist $u, v \in S$ such that $s = esf = euefvf$. If $u \not\leq_{\mathscr{J}} s$, then

$$\begin{aligned} s = esf &= efesf \\ &= efesef \quad \text{by Lemma 4.3} \\ &= esefef \quad \text{since } eSe \text{ is commutative} \\ &= esef, \quad \text{a contradiction.} \end{aligned}$$

Hence $u \leq_{\mathscr{J}} s$. It follows by Lemma 4.3 that $s \leq_{\mathscr{J}} sef$ and so $s \mathscr{J} sef$. Therefore, $s \mathscr{R} sef$, so that there exists $w \in S$ such that $s = sefwf$. If $w \leq_{\mathscr{J}} s$, then $s$ is regular, contradicting a previous assumption. Else, $s = sefef = sef$, again a contradiction. Hence, $ef \in I$, whence $sef \in I$ and we may assume $I = \{0\}$.

If $s \leq_{\mathscr{J}} fe$, then there exist $u, v \in S$ such that $s = es = eufev = euefev$, the last step because of Lemma 4.3. Since $ef = 0$, we conclude that $s = 0$, contradicting the definition of $I$. Hence, $fe = 0$.

Finally, $se = sfe = 0 = fes = fs$. Hence, $S \simeq Y$ and $Y \in \underline{V}$, a contradiction. This completes the proof of Lemma 4.4.

To finish the proof of Theorem 4.1, we just need to notice that, by Theorem 5.3 [**1**] and Lemmas 4.3 and 4.4, the conditions of Theorem 3.9(iv) [**1**] are satisfied for any $S \in \underline{V}$. Hence, $\underline{V}$ is permutative.

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