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**CHARACTER IDENTITIES AND ASYMPTOTIC BEHAVIOR OF
MATRIX COEFFICIENTS OF DISCRETE SERIES**

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Using Schmid's character identities we give a new proof of a result of Trombi and Varadarajan which describes a necessary condition for a discrete series representation of a semisimple Lie group to have a specific asymptotic decay.

In this note we want to explain a simple relationship between Schmid's character identities [7] and the results of Trombi and Varadarajan on the rate of decay of the K -finite matrix coefficients of the discrete series [8].

Let G be a connected semisimple Lie group with finite center. We assume that G has nonempty discrete series, therefore by a result of Harish-Chandra it has a compact Cartan subgroup H . Let K be a maximal compact subgroup of G containing H . Let \mathfrak{g}_0 , \mathfrak{k}_0 and \mathfrak{h}_0 be the Lie algebras of G , K and H respectively, and \mathfrak{g} , \mathfrak{k} and \mathfrak{h} their complexifications. Denote by Φ the root system of $(\mathfrak{g}, \mathfrak{h})$. A root $\alpha \in \Phi$ is called *compact* if its root subspace is contained in \mathfrak{k} and *noncompact* otherwise. Let W be the Weyl group of $(\mathfrak{g}, \mathfrak{h})$ and W_k its subgroup generated by the reflections with respect to the compact roots. The Killing form of \mathfrak{g} induces an inner product (\mid) on $i\mathfrak{h}_0^*$, the space of all linear forms on \mathfrak{h} which assume imaginary values on \mathfrak{h}_0 . An element λ of $i\mathfrak{h}_0^*$ is singular if it is orthogonal to at least one root of Φ and nonsingular otherwise. The differentials of the characters of H form a lattice Λ in $i\mathfrak{h}_0^*$. Let ρ be the half-sum of positive roots in Φ with respect to some ordering on $i\mathfrak{h}_0^*$. Then $\Lambda + \rho$ is independent of the choice of this ordering.

Harish-Chandra has shown that to each nonsingular $\lambda \in \Lambda + \rho$ we can attach a discrete series representation π_λ , π_λ is equal to π_μ if and only if λ and μ are conjugate under W_k , and the discrete series are exhausted in this way ([3], [9, 10.2.4.1]).

Choose an Iwasawa decomposition $G = KAN$ of G . Let \mathfrak{a}_0 and \mathfrak{n}_0 be the Lie algebras of A and N respectively, and \mathfrak{a} and \mathfrak{n} their complexifications. Let Σ be the root system of $(\mathfrak{g}, \mathfrak{a})$. Then \mathfrak{n} determines a set of positive roots Σ^+ and its negative Weyl chamber A^- in A . Let δ be the modular function of the group AN ; it is evidently rapidly decreasing on A^- .

Put

$$k(\alpha) = \frac{1}{4} \sum_{\beta \in \Phi} |(\alpha|\beta)|, \quad \alpha \in \Phi.$$

By the Cartan decomposition of G we know that the growth of K -finite matrix coefficients of a representation at infinity on G is completely determined by their behavior on $\text{Cl}(A^-)$. Trombi and Varadarajan proved in [8]:

THEOREM. *For a nonsingular $\lambda \in \Lambda + \rho$ and $\kappa > 0$, the assertion (i) for every K -finite matrix coefficient c of the discrete series representation π_λ there exist $C, r \geq 0$ such that*

$$|c(a)| \leq C\delta^{(1+\kappa)/2}(a)(1 + \|\log a\|)^r$$

for all $a \in A^-$; implies

(ii) $|(\lambda|\alpha)| \geq \kappa k(\alpha)$ *for every noncompact root $\alpha \in \Phi$.*

In [5] one of us proved that the converse holds, i.e. (i) and (ii) are equivalent.

The original argument of Trombi and Varadarajan is based on a long and complicated analysis of differential equations satisfied by the matrix coefficients and the argument in [5] to prove (i) \Rightarrow (ii) is based on the Langlands classification of representations.

Therefore, it seems worthwhile to point out the real simplicity of this result—it is more or less just a formal consequence of the character identities. Roughly speaking, the growth estimate (i) is the best possible estimate which does not contradict the character identities involving the character of π_λ (as was remarked in [7], these identities can be deduced from Harish-Chandra's matching conditions, which were also used in [8] to determine the character of π_λ on Cartan subgroups with one dimensional split part). We hope that our argument clarifies the meaning of the condition (ii).

In the first section we interpret the character identities as a statement about an imbedding of π_λ into a principal series representation corresponding to a maximal parabolic subgroup. Evidently this is just a part of information one can extract from the character identities, but we want to stress the fact that this part follows easily. In the second section we give a growth estimate for the matrix coefficients of this principal series representation and show, via an orthogonality argument, that it contradicts any growth condition better than (i).

Finally, let us remark that the companion result to the theorem (for $1 \leq p < 2$, π_λ is p -integrable implies that $|\langle \lambda | \alpha \rangle| > (2/p - 1)k(\alpha)$ for all noncompact roots $\alpha \in \Phi$ [8]) follows immediately from it using 8.11 and 8.14 of [2].

1. Representation-theoretic interpretation of the character identities.

Let Ψ be a system of positive roots in Φ . In [7] Schmid constructed a “coherent” family of eigendistributions $\Theta(\Psi, \lambda)$, $\lambda \in \Lambda + \rho$, which “extends” characters of discrete series. More precisely:

- (i) If Z belongs to the center $\mathcal{Z}(\mathfrak{g})$ of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of \mathfrak{g} , then

$$Z\Theta(\Psi, \lambda) = \chi_\lambda(Z)\Theta(\Psi, \lambda)$$

where χ_λ is the character of $\mathcal{Z}(\mathfrak{g})$ corresponding to $\lambda \in \mathfrak{h}^*$ via the (twisted) Harish-Chandra isomorphism of $\mathcal{Z}(\mathfrak{g})$ onto the Weyl group invariants in $\mathcal{U}(\mathfrak{h})$.

- (ii) Let φ be the character of an irreducible finite-dimensional representation τ of G . Then

$$\varphi \cdot \Theta(\Psi, \lambda) = \sum_{\nu} \Theta(\Psi, \lambda + \nu)$$

where the sum is taken over the set of all weights of τ , counted with multiplicities.

- (iii) If λ is nonsingular and dominant with respect to Ψ then $\Theta(\Psi, \lambda)$ is the character of π_λ .

The character identities relate the eigendistributions $\Theta(\Psi, \lambda)$ corresponding to different positive root systems Ψ . Fix a simple noncompact root $\alpha \in \Psi$. We decompose \mathfrak{h} as

$$\mathfrak{h} = \mathfrak{c} \oplus \mathbb{C}H_\alpha$$

where $\mathfrak{c} = \ker \alpha$ and $H_\alpha \in \mathfrak{h}$ is the dual root of α . We attach to α the Cayley transform c_α [7]. It is an inner automorphism of \mathfrak{g} which acts as an identity on \mathfrak{c} and maps H_α into $\tilde{H}_\alpha \in \mathfrak{g}_0$. Thus

$$c_\alpha(\mathfrak{h}) = \mathfrak{c} \oplus \mathbb{C}\tilde{H}_\alpha$$

is the complexification of a real Cartan subalgebra of \mathfrak{g}_0 , with one-dimensional split part. Let P_α be the corresponding maximal cuspidal parabolic subgroup of G and

$$P_\alpha = M_\alpha A_\alpha N_\alpha$$

its Langlands decomposition. Denote by $\mathfrak{m}_\alpha, \mathfrak{a}_\alpha, \mathfrak{n}_\alpha$ the complexified Lie algebras of $M_\alpha, A_\alpha, N_\alpha$, respectively. Then $H \cap M_\alpha$ becomes a compact Cartan subgroup of M_α and \mathfrak{c} is its complexified Lie algebra. The root system of $(\mathfrak{m}_\alpha, \mathfrak{c})$ can be identified with

$$\Phi_\alpha = \{ \beta \in \Phi \mid (\alpha|\beta) = 0 \}.$$

In particular, $\Psi_\alpha = \Psi \cap \Phi_\alpha$ is a system of positive roots in Φ_α . Let $\lambda_\alpha = \lambda|_{\mathfrak{c}}$. Then Ψ_α and λ_α determine an eigendistribution $\Theta_{M_\alpha^0}(\Psi_\alpha, \lambda_\alpha)$ on the identity component M_α^0 of M_α , which in turn (as explained in [7]) determines an eigendistribution $\Theta_{M_\alpha}(\Psi_\alpha, \lambda_\alpha)$ on M_α . On the other hand, λ determines a linear form ν on $\mathfrak{a}_\alpha = \mathbb{C}\tilde{H}_\alpha$ by $\nu(\tilde{H}_\alpha) = \lambda(H_\alpha)$. Now, by the normalized induction, $\Theta_{M_\alpha}(\Psi_\alpha, \lambda_\alpha)$ and ν determine an eigendistribution $\Theta_{\Psi, \alpha, \lambda}$ on G .

The character identity we shall use asserts that

$$(1.1) \quad \Theta(\Psi, \lambda) + \Theta(s_\alpha \Psi, \lambda) = \Theta_{\Psi, \alpha, \lambda}.$$

The eigendistributions $\Theta(\Psi, \lambda)$ are, in general, only virtual characters; and, a priori, it is difficult to attach a representation-theoretic meaning to (1.1). However, in the special case we shall need, the situation turns out to be much nicer. Assume that λ is nonsingular and dominant with respect to Ψ . Then λ_α is nonsingular and dominant with respect to Ψ_α and $\Theta(\Psi, \lambda), \Theta_{M_\alpha}(\Psi_\alpha, \lambda_\alpha)$ are the characters of π_λ and a discrete series representation σ of M_α , respectively. The eigendistribution $\Theta_{\Psi, \alpha, \lambda}$ becomes therefore the character of the induced representation $\text{Ind}_{P_\alpha}^G(\sigma, \nu)$.

If

$$\rho_\alpha(H) = \frac{1}{2} \text{tr}(\text{ad } H|_{\mathfrak{n}_\alpha}), \quad H \in \mathfrak{a}_\alpha,$$

then

$$\rho_\alpha(\tilde{H}_\alpha) = \frac{1}{4} \sum_{\beta \in \Phi} |\beta(H_\alpha)| = \frac{1}{4} \sum_{\beta \in \Phi} \left| \frac{2(\alpha|\beta)}{(\alpha|\alpha)} \right| = \frac{2k(\alpha)}{(\alpha|\alpha)}.$$

Therefore, if we put

$$t = \frac{(\lambda|\alpha)}{k(\alpha)}$$

we have

$$\nu = t\rho_\alpha.$$

The interpretation of (1.1), we alluded to in the Introduction, is contained in the following proposition.

1.1. PROPOSITION. *Let $\lambda \in \Lambda + \rho$ be nonsingular and dominant with respect to Ψ . Then the discrete series representation π_λ is infinitesimally equivalent to a subquotient of $\text{Ind}_{P_\alpha}^G(\sigma, \tau_{\rho_\alpha})$.*

A more complete representation-theoretic explanation of the character identity (1.1) was given by Schmid [6]; however, 1.1 is all we need in this paper, and we give a simple proof of it below. Since in this case $\Theta(\Psi, \lambda)$ and $\Theta_{\Psi, \alpha, \lambda}$ are characters, it is enough to show the following lemma.

1.2. LEMMA. *Let $\lambda \in \Lambda + \rho$ be dominant and nonsingular with respect to Ψ . Then $\Theta(s_\alpha \Psi, \lambda)$ is a character.*

Proof. Let w_0 be the longest element in W . Let μ be the highest weight of an irreducible finite-dimensional representation τ_μ of G such that $s_\alpha \lambda + \mu$ is dominant. Then the contragredient representation $\tilde{\tau}_\mu$ of τ_μ has the highest weight $-w_0 \mu$. Let φ be the character of $\tilde{\tau}_\mu$.

Now we can compute the character of the component of $\tilde{\tau}_\mu \otimes \pi_{\lambda + s_\alpha \mu}$ with generalized infinitesimal character χ_λ ; it is equal to

$$[\varphi \cdot \Theta(s_\alpha \Psi, \lambda + s_\alpha \mu)]_{(\lambda)} = \sum \Theta(s_\alpha \Psi, \lambda + s_\alpha \mu + \nu)$$

where the sum is taken over all weights ν of $\tilde{\tau}_\mu$ such that

$$\lambda + s_\alpha \mu + \nu = w\lambda$$

for some $w \in W$, counting multiplicities. This implies that

$$w\lambda - \lambda = s_\alpha(\mu + s_\alpha \nu)$$

and, $-\mu$ being the lowest weight of $\tilde{\tau}_\mu$,

$$s_\alpha(w\lambda - \lambda) = \mu + s_\alpha \nu$$

is a sum of positive roots. But λ being dominant, $w\lambda - \lambda$ is a combination of simple roots with negative coefficients, therefore we conclude that

$$s_\alpha(w\lambda - \lambda) = \mu + s_\alpha \nu = n_\alpha \alpha$$

where $n_\alpha \in \mathbb{Z}_+$. Now,

$$\|\lambda\|^2 = \|s_\alpha w\lambda\|^2 = \|s_\alpha \lambda + n_\alpha \alpha\|^2 = \|\lambda\|^2 - 2n_\alpha(\lambda|\alpha) + n_\alpha^2 \|\alpha\|^2,$$

which implies that either

$$n_\alpha = 0 \quad \text{or} \quad n_\alpha = \frac{2(\lambda|\alpha)}{(\alpha|\alpha)}.$$

Assume first that

$$\frac{2(\lambda|\alpha)}{(\alpha|\alpha)} \notin \mathbb{Z}.$$

Then the only possibility is $n_\alpha = 0$, which implies $\nu = -s_\alpha\mu$ and $w = 1$ by the nonsingularity of λ . Therefore

$$[\varphi \cdot \Theta(s_\alpha\Psi, \lambda + s_\alpha\mu)]_{(\lambda)} = \Theta(s_\alpha\Psi, \lambda)$$

and $\Theta(s_\alpha\Psi, \lambda)$ is a character in this case.

Now assume that

$$\frac{2(\lambda|\alpha)}{(\alpha|\alpha)} \in \mathbb{Z}.$$

Then $s_\alpha\lambda \in \Lambda + \rho$. For $n_\alpha \neq 0$, we have

$$s_\alpha w\lambda = s_\alpha\lambda + n_\alpha\alpha = \lambda,$$

so by the nonsingularity of λ , it follows that $w = s_\alpha$. Also

$$\begin{aligned} \nu &= -s_\alpha\mu - n_\alpha\alpha = -\mu + \frac{2(\mu|\alpha)}{(\alpha|\alpha)}\alpha - \frac{2(\lambda|\alpha)}{(\alpha|\alpha)}\alpha \\ &= -\mu + \frac{2(\mu - \lambda|\alpha)}{(\alpha|\alpha)}\alpha. \end{aligned}$$

By our assumption $s_\alpha\lambda + \mu$ is dominant, therefore,

$$0 \leq (s_\alpha\lambda + \mu|\alpha) = -(\lambda|\alpha) + (\mu|\alpha) = (\mu - \lambda|\alpha) \leq (\mu|\alpha),$$

which implies that ν is a weight of $\tilde{\tau}_\mu$ lying between $-\mu$ and $-s_\alpha\mu$. Hence the multiplicity of ν is equal to one.

Therefore, in this case,

$$[\varphi \cdot \Theta(s_\alpha\Psi, \lambda + s_\alpha\mu)]_{(\lambda)} = \Theta(s_\alpha\Psi, \lambda) + \Theta(s_\alpha\Psi, s_\alpha\lambda).$$

Thus, $\Theta(s_\alpha\Psi, \lambda)$ is a difference of a character and a character of the discrete series representation $\pi_{s_\alpha\lambda}$. The character identity (1.1) implies that it is a difference of a character and the character of π_λ . The fact that π_λ and $\pi_{s_\alpha\lambda}$ are not equivalent and the linear independence of irreducible characters show that $\Theta(s_\alpha\Psi, \lambda)$ is a character. \square

2. Asymptotic behavior of discrete series representations. Now we want to show how the results of the previous section imply the theorem we stated in the Introduction.

Let π_λ be a discrete series representation, Ψ a system of positive roots in Φ such that λ is dominant with respect to it and $\alpha \in \Psi$ a simple noncompact root. By 1.1 we know that there exists a discrete series representation σ of M_α such that π_λ is infinitesimally equivalent to a subquotient of $\text{Ind}_P^G(\sigma, t\rho_\alpha)$.

First we need a rough estimate for the K -finite matrix coefficients of $\text{Ind}_{P_\alpha}^G(\sigma, s\rho_\alpha)$, $s \geq 0$, which is analogous to 3.8 in [4]. We include a proof for the sake of completeness.

Let $K_\alpha = K \cap P_\alpha$ be a maximal compact subgroup of M_α . The restriction to K defines an isometric isomorphism of $\text{Ind}_{P_\alpha}^G(\sigma, s\rho_\alpha)$ onto $\text{Ind}_{K_\alpha}^K(\sigma|_{K_\alpha})$ as unitary representations of K for all $s \in \mathbb{C}$, so we can view all representations $\text{Ind}_{P_\alpha}^G(\sigma, s\rho_\alpha)$ as acting on the same space.

Let P be a minimal parabolic subgroup of G contained in P_α and $P = MAN$ its Langlands decomposition.

2.1. LEMMA. *Let v be a K -finite vector in $\text{Ind}_{P_\alpha}^G(\sigma, s\rho_\alpha)$, \tilde{v} a K -finite linear form on $\text{Ind}_{P_\alpha}^G(\sigma, s\rho_\alpha)$ and $c_{v,\tilde{v}}^s$ the corresponding matrix coefficient. Then,*

- (i) *the function $(s, x) \rightarrow c_{v,\tilde{v}}^s(x)$ is continuous on $\mathbb{C} \times G$;*
- (ii) *if $s_0 \geq 0$, then there exist constants $C, r \geq 0$, depending on v, \tilde{v} and s_0 only, such that*

$$|c_{v,\tilde{v}}^s(a)| \leq C\delta^{(1-s_0)/2}(a)(1 + \|\log a\|)^r$$

for all $a \in A^-$ and $0 \leq s \leq s_0$.

Proof. By the Iwasawa decomposition, there exist unique smooth maps $\kappa: G \rightarrow K$ and $H: G \rightarrow \mathfrak{a}$ such that

$$x \in \kappa(x)\exp H(x) \cdot N$$

for all $x \in G$. The subgroup ${}^*P = M_\alpha \cap P$ is a minimal parabolic subgroup of M_α . Let ${}^*P = M{}^*A{}^*N$ be its Langlands decomposition. Denote by ${}^*\mathfrak{a}$ the complexified Lie algebra of *A . Then $\mathfrak{a} = {}^*\mathfrak{a} + \mathfrak{a}_\alpha$. Let ${}^*H: G \rightarrow {}^*\mathfrak{a}$ and $H_\alpha: G \rightarrow \mathfrak{a}_\alpha$ be such smooth maps that $H = {}^*H + H_\alpha$. The Iwasawa decomposition $M_\alpha = K_\alpha{}^*A{}^*N$ allows us to define a unique smooth map $\mu: G \rightarrow {}^*A{}^*N$ such that

$$x \in \kappa(x)\mu(x)\exp H_\alpha(x) \cdot N_\alpha$$

for all $x \in G$. Evidently, for $m \in M_\alpha$, we have

$$(2.1) \quad \mu(xm) = \mu(\mu(x)m)$$

and

$$(2.2) \quad H_\alpha(xm) = H_\alpha(x)$$

for all $x \in G$. Also, $\mu(x) \in \exp{}^*H(x){}^*N$, so

$${}^*H(\mu(x)) = {}^*H(x);$$

and by (2.1)

$$(2.3) \quad *H(\mu(x)m) = *H(\mu(\mu(x)m)) = *H(\mu(xm)) = *H(xm)$$

for all $x \in G$ and $m \in M_\alpha$.

Let $f, g \in \text{Ind}_{K_\alpha}^K(\sigma|K_\alpha)$ be such that f corresponds to v and g defines the linear form \tilde{v} . Then, f and g are K -finite. Assume that (f_1, f_2, \dots, f_p) and (g_1, g_2, \dots, g_q) are bases of the finite-dimensional subspaces spanned by K -orbits of f and g respectively. Then

$$f(kh) = \sum_{i=1}^p a_i(h) f_i(k), \quad h, k \in K;$$

and

$$g(kh) = \sum_{j=1}^q b_j(h) g_j(k), \quad h, k \in K;$$

where $a_i, 1 \leq i \leq p$, and $b_j, 1 \leq j \leq q$, are smooth functions on K . In particular,

$$f(k) = \sum_{i=1}^p a_i(k) f_i(1),$$

$$g(k) = \sum_{j=1}^q b_j(k) g_j(1),$$

for $k \in K$, and $f_i(1), 1 \leq i \leq p, g_j(1), 1 \leq j \leq q$, are K_α -finite.

A short calculation shows that

$$c_{v, \tilde{v}}^s(x) = \int_K e^{-(s+1)\rho_\alpha(H_\alpha(x^{-1}k))} \left(\sigma(\mu(x^{-1}k)^{-1}) f(\kappa(x^{-1}k)^{-1}) |g(k^{-1}) \right) dk$$

for all $x \in G$ and $s \in \mathbb{C}$, which immediately implies (i); and therefore for real s

$$|c_{v, \tilde{v}}^s(x)| \leq \sum_{i=1}^p \sum_{j=1}^q \left| a_i(\kappa(x^{-1}k)^{-1}) \right| \left| b_j(k^{-1}) \right|$$

$$\cdot \int_K e^{-(s+1)\rho_\alpha(H_\alpha(x^{-1}k))} \left| \left(\sigma(\mu(x^{-1}k)^{-1}) f_i(1) |g_j(1) \right) \right| dk$$

for $x \in G$.

Let Ξ, Ξ_α be the zonal spherical functions on G, M_α corresponding to the zero linear forms on $\mathfrak{a}, * \mathfrak{a}$, respectively. Then, by a result of Harish-Chandra ([3], [9, 9.3.1.5]; this also follows directly from [2, 7.7] and [9, 8.3.7.3]), any K -finite matrix coefficient of σ is majorized by a

multiple of Ξ_α . Therefore

$$|c_{\bar{v}, \bar{v}}^s(x)| \leq c_1 \cdot \int_K e^{-(s+1)\rho_\alpha(H_\alpha(x^{-1}k))} \Xi_\alpha(\mu(x^{-1}k)^{-1}) dk$$

for all $x \in G$.

Let

$$*\rho(H) = \frac{1}{2} \text{tr}(\text{ad } H | *\mathfrak{n}), \quad H \in *\mathfrak{a};$$

and

$$\rho(H) = \frac{1}{2} \text{tr}(\text{ad } H | \mathfrak{n}), \quad H \in \mathfrak{a};$$

where $*\mathfrak{n}$ and \mathfrak{n} are the complexified Lie algebras of $*N$ and N , respectively. Considering $*\rho$ and ρ_α as linear forms on \mathfrak{a} equal to zero on α_α and $*\mathfrak{a}$, respectively, we see that $\rho = *\rho + \rho_\alpha$.

By [9, 6.2.2.1], we have

$$\Xi_\alpha(y^{-1}) = \Xi_\alpha(y) = \int_{K_\alpha} e^{-*\rho(*H(yh))} dh;$$

and using (2.2) and (2.3) it follows that

$$\begin{aligned} |c_{\bar{v}, \bar{v}}^s(x)| &\leq c_1 \int_K e^{-(s+1)\rho_\alpha(H_\alpha(x^{-1}k))} \left(\int_{K_\alpha} e^{-*\rho(*H(\mu(x^{-1}k)h))} dh \right) dk \\ &= c_1 \int_K e^{-(s\rho_\alpha + \rho)(H(x^{-1}k))} dk \end{aligned}$$

for all $x \in G$.

Let w_0 be the longest element in the Weyl group of $(\mathfrak{g}, \mathfrak{a})$. Then, by [9, 3.3.2.3],

$$\rho(\log a) \leq -(w_0\rho_\alpha)(\log a) \leq \rho_\alpha(H(a^{-1}k))$$

for all $a \in A^-$ and $k \in K$. Hence, for $0 \leq s \leq s_0$, we have

$$\begin{aligned} |c_{\bar{v}, \bar{v}}^s(a)| &\leq c_1 e^{-s\rho(\log a)} \int_K e^{-\rho(H(a^{-1}k))} dk \\ &= c_1 \delta^{-s_0/2}(a) \Xi(a), \end{aligned}$$

for all $a \in A^-$. The assertion (ii) now follows from [9, 8.3.7.4]. □

Assume now that the K -finite matrix coefficients of π_λ satisfy the condition (i) from Introduction for some $\kappa > 0$. Then by 2.1 and the well-known integral formula associated to the Cartan decomposition, the products of K -finite matrix coefficients of π_λ with K -finite matrix coefficients of $\text{Ind}_P^G(\sigma, s\rho_\alpha)$ are integrable on G for all $s < \kappa$.

Now we need a kind of “orthogonality relations” for K -finite matrix coefficients. The next lemma is an “infinitesimal” version of 5.21 in [1].

Let (π_i, V_i) , $1 \leq i \leq 2$, be two admissible representations of (\mathfrak{g}, K) ; $(\tilde{\pi}_i, \tilde{V}_i)$, $1 \leq i \leq 2$, their contragredient representations and $c^{(i)}: \tilde{V}_i \otimes V_i \rightarrow C^\infty(G)$, $1 \leq i \leq 2$, their matrix coefficient maps (see §8 of [2]).

2.2. LEMMA. *Assume that the functions $x \rightarrow c_{v,\tilde{v}}^{(1)}(x)c_{w,\tilde{w}}^{(2)}(x^{-1})$ are integrable on G for all $v \in V_1$, $\tilde{v} \in \tilde{V}_1$, $w \in V_2$, $\tilde{w} \in \tilde{V}_2$; and that $\text{Hom}_{(\mathfrak{g},K)}(V_1, V_2) = 0$. Then*

$$\int_G c_{v,\tilde{v}}^{(1)}(x)c_{w,\tilde{w}}^{(2)}(x^{-1}) dx = 0,$$

for all $v \in V_1$, $\tilde{v} \in \tilde{V}_1$, $w \in V_2$, $\tilde{w} \in \tilde{V}_2$.

Proof. Fix $\tilde{v} \in \tilde{V}_1$ and $w \in V_2$ and define

$$T(v, \tilde{w}) = \int_G c_{v,\tilde{v}}^{(1)}(x)c_{w,\tilde{w}}^{(2)}(x^{-1}) dx,$$

for $v \in V_1$ and $\tilde{w} \in \tilde{V}_2$. We claim that this bilinear form on $V_1 \times \tilde{V}_2$ is (\mathfrak{g}, K) -invariant.

Denote by R, L the right and left regular actions of G and \mathfrak{g} on $C^\infty(G)$ respectively.

Let $X \in \mathfrak{g}$. Then

$$\begin{aligned} & \frac{d}{dt} \left(c_{v,\tilde{v}}^{(1)}(x \exp tX) c_{w,\tilde{w}}^{(2)}(\exp(-tX)x^{-1}) \right) \\ &= (R_X c_{v,\tilde{v}}^{(1)})(x \exp tX) c_{w,\tilde{w}}^{(2)}(\exp(-tX)x^{-1}) \\ & \quad + c_{v,\tilde{v}}^{(1)}(x \exp tX) (L_X c_{w,\tilde{w}}^{(2)})(\exp(-tX)x^{-1}) \\ &= c_{\pi_1(X)v,\tilde{v}}^{(1)}(x \exp tX) c_{w,\tilde{w}}^{(2)}(\exp(-tX)x^{-1}) \\ & \quad + c_{v,\tilde{v}}^{(1)}(x \exp tX) c_{w,\tilde{\pi}_2(X)\tilde{w}}^{(2)}(\exp(-tX)x^{-1}), \end{aligned}$$

so by the assumption,

$$\int_G \left| \frac{d}{dt} \left(c_{v,\tilde{v}}^{(1)}(x \exp tX) c_{w,\tilde{w}}^{(2)}(\exp(-tX)x^{-1}) \right) \right| dx < +\infty;$$

and by the invariance of the Haar measure it is independent of $t \in \mathbb{R}$. By Fubini's theorem and the invariance of Haar measure

$$\begin{aligned} 0 &= \int_G \left[\int_0^1 \frac{d}{dt} \left(c_{v,\tilde{v}}^{(1)}(x \exp tX) c_{w,\tilde{w}}^{(2)}(\exp(-tX)x^{-1}) \right) dt \right] dx \\ &= \int_0^1 \left[\int_G c_{\pi_1(X)v,\tilde{v}}^{(1)}(x \exp tX) c_{w,\tilde{w}}^{(2)}(\exp(-tX)x^{-1}) dx \right. \\ & \quad \left. + \int_G c_{v,\tilde{v}}^{(1)}(x \exp tX) c_{w,\tilde{\pi}_2(X)\tilde{w}}^{(2)}(\exp(-tX)x^{-1}) dx \right] dt \\ &= T(\pi_1(X)v, \tilde{w}) + T(v, \tilde{\pi}_2(X)\tilde{w}); \end{aligned}$$

so

$$T(\pi_1(X)v, \tilde{w}) = -T(v, \tilde{\pi}_2(X)\tilde{w})$$

for all $X \in \mathfrak{g}$, $v \in V_1$ and $\tilde{w} \in \tilde{V}_2$; i.e. T is \mathfrak{g} -invariant. Also

$$\begin{aligned} T(\pi_1(k)v, \tilde{w}) &= \int_G c_{\pi_1(k)v, \tilde{v}}^{(1)}(x) c_{\tilde{w}, \tilde{w}}^{(2)}(x^{-1}) dx \\ &= \int_G c_{v, \tilde{v}}^{(1)}(xk) c_{\tilde{w}, \tilde{w}}^{(2)}(x^{-1}) dx = \int_G c_{v, \tilde{v}}^{(1)}(x) c_{\tilde{w}, \tilde{w}}^{(2)}(kx^{-1}) dx \\ &= \int_G c_{v, \tilde{v}}^{(1)}(x) c_{\tilde{w}, \tilde{\pi}_2(k^{-1})\tilde{w}}^{(2)}(x^{-1}) dx = T(v, \tilde{\pi}_2(k^{-1})\tilde{w}), \end{aligned}$$

so

$$T(\pi_1(k)v, \tilde{w}) = T(v, \tilde{\pi}_2(k^{-1})\tilde{w})$$

for all $k \in K$, $v \in V_1$ and $\tilde{w} \in \tilde{V}_2$; i.e. T is K -invariant. Therefore the linear map $S: V_1 \rightarrow V_2$ defined by

$$\langle S(v), \tilde{w} \rangle = T(v, \tilde{w})$$

for $v \in V_1$ and $\tilde{w} \in \tilde{V}_2$, is an element of $\text{Hom}_{(\mathfrak{g}, K)}(V_1, V_2)$, and by our assumption equal to zero. \square

It is evident (for example, by considering the infinitesimal characters) that π_λ is not infinitesimally equivalent to a subquotient of $\text{Ind}_{P_\alpha}^G(\sigma, s\rho_\alpha)$ for “almost all” $s < \kappa$. Therefore, 2.1 and 2.2 imply that for any K -finite matrix coefficients c of π_λ and d of $\text{Ind}_{P_\alpha}^G(\sigma, s\rho_\alpha)$, $0 \leq s < \kappa$, we have

$$\int_G c(x)d(x^{-1}) dx = 0.$$

Because of 1.1, this is evidently impossible for $s = t$, so $t \geq \kappa$.

Therefore, if π_λ satisfies the condition (i) we have that

$$(\lambda|\alpha) \geq \kappa k(\alpha)$$

for all simple noncompact roots $\alpha \in \Psi$, what is apparently a condition weaker than (ii). To end the proof of the theorem we need only to establish the following result.

2.3. LEMMA. *Let λ be dominant and regular. If the function*

$$\beta \rightarrow \frac{(\lambda|\beta)}{k(\beta)}$$

on positive noncompact roots attains its minimum at α , then α is a simple root.

Proof. Firstly, it is evident that k is a convex function constant on Weyl group orbits and $k(\gamma)$ depends only on the irreducible component of Φ in which γ lies. This implies, in particular, that if γ and δ are in the same irreducible component and $\|\gamma\| = \|\delta\|$, then $k(\gamma) = k(\delta)$.

Assume that α is not simple. Then there exist $\gamma, \delta \in \Psi$ such that $\alpha = \gamma + \delta$. We can assume that one of them is simple. Also they cannot both be either compact or noncompact, therefore we can assume that γ is noncompact and δ compact. Evidently

$$(\lambda|\alpha) = (\lambda|\gamma) + (\lambda|\delta),$$

which implies

$$(\lambda|\gamma) < (\lambda|\alpha),$$

and by the choice of α

$$\frac{(\lambda|\alpha)}{k(\alpha)} \leq \frac{(\lambda|\gamma)}{k(\gamma)}.$$

Therefore

$$\frac{k(\gamma)}{k(\alpha)} \leq \frac{(\lambda|\gamma)}{(\lambda|\alpha)} < 1,$$

i.e. $k(\gamma) < k(\alpha)$, so by the above remark $\|\gamma\| \neq \|\alpha\|$.

Let S be the δ -string containing γ and α . It cannot have length 2, so its length is either 3 or 4. The reflection s_δ maps S into itself and

$$s_\delta\alpha = s_\delta\gamma - \delta;$$

therefore we have the following two possibilities: either the length of S is 3 and

$$S = \{s_\delta\alpha, \gamma, \alpha\}$$

or the length of S is 4 and

$$S = \{s_\delta\alpha, s_\delta\gamma, \gamma, \alpha\}.$$

Assume that δ is simple. Then S consists of positive noncompact roots and

$$(\lambda|s_\delta\alpha) < (\lambda|\alpha);$$

also $k(s_\delta\alpha) = k(\alpha)$, so

$$\frac{(\lambda|s_\delta\alpha)}{k(s_\delta\alpha)} < \frac{(\lambda|\alpha)}{k(\alpha)}$$

contrary to our assumption.

Assume that γ is simple. Then in both cases $\alpha - 2\delta \notin \Psi$. Therefore

$$(\lambda|\alpha) - 2(\lambda|\delta) < 0$$

and

$$(\lambda|\gamma) = (\lambda|\alpha) - (\lambda|\delta) < \frac{1}{2}(\lambda|\alpha).$$

Also, in both cases $\|\gamma\| = \|\delta\|$ and therefore $k(\gamma) = k(\delta)$, and

$$k(\alpha) \leq k(\gamma) + k(\delta) \leq 2k(\gamma).$$

Finally, this implies

$$\frac{(\lambda|\gamma)}{k(\gamma)} < \frac{(\lambda|\alpha)}{k(\alpha)},$$

which again contradicts our assumption. □

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