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 $\bar{\partial}$ -EQUATION IN THE PRODUCT OF STRICTLY  
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# THE BOUNDARY REGULARITY OF THE SOLUTION OF THE $\bar{\partial}$ -EQUATION IN THE PRODUCT OF STRICTLY PSEUDOCONVEX DOMAINS

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Let  $D$  be a strictly pseudoconvex domain in  $\mathbf{C}^n$ . We prove that for every  $\bar{\partial}$ -closed differential  $(0, q)$ -form  $f$ ,  $q \geq 1$ , with coefficients of class  $\mathcal{C}^\infty(D \times D)$ , and continuous in the set  $\bar{D} \times \bar{D} \setminus \Delta(D)$ , the equation  $\bar{\partial}u = f$  admits a solution  $u$  with the same boundary regularity properties. As an application, we prove that certain ideals of analytic functions in strictly pseudoconvex domains are finitely generated.

**1. Introduction.** Let  $D$  be a bounded strictly pseudoconvex domain in  $\mathbf{C}^n$  with  $\mathcal{C}^2$  boundary. It is known ([2], Theorem 2) that given a  $(0, q)$ -form  $f$  in  $D$  with coefficients of class  $\mathcal{C}^\infty(D \times D)$  and continuous in  $\bar{D} \times \bar{D}$ , such that  $\bar{\partial}f = 0$ ,  $q = 1, \dots, 2n$ , there exists a  $(0, q - 1)$ -form  $u$  in  $D \times D$  such that the coefficients of  $u$  are also of class  $\mathcal{C}^\infty(D \times D)$  and continuous in  $\bar{D} \times \bar{D}$ , and such that  $\bar{\partial}u = f$ .

In this paper, using the results from [2], and the method of [6], we prove the following theorem:

**THEOREM 1.** *Let  $D$  be a bounded strictly pseudoconvex domain in  $\mathbf{C}^n$  with  $\mathcal{C}^2$  boundary. Set  $Q = (\bar{D} \times \bar{D}) \setminus \{(z, z) | z \in \partial D\}$ . Suppose that  $f$  is a  $(0, q)$   $\bar{\partial}$ -closed differential form with coefficients in  $\mathcal{C}^\infty(D \times D) \cap \mathcal{C}(Q)$ . Then there exists a  $(0, q - 1)$ -form  $u$  with coefficients in  $\mathcal{C}^\infty(D \times D) \cap \mathcal{C}(Q)$ , such that  $\bar{\partial}u = f$ .*

As an application, we prove a following theorem on the existence of the decomposition operators in some spaces of holomorphic functions in the product domain  $D \times D$ : Let  $D$  and  $Q$  be as above. Denote by  $A_Q(D \times D)$  the space of all functions holomorphic in  $D \times D$ , which are continuous in  $Q$ . Let  $(A_Q)_0(D \times D)$  be the subspace of  $A_Q(D \times D)$ , consisting of all functions which vanish on  $\Delta(D)$ , the diagonal in  $D \times D$ .

**THEOREM 2.** *Let  $g_1, \dots, g_N \in (A_Q)_0(D \times D)$  satisfy the following properties: (i)  $\{(z, s) \in Q | g_1(z, s) = \dots = g_N(z, s) = 0\} = \Delta(D)$ ; (ii) for every  $z \in D$ , the germs at  $(z, z)$  of the functions  $g_i$ ,  $i = 1, \dots, N$ , generate*

the ideal of germs at  $(z, z)$  of holomorphic functions which vanish on  $\Delta(D)$ . Then for every  $f \in (A_Q)_0(D \times D)$  there exist functions  $f_1, \dots, f_N \in A_Q(D \times D)$ , such that  $f = \sum_{i=1}^N g_i f_i$ .

This theorem is an improvement of several results, obtained previously by different authors. Namely, Ahern and Schneider proved in [1], that if  $f \in A(D)$ , then there exist functions  $f_i(z, s) \in A_Q(D \times D)$ , such that

$$f(z) - f(s) = \sum_{i=1}^n (z_i - s_i) f_i(z, s), \quad z, s \in D.$$

Øvrelid showed in [5], that if  $s \in D$  is fixed and  $g_1, \dots, g_N \in A(D)$  are such that  $\{z \in \bar{D} \mid g_1(z) = \dots = g_N(z) = 0\} = \{s\}$  and the germs of the functions  $g_i$  at  $s$  generate the ideal of germs at  $s$  of holomorphic functions which vanish at  $s$ , then every  $f \in A(D)$  with  $f(s) = 0$  can be written in the form

$$f(z) = \sum_{i=1}^N g_i(z) f_i(z), \quad z \in D,$$

for some  $f_i \in A(D)$ . In [4], the validity of Theorem 2 was shown in the special case, when  $D = U$ —the unit disc in  $\mathbf{C}$ —and under the additional assumption, that there exists a neighborhood  $V$  of  $\Delta(\partial U)$  in  $\bar{U} \times \bar{U}$  such that  $g_1, \dots, g_N$  have no zeros in  $V \cap (Q \setminus \Delta(U))$ . The proof given in [4] is different from that in the present paper.

It could seem unnatural to omit the boundary diagonal  $\Delta(\partial D)$  from study. However, when  $f \in A(D \times D)$  and  $f|_{\Delta(\bar{D})} \equiv 0$ ,  $g_i(z, s) = z_i - s_i$ ,  $i = 1, \dots, n$ , and

$$f(z, s) = \sum_{i=1}^n (z_i - s_i) f_i(z, s),$$

then, as in [1],

$$\frac{\partial f}{\partial z_k}(z, s) = f_k(z, s) + \sum_{i=1}^n (z_i - s_i) \frac{\partial f_i}{\partial z_k}(z, s),$$

and so, setting  $s = z$ , we obtain

$$\frac{\partial f}{\partial z_k}(z, z) = f_k(z, z), \quad z \in D;$$

therefore, the functions  $f_i$  need not be in  $A(D \times D)$ , even if  $f \in A(D \times D)$ . In the sequel we will always assume that the considered domains are bounded. We will also use the following notations:

Given a domain  $D \subset \mathbf{C}^n$ , we denote by  $\mathcal{O}(D)$  the space of holomorphic functions in  $D$ , and by  $A(D)$  the algebra of all functions holomorphic in  $D$  and continuous in  $\bar{D}$ .

If  $F(D)$  is a function space in the domain  $D$ , and  $q = 1, 2, \dots, F_{0,q}(D)$  denotes the space of all differential forms of type  $(0, q)$  with coefficients in  $F(D)$ .

Given a set  $X$ ,  $\Delta(X)$  is a diagonal in the Cartesian product  $X \times X$ .

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**2. The solution of the  $\bar{\partial}$ -equation.** In this section we give the proof of Theorem 1. Let  $D \subset \mathbf{C}^n$  be a strictly pseudoconvex domain, with the defining function  $\sigma$ , i.e.  $\sigma$  is of class  $\mathcal{C}^2$  and strictly plurisubharmonic in some neighborhood  $\tilde{D}$  of  $\bar{D}$ ,  $D = \{z \in \tilde{D} | \sigma(z) < 0\}$ , and  $d\sigma(z) \neq 0$  for  $z \in \partial D$ . For  $\varepsilon > 0$ , set  $\tau_\varepsilon(z, w) = \sigma(z) + \sigma(w) - \varepsilon|z - w|^2$ ,  $(z, w) \in \tilde{D} \times \tilde{D}$ . Then, if  $\varepsilon$  is sufficiently close to zero, the domain  $G_\varepsilon = \{(z, w) \in \tilde{D} \times \tilde{D} | \tau_\varepsilon(z, w) < 0\}$  is strictly pseudoconvex in  $\mathbf{C}^{2n}$  with the defining function  $\tau_\varepsilon$ . Moreover,  $\bar{D} \times \bar{D} \subset \bar{G}_\varepsilon$ , and  $\partial(\bar{D} \times \bar{D}) \cap \partial G_\varepsilon = \Delta(\partial D)$ ; therefore  $Q \subset G_\varepsilon$  (we recall that  $Q = (\bar{D} \times \bar{D}) \setminus \Delta(\partial D)$ ). It follows, that if  $t < 0$  is sufficiently close to 0, the sets  $G_{\varepsilon,t} = \{(z, w) \in \tilde{D} \times \tilde{D} | \tau_\varepsilon(z, w) < t\}$  are strictly pseudoconvex with  $\mathcal{C}^2$  boundary, and  $G_{\varepsilon,t} \subset G_{\varepsilon,t'} \subset G_\varepsilon$  for  $t < t' < 0$ . Set  $E_{\varepsilon,t} = G_{\varepsilon,t} \cap (D \times D)$ .

We want to apply [2], Theorem 2 to the domains  $E_{\varepsilon,t}$ . Note first, that if we define the mappings  $\chi^i: \mathbf{C}^n \times \mathbf{C}^n \rightarrow \mathbf{C}^n$ ,  $i = 1, 2$ , and  $\chi^3: \mathbf{C}^n \times \mathbf{C}^n \rightarrow \mathbf{C}^n \times \mathbf{C}^n$  by  $\chi^1(z, w) = z$ ,  $\chi^2(z, w) = w$ , and  $\chi^3(z, w) = (z, w)$ , and set, for fixed  $\varepsilon > 0$  and  $t < 0$ ,  $\rho_1 = \rho_2 = \sigma$ , and  $\rho_3 = \tau_\varepsilon - t$ , then

$$E_{\varepsilon,t} = \left\{ (z, w) \in \tilde{D} \times \tilde{D} | \rho_i(\chi^i(z, w)) < 0, i = 1, 2, 3 \right\}.$$

Therefore,  $E_{\varepsilon,t}$  is a pseudoconvex polyhedron in the sense of [2]. We must also verify, that  $E_{\varepsilon,t}$  satisfies the assumptions (C) and (CR) from [2], p. 523. Set

$$\text{grad}_{\mathbf{C}} f = {}^t \left( \frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n}, \frac{\partial f}{\partial w_1}, \dots, \frac{\partial f}{\partial w_n} \right)$$

and

$$\text{grad}_{\mathbf{R}} f = {}^t \left( \frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial w_n}, \frac{\partial f}{\partial \bar{z}_1}, \dots, \frac{\partial f}{\partial \bar{w}_n} \right).$$

The condition (C) says, that for every ordered subset  $A \subset \{1, 2, 3\}$ ,  $A = \{\alpha_1, \dots, \alpha_s\}$ , the number  $m_A = \text{rank}(\text{grad}_{\mathbf{C}} \chi^{\alpha_1}, \dots, \text{grad}_{\mathbf{C}} \chi^{\alpha_s})$  is constant in the neighborhood of the set

$$S_A = \left\{ (z, w) \in \partial E_{\varepsilon,t} | \rho_{\alpha_1}(\chi^{\alpha_1}(z, w)) = \dots = \rho_{\alpha_s}(\chi^{\alpha_s}(z, w)) = 0 \right\}.$$

This condition is trivially satisfied, since the mappings  $\chi^i$  are linear. It rests to verify the condition (CR): For every pair of ordered subsets  $A, B \subset \{1, 2, 3\}$ ,  $A = \{\alpha_1, \dots, \alpha_s\}$ ,  $B = \{\beta_1, \dots, \beta_t\}$ , such that for every  $\beta_i \in B$ ,

$$(2.1) \quad \text{rank}(\text{grad}_C \chi^{\beta_i}, \text{grad}_C \chi^{\alpha_1}, \dots, \text{grad}_C \chi^{\alpha_s}) > m_A,$$

it follows that

$$(2.2) \quad \text{rank}(\text{grad}_R(\rho_{\beta_1} \circ \chi^{\beta_1}), \dots, \text{grad}_R(\rho_{\beta_t} \circ \chi^{\beta_t}), \text{grad}_R \chi^{\alpha_1}, \dots, \text{grad}_R \chi^{\alpha_s}, \text{grad}_R \overline{\chi^{\alpha_1}}, \dots, \text{grad}_R \overline{\chi^{\alpha_s}}) = t + 2m_A$$

in a neighborhood of the set  $S_{A \cup B}$ . Note that if  $3 \in A$  or  $A = \{1, 2\}$ , then  $m_A = 2n$ , and hence for any  $\beta$ ,

$$\text{rank}(\text{grad}_C \chi^\beta, \text{grad}_C \chi^{\alpha_1}, \dots, \text{grad}_C \chi^{\alpha_s}) = 2n = m_A,$$

and so (2.1) is not satisfied. On the other hand, if  $A = \{1\}$  or  $A = \{2\}$ , one can show that for every  $B \subset \{1, 2, 3\}$  such that  $A \cap B = \emptyset$ , (2.1) holds. Therefore, in all those cases, we should verify (2.2).

Consider first the case  $A = \{1\}$  and  $B = \{2, 3\}$ . Then

$$S_{123} = \{ (z, w) \in \tilde{D} \times \tilde{D} \mid z, w \in \partial D, -\varepsilon |z - w|^2 = t \},$$

and the matrix

$$(\text{grad}_R(\rho_2 \circ \chi^2), \text{grad}_R(\rho_3 \circ \chi^3), \text{grad}_R \chi^1, \text{grad}_R \overline{\chi^1})$$

at a point  $(z, w) \in \tilde{D} \times \tilde{D}$  has the form

	$\sigma_1(z) - 2\varepsilon(\bar{z}_1 - \bar{w}_1)$	1	$\dots$	
	$\vdots$		$\dots$	
	$\sigma_n(z) - 2\varepsilon(\bar{z}_n - \bar{w}_n)$		1	
$\sigma_1(w)$	$\sigma_1(w) - 2\varepsilon(\bar{w}_1 - \bar{z}_1)$			
$\vdots$	$\vdots$			
$\sigma_n(w)$	$\sigma_n(w) - 2\varepsilon(\bar{w}_n - \bar{z}_n)$			
	$\sigma_{\bar{1}}(z) - 2\varepsilon(z_1 - w_1)$			1
	$\vdots$		$\dots$	
	$\sigma_{\bar{n}}(z) - 2\varepsilon(z_n - w_n)$			1
$\sigma_{\bar{1}}(w)$	$\sigma_{\bar{1}}(w) - 2\varepsilon(w_1 - z_1)$			
$\vdots$	$\vdots$			
$\sigma_{\bar{n}}(w)$	$\sigma_{\bar{n}}(w) - 2\varepsilon(w_n - z_n)$			

where we have set  $\sigma_i = \partial\sigma/\partial\bar{\zeta}_i$  and  $\sigma_{\bar{i}} = \partial\sigma/\partial\bar{\zeta}_{\bar{i}}$ . Since  $z \neq w$  for  $(z, w) \in S_{123}$ , this is true also for some neighborhood of  $S_{123}$ . Moreover,  $(\sigma_1(w), \dots, \sigma_n(w), \sigma_{\bar{1}}(w), \dots, \sigma_{\bar{n}}(w)) \neq 0$  for  $w \in \partial D$ , since  $d\sigma(w) \neq 0$  there. Therefore, in order to prove that the above matrix has rank  $2 + 2m_A = 2 + 2n$ , it is sufficient to show, that the vectors

$$u = (\sigma_1(w), \dots, \sigma_n(w), \sigma_{\bar{1}}(w), \dots, \sigma_{\bar{n}}(w)) = (u_1, \bar{u}_1)$$

and

$$v = (\bar{w}_1 - \bar{z}_1, \dots, \bar{w}_n - \bar{z}_n, w_1 - z_1, \dots, w_n - z_n) = (v_1, \bar{v}_1),$$

are linearly independent (over  $\mathbb{C}$ ) in some neighborhood of  $S_{123}$ . But if  $z, w \in \partial D$  and  $u = \alpha v$  for some  $\alpha \in \mathbb{C}$ ,  $\alpha \neq 0$ , then  $u_1 = \alpha v_1$  and  $\bar{u}_1 = \alpha \bar{v}_1$ . Hence  $\alpha$  is real. Therefore the vectors  $z - w$  and  $v(w) = (\sigma_1(w), \dots, \sigma_{\bar{n}}(w))$  (the normal vector to  $\partial D$  at  $w$ ) are linearly dependent over  $\mathbb{R}$ , as the vectors in  $\mathbb{R}^{2n}$ . This is impossible, if  $z, w \in \partial D$  and  $z$  is sufficiently close to  $w$ , i.e. if  $t$  is sufficiently near 0. Hence, if we choose  $t$  sufficiently close to 0, vectors  $u$  and  $v$  are linearly independent over  $\mathbb{C}$ , for  $(z, w)$  in some neighborhood of  $S_{123}$ , and thus the condition (CR) is satisfied.

In order to prove (2.2) in the case  $A = \{1\}$  and  $B = \{2\}$  (resp.  $B = \{3\}$ ), it suffices to note, that  $S_{12} = \{(z, w) \in \partial E_{\varepsilon, t} | \sigma(z) = \sigma(w) = 0\}$  and  $(\sigma_1(w), \dots, \sigma_n(w), \sigma_{\bar{1}}(w), \dots, \sigma_{\bar{n}}(w)) \neq 0$  for  $w$  in a neighborhood of  $\partial D$  (resp. that since

$$S_{13} = \{(z, w) \in \partial E_{\varepsilon, t} | \sigma(z) = 0, \sigma(w) - \varepsilon|z - w|^2 = t\},$$

then also

$$(\sigma_1(w) - 2\varepsilon(\bar{w}_1 - \bar{z}_1), \dots, \sigma_n(w) - 2\varepsilon(\bar{w}_n - \bar{z}_n), \sigma_{\bar{1}}(w) - 2\varepsilon(w_1 - z_1), \dots, \sigma_{\bar{n}}(w) - 2\varepsilon(w_n - z_n)) \neq 0 \quad \text{for } (z, w)$$

in some neighborhood of  $S_{13}$ , provided that  $\varepsilon$  and  $t$  are sufficiently close to 0).

The verification of the condition (CR) for  $A = \{2\}$  is similar. We obtain therefore the following corollary, which is Theorem 2 from [2] in this special situation:

**COROLLARY 2.1.** *If  $D$  is as above, then there exist  $\varepsilon > 0$  and  $t_0 < 0$  such that for every  $t$  with  $t_0 < t < 0$ , for every  $q = 1, \dots, 2n$ , for every  $f \in \mathcal{C}_{0q}^\infty(E_{\varepsilon, t}) \cap \mathcal{C}_{0q}(\overline{E_{\varepsilon, t}})$  with  $\bar{\partial}f = 0$ , there exists  $u \in \mathcal{C}_{0q-1}^\infty(E_{\varepsilon, t}) \cap \mathcal{C}_{0q-1}^\infty(\overline{E_{\varepsilon, t}})$  such that  $\bar{\partial}u = f$ .*

In the next part of the proof of Theorem 1 we apply a method used in [6]. Consider first the case  $q \geq 2$ . Let  $f \in \mathcal{C}_{0q}^\infty(D \times D) \cap \mathcal{C}_{0q}(Q)$  with  $\bar{\partial}f = 0$ . Take a strictly increasing sequence  $\{t_n\}_{n=1}^\infty$  of negative real numbers, such that  $\lim_{n \rightarrow \infty} t_n = 0$ , and  $t_1 > t_0$ . Set  $E_n = E_{\varepsilon, t_n}$  for simplicity. We shall construct a sequence  $\{u_n\}_{n=1}^\infty$  of differential forms such that

$$(2.3) \quad u_n \in \mathcal{C}_{0q-1}^\infty(E_n) \cap \mathcal{C}_{0q-1}(\overline{E_n}), \quad \bar{\partial}u_n = f \text{ in } \overline{E_n}, \quad \text{and} \\ u_{n+1}|_{\overline{E_{n-1}}} = u_n|_{\overline{E_{n-1}}}.$$

Suppose that  $u_1, \dots, u_m$  are constructed. By Corollary 2.1, there exists  $v \in \mathcal{C}_{0q-1}^\infty(E_{m+1}) \cap \mathcal{C}_{0q-1}(\overline{E_{m+1}})$  such that  $\bar{\partial}v = f$  in  $\overline{E_{m+1}}$ . Then

$$\bar{\partial}(u_m - v) = 0 \quad \text{on } \overline{E_m}.$$

Hence, by Corollary 2.1, there exists  $w \in \mathcal{C}_{0q-2}^\infty(E_m) \cap \mathcal{C}_{0q-2}(\overline{E_m})$  such that  $\bar{\partial}w = u_m - v$ . Let  $\chi$  be a  $\mathcal{C}^\infty$  function on  $C^{2n}$ , such that  $0 \leq \chi \leq 1$ ,  $\chi \equiv 1$  on  $\overline{E_{m-1}}$ ,  $\chi \equiv 0$  on  $(\overline{D} \times \overline{D}) \setminus E_m$ . Then the form  $\chi w$ , extended trivially by 0, is in  $\mathcal{C}_{0q-2}^\infty(E_{m+1}) \cap \mathcal{C}_{0q-2}(\overline{E_{m+1}})$ , and

$$\bar{\partial}(\chi w) = (\bar{\partial}\chi)w + \chi(u_m - v) \in \mathcal{C}_{0q-1}^\infty(E_{m+1}) \cap \mathcal{C}_{0q-1}(\overline{E_{m+1}}).$$

Define  $u_{m+1}$  on  $\overline{E_{m+1}}$  by  $u_{m+1} = v + \bar{\partial}(\chi w)$ . Then  $u_{m+1}$  satisfies (2.3). Since  $\bigcup_{n=1}^\infty \overline{E_n} = Q$ , the desired solution  $u$  is defined by setting  $u = u_n$  on  $\overline{E_n}$ . Now let  $q = 1$ . We need some auxiliary approximation lemmas:

**LEMMA 2.2.** *Let  $D, \varepsilon, t_0$  be as in Corollary 2.1. Let  $t, t' \in \mathbf{R}$  satisfy the condition  $t_0 < t' < t < 0$ . Then there exists a neighborhood  $U$  of  $\overline{E_{\varepsilon, t}}$  such that every function  $f$  holomorphic in a neighborhood of  $\overline{E_{\varepsilon, t'}}$  can be approximated on  $\overline{E_{\varepsilon, t'}}$  by functions holomorphic in  $U$ .*

*Proof.* By Theorems 4.3.2 and 4.3.4 of [3], it is sufficient to find a neighborhood  $U$  of  $\overline{E_{\varepsilon, t}}$  such that  $(\overline{E_{\varepsilon, t'}})_U^\wedge = \overline{E_{\varepsilon, t'}}$ , where  $(\overline{E_{\varepsilon, t'}})_U^\wedge$  denotes the holomorphic convex hull of  $\overline{E_{\varepsilon, t'}}$  in  $U$ . Fix  $t''$  such that  $t < t'' < 0$ , and let  $D_\eta = \{z \in \tilde{D} \mid \sigma(z) < \eta\}$ . If  $\eta > 0$  is sufficiently small, then  $(\overline{D})_{D_\eta}^\wedge = \overline{D}$ , and hence

$$(2.4) \quad (\overline{D} \times \overline{D})_{D_\eta \times D_\eta}^\wedge = \overline{D} \times \overline{D}.$$

Moreover,

$$(2.5) \quad (\overline{G_{\varepsilon, t'}})_{G_{\varepsilon, t''}}^\wedge = \overline{G_{\varepsilon, t'}}.$$

Set  $U = (D_\eta \times D_\eta) \cap G_{\varepsilon, t''}$ . Then  $U$  is a neighborhood of  $\overline{E_{\varepsilon, t}}$  and it follows from (2.4) and (2.5) that

$$(\overline{E_{\varepsilon, t'}})_U^\wedge = ((\overline{D} \times \overline{D}) \cap \overline{G_{\varepsilon, t'}})_U^\wedge = (\overline{D} \times \overline{D}) \cap \overline{G_{\varepsilon, t'}} = \overline{E_{\varepsilon, t'}}.$$

LEMMA 2.3. *Let  $D$ ,  $\varepsilon$ ,  $t_0$ ,  $t$  and  $t'$  be as in Lemma 2.2. Then every function  $f \in A(E_{\varepsilon,t})$  can be uniformly approximated on  $\overline{E_{\varepsilon,t'}}$  by functions which are holomorphic in a neighborhood of  $\overline{E_{\varepsilon,t'}}$ .*

*Proof.* We prove first one result on the separation of singularities:

LEMMA 2.4. *Given  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  and the strictly pseudoconvex domains  $D_i \subset \mathbb{C}^n$ ,  $i = 1, \dots, N$ , such that  $D \subset D_i$ ,  $\text{diam}(\partial D \setminus D_i) < \varepsilon$ , and such that for every  $f \in A(E_{\varepsilon,t})$  there exist functions  $L_i f \in A((D_i \times D) \cap G_{\varepsilon,t})$ , such that  $f = \sum_{i=1}^N L_i f$ .*

*Proof.* While  $\bar{D}$  is compact, there exist a positive integer  $N$ , and points  $z_1, \dots, z_N \in \partial D$ , such that  $\partial D \subset \bigcup_{i=1}^N B(z_i, \varepsilon/4)$ . Let  $f \in A(E_{\varepsilon,t})$ . Choose a function  $\varphi_1 \in \mathcal{C}^\infty(\mathbb{C}^n)$ , such that  $0 \leq \varphi_1 \leq 1$ ,  $\varphi_1|_{\partial D \cap B(z_1, \varepsilon/2)} = 1$ ,  $\varphi_1|_{\partial D \setminus B(z_1, 3\varepsilon/4)} = 0$ . Set  $w_1 = f \bar{\partial} \varphi_1$ . Then there exist strictly pseudoconvex domains  $D_1$  and  $D'_1$  in  $\mathbb{C}^n$  such that  $D \cup (\partial D \setminus B(z_1, \varepsilon)) \subset D_1$ ,  $D \cup (\partial D \cap \overline{B(z_1, \varepsilon/4)}) \subset D'_1$ ,  $D''_1 = D_1 \cup D'_1$  is strictly pseudoconvex, and  $w_1$ , extended trivially by 0, is in

$$\mathcal{C}^\infty((D''_1 \times D) \cap G_{\varepsilon,t}) \cap \mathcal{C}_{01}((\overline{D''_1} \times \bar{D}) \cap \overline{G_{\varepsilon,t}}),$$

and  $\bar{\partial} w = 0$  there. Moreover, if  $D''_1$  is sufficiently close to  $D$ , then the domain  $(D''_1 \times D) \cap G_{\varepsilon,t}$  satisfies the assumptions (C) and (CR) from [2]. Therefore, by [2], Theorem 2, there exists  $\alpha_1 \in \mathcal{C}^\infty((D''_1 \times D) \cap G_{\varepsilon,t}) \cap \mathcal{C}((D''_1 \times \bar{D}) \cap \overline{G_{\varepsilon,t}})$  such that  $\bar{\partial} \alpha_1 = w_1$ . Set

$$L_1 f = \varphi_1 f - \alpha_1, \quad L'_1 f = (1 - \varphi_1) f + \alpha_1.$$

Then

$$L_1 f \in A((D_1 \times D) \cap G_{\varepsilon,t}), \quad L'_1 f \in A((D'_1 \times D) \cap G_{\varepsilon,t}),$$

and  $f = L_1 f + L'_1 f$ .

Suppose that for  $k < N - 1$  we have constructed the strictly pseudoconvex domains  $D_1, \dots, D_k$  and  $D'_k$  in  $\mathbb{C}^n$  such that  $D \cup (\partial D \setminus B(z_i, \varepsilon)) \subset D_i$ ,  $i = 1, \dots, k$ , and  $D \cup (\partial D \cap \bigcup_{i=1}^k \overline{B(z_i, \varepsilon/4)}) \subset D'_k$ , and the functions  $L_1 f, \dots, L_k f$  and  $L'_k f$  such that  $L_i f \in A((D_i \times D) \cap G_{\varepsilon,t})$ ,  $i = 1, \dots, k$ ,  $L'_k f \in A((D'_k \times D) \cap G_{\varepsilon,t})$ , and  $f = \sum_{i=1}^k L_i f + L'_k f$ . Choose a function  $\varphi_{k+1} \in \mathcal{C}^\infty(\mathbb{C}^n)$  such that  $0 \leq \varphi_{k+1} \leq 1$ ,  $\varphi_{k+1} = 1$  on  $\partial D \cap B(z_{k+1}, \varepsilon/4)$ , and  $\varphi_{k+1} = 0$  on  $\partial D \setminus B(z_{k+1}, 3\varepsilon/4)$ . Set  $w_{k+1} = (L'_k f) \bar{\partial} \varphi_{k+1}$ . Then there exist strictly pseudoconvex domains  $D_{k+1}$  and  $D'_{k+1} \subset \mathbb{C}^n$ , such that

$$D'_k \cup (\partial D \setminus B(z_{k+1}, \varepsilon)) \subset D_{k+1}, \quad D'_k \cup (\partial D \cap \overline{B(z_{k+1}, \varepsilon/4)}) \subset D'_{k+1},$$

$$D''_{k+1} = D_{k+1} \cup D'_{k+1}$$



is strictly pseudoconvex, the domain  $(D''_{k+1} \times D) \cap G_{\varepsilon,t}$  satisfies the assumptions (C) and (CR) from [2], the form  $w_{k+1}$  extended trivially by 0, is in

$$\mathcal{C}_{01}^\infty((D'_{k+1} \times D) \cap G_{\varepsilon,t}) \cap \mathcal{C}_{01}(\overline{D''_{k+1} \times D} \cap \overline{G_{\varepsilon,t}}),$$

and  $\bar{\partial}w_{k+1} = 0$  there. By [2], Theorem 2, there exists

$$\alpha_{k+1} \in \mathcal{C}^\infty((D''_{k+1} \times D) \cap G_{\varepsilon,t}) \cap \mathcal{C}(\overline{D''_{k+1} \times D} \cap \overline{G_{\varepsilon,t}})$$

such that  $\bar{\partial}\alpha_{k+1} = w_{k+1}$ . Set

$$L_{k+1}f = \varphi_{k+1}L'_kf - \alpha_{k+1}, \quad L'_{k+1}f = (1 - \varphi_{k+1})L'_kf + \alpha_{k+1}.$$

Then

$$L_{k+1}f \in A((D_{k+1} \times D) \cap G_{\varepsilon,t}), \quad L'_{k+1}f \in A((D'_{k+1} \times D) \cap G_{\varepsilon,t}),$$

$$f = \sum_{i=1}^{k+1} L_i f + L'_{k+1}f, \quad D \cup (\partial D \setminus B(z_i, \varepsilon)) \subset D_i, \quad i = 1, \dots, k + 1,$$

and

$$D \cup \left( \partial D \cap \bigcup_{i=1}^{k+1} \overline{B(z_i, \varepsilon/4)} \right) \subset D'_{k+1}.$$

After  $N - 1$  steps, we obtain the decomposition  $f = \sum_{i=1}^{N-1} L_i f + L'_{N-1}f$ . It remains to put  $D_N = D'_{N-1}$  and  $L_N f = L'_{N-1}f$ .

We return to the proof of Lemma 2.3. We can choose  $N$ ,  $\varepsilon$  and the domains  $D_i$  in Lemma 2.4 in such a way, that there exist  $t_1 \in \mathbf{R}$  with  $t' < t_1 < t$  (where  $t$  and  $t'$  are as in the assumption of Lemma 2.3),  $\delta > 0$ , and  $v_1, \dots, v_N \in \mathbf{C}^n$ , such that for every  $i = 1, \dots, N$ , for every  $s$  such that  $0 < s < \delta$ , the set  $\{(z + sv_i, w) \mid (z, w) \in (D_i \times \bar{D}) \cap G_{\varepsilon,t}\}$  contains  $(\bar{D} \times \bar{D}) \cap \overline{G_{\varepsilon,t_1}}$ . Therefore, if  $f \in A(E_{\varepsilon,t})$  and  $f = \sum_{i=1}^N L_i f$  is the decomposition of  $f$  according to Lemma 2.4, then the functions  $g_{i,s}(z, w) = L_i f(z - sv_i, w)$  are defined in the set  $(D'_s \times \bar{D}) \cap \overline{G_{\varepsilon,t_1}}$ , where  $D'_s$  is some neighborhood of  $\bar{D}$ , and  $g_{i,s} \rightarrow f$  uniformly on  $(\bar{D} \times \bar{D}) \cap \overline{G_{\varepsilon,t_1}}$  as  $s \rightarrow 0$ . Given a function  $g_{i,s}$ , we can apply the decomposition procedure, described in Lemma 2.4, but now with respect to the second group of variables, and with respect to the domain  $(D'_s \times D) \cap G_{\varepsilon,t_1}$ . We obtain then for some  $N' = N'(g_{i,s})$  the decomposition  $g_{i,s} = \sum_{j=1}^{N'} L_j g_{i,s}$ , where for  $j = 1, \dots, N'$ ,  $L_j g_{i,s} \in A((D'_s \times D_j) \cap G_{\varepsilon,t_1})$ , and  $D_j \subset \mathbf{C}^n$  is a strictly pseudoconvex domain such that  $D \subset D_j$  and  $\text{diam}(\partial D \setminus D_j) < \varepsilon$ . We choose then  $t_2$  with  $t' < t_2 < t_1$  and shift the functions  $L_j g_{i,s}$  similarly as

above, but now with respect to the second group of variables, in order to approximate every function  $L_j g_{i,s}$  uniformly on  $(\bar{D} \times \bar{D}) \cap \overline{G_{\varepsilon,t_2}}$  by functions of the form  $h_{i,j,s,r}(z,w) = L_j g_{i,s}(z,w - ru_j)$ ,  $u_j \in \mathbf{C}^n$ ,  $r > 0$ , defined in a set  $(D'_s \times D''_r) \cap G_{\varepsilon,t_2}$ , where  $D''_r$  is some neighborhood of  $\bar{D}$ . Since  $\overline{E_{\varepsilon,t'}} \subset (\bar{D} \times \bar{D}) \cap \overline{G_{\varepsilon,t_2}}$ , we obtain the conclusion of Lemma 2.3.

Having proved Lemmas 2.2 and 2.3, we can finish the proof of Theorem 1 for  $q = 1$ . Choose two sequences  $\{t_n\}$  and  $\{s_n\}$  of negative real numbers, such that  $t_n, s_n \rightarrow 0$  as  $n \rightarrow \infty$ ,  $t_0 < s_1$ , and  $s_n < t_n < s_{n+1}$ ,  $n = 1, 2, \dots$ . Set  $E_n = E_{\varepsilon,t_n}$ ,  $F_n = E_{\varepsilon,s_n}$ . We shall construct a sequence of functions  $\{u_n\}$  such that  $u_n \in \mathcal{C}^\infty(E_n) \cap \mathcal{C}(\overline{E_n})$ ,  $\bar{\partial}u_n = f$  on  $\overline{E_n}$ , and the uniform norm  $\|u_{n+1} - u_n\|_{\overline{F}} \leq 2^{-n}$ . Suppose that  $u_1, \dots, u_m$  are constructed. By [2], Theorem 2, there exists  $v \in \mathcal{C}^\infty(E_{m+1}) \cap \mathcal{C}(\overline{E_{m+1}})$ , such that  $\bar{\partial}v = f$  in  $\overline{E_{m+1}}$ . Then  $u_m - v \in A(E_m)$ . By Lemma 2.3 there exists a function  $w$  holomorphic in a neighborhood of  $\overline{F_m}$ , such that

$$\|w - (u_m - v)\|_{\overline{F_m}} \leq 2^{-(m+1)}.$$

By Lemma 2.2 there exists a neighborhood  $U$  of  $\overline{E_{m+1}}$  and a function  $t$  holomorphic in  $U$ , such that  $\|t - w\|_{\overline{F}} \leq 2^{-(m+1)}$ . Let  $u_{m+1} = t + v$  on  $\overline{E_{m+1}}$ . Then  $u_{m+1} \in \mathcal{C}^\infty(E_{m+1}) \cap \mathcal{C}(\overline{E_{m+1}})$ ,  $\bar{\partial}u_{m+1} = f$  on  $\overline{E_{m+1}}$ , and  $\|u_{m+1} - u_m\|_{\overline{F_m}} \leq 2^{-m}$ . Since  $u_{m+1} - u_m$  is holomorphic in  $E_m$ , it follows that the sequence  $\{u_n\}$  converges to the function  $u \in \mathcal{C}^\infty(D \times D) \cap \mathcal{C}(Q)$ , such that  $\bar{\partial}u = f$ .

**3. The decomposition in the algebra  $A_Q(D \times D)$ .** We prove here Theorem 2. The method of the proof is that used by Øvrelid [5], therefore we give only the necessary modifications. It follows from the assumptions that at every point  $(z, s) \in D \times D$ , the germs at  $(z, s)$  of the functions  $g_i$  generate the ideal of germs at  $(z, s)$  of holomorphic functions vanishing on  $\Delta(D)$ . Therefore, by [3], Theorem 7.2.9, for every  $f \in (A_Q)_0(D \times D)$  there exist functions  $(Rf)_1, \dots, (Rf)_N \in \mathcal{O}(D \times D)$  such that  $f = \sum_{i=1}^N g_i (Rf)_i$ . Let  $N_i = \{(z, s) \in Q \setminus \Delta(D) \mid g_i(z, s) = 0\}$ ,  $i = 1, \dots, N$ . Since the sets  $N_i$  are closed in  $\mathbf{C}^{2n} \setminus \Delta(\bar{D})$ , there exist functions  $\tilde{\varphi}_i \in \mathcal{C}^\infty(\mathbf{C}^{2n} \setminus \Delta(\bar{D}))$  such that  $0 \leq \tilde{\varphi}_i \leq 1$ ,  $\sum_{i=1}^N \tilde{\varphi}_i = 1$ , and  $\tilde{\varphi}_i$  vanishes in a neighborhood of  $N_i$  in  $\mathbf{C}^{2n} \setminus \Delta(\bar{D})$ ,  $i = 1, \dots, N$ .

Choose  $\varphi_0 \in \mathcal{C}^\infty(\mathbf{C}^{2n} \setminus \Delta(\partial D))$ , such that  $0 \leq \varphi_0 \leq 1$ ,  $\varphi_0 = 1$  in some neighborhood  $W$  of  $\Delta(D)$  in  $D \times D$ , and  $\varphi_0 = 0$  in a neighborhood of  $\partial(D \times D) \setminus \Delta(\partial D)$ . Set  $\varphi_i = (1 - \varphi_0)\tilde{\varphi}_i$ , and define  $(Sf)_i = \varphi_0(Rf)_i + \varphi_i f/g_i$ ,  $i = 1, \dots, N$ . Then  $\sum_{i=1}^N g_i (Sf)_i = f$  in  $Q$ . Choose the neighborhoods  $W_1$  and  $W_2$  of  $\Delta(D)$  in  $D \times D$  such that  $\overline{W_1} \subset W$  and  $\overline{W_2} \subset W_1$

(the closures in  $Q$ ), and let  $\varphi$  be a function in  $\mathcal{C}^\infty(\mathbf{C}^{2n} \setminus \Delta(\partial D))$  such that  $0 \leq \varphi \leq 1$ ,  $\varphi = 1$  outside  $W_1$ , and  $\varphi = 0$  in  $W_2$ . Set  $L_r = \{u \in \mathcal{C}_{0,r}^\infty(D \times D) \cap \mathcal{C}_{0,r}(Q) \mid \bar{\partial}f \in \mathcal{C}_{0,r}(Q)\}$ . Define  $L_r^s$ ,  $0 \leq r, s$ , the operators  $\bar{\partial}$  and  $P_r$  on  $L_r^s$  similarly as in [5], and let  $M_r^s = \{k \in L_r^s \mid k|_{W_1} \equiv 0\}$ , and  $k_0 = \sum_{i=1}^N \varphi \tilde{\varphi}_i / g_i \otimes e_i \in L_0^1$  (here  $e_1, \dots, e_N$  is some basis of  $\mathbf{C}^N$ ). Using then Theorem 1 in place of Lemma 1 from [5], we end the proof similarly as in [5] (of course, after the suitable change of notations, according to that given above; in particular, the functions  $g'_i$  and  $h_i$  from the final part of the proof of [5], Theorem 1 should be replaced by  $(Sf_i)$  and  $f_i$  respectively).

*Note.* The operator  $f \rightarrow (f_1, \dots, f_N)$  from Theorem 2 is in general neither linear nor continuous. Nevertheless, if  $n = 1$  (i.e.  $D \subset \mathbf{C}$ ), then every  $f \in (A_Q)_0(D \times D)$  can be represented as  $f(z, s) = (z - s)(Rf)(z, s)$  with (uniquely determined)  $Rf \in A_Q(D \times D)$ , and the mapping  $f \rightarrow Rf$  is linear and continuous (where  $A_Q(D \times D)$  is equipped with the topology of uniform convergence on compact subsets of  $Q$ ). Moreover, by Theorem 2, the function  $z - s$  can be decomposed with respect to the functions  $g_i$  in the form  $z - s = \sum_{i=1}^N g_i h_i$  with some  $h_i \in A_Q(D \times D)$ . Therefore, setting  $f_i = (Rf)h_i$ ,  $i = 1, \dots, N$ , we obtain the continuous and linear operator

$$(A_Q)_0(D \times D) \ni f \rightarrow ((Rf)h_1, \dots, (Rf)h_N) \in [A_Q(D \times D)]^N,$$

such that

$$(3.1) \quad f = \sum_{i=1}^N g_i f_i.$$

We obtain therefore the full generalization of Theorem 2 in [4]. Similarly, if  $D \subset \mathbf{C}^n$  is strictly pseudoconvex with  $\mathcal{C}^2$  boundary, then, by a theorem of Ahern and Schneider [1], every function

$$f \in A_0(D \times D) = \{f \in A(D \times D) \mid f|_{\Delta(D)} = 0\}$$

can be decomposed with respect to the functions  $z_1 - s_1, \dots, z_n - s_n$  into  $f(z, s) = \sum_{i=1}^n (z_i - s_i) \tilde{f}_i(z, s)$  with some functions  $f_i \in A_Q(D \times D)$ , and the operator  $A_0(D \times D) \ni f \rightarrow (\tilde{f}_1, \dots, \tilde{f}_n) \in [A_Q(D \times D)]^n$  is linear and continuous. Applying Theorem 2 to the functions  $z_i - s_i$ ,  $i = 1, \dots, n$ , and proceeding as above, we obtain the linear and continuous operator  $A_0(D \times D) \ni f \rightarrow (f_1, \dots, f_N) \in [A_Q(D \times D)]^N$ , satisfying (3.1).

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