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Let K, H be real Banach spaces with strictly convex duals, and let X, Y be any real Banach spaces. In this paper we find a general form of isometries between the Banach spaces $X \check{\otimes} K$ and $Y \check{\otimes} H$. As a consequence we obtain that $X \check{\otimes} K$ and $Y \check{\otimes} K$ are isometric if and only if X and Y are isometric. We also derive a theorem characterizing Banach spaces with a trivial centralizer.

0. Introduction. Let X, Y, K, H be real Banach spaces. The purpose of this note is to study the isometries between the injective tensor products $X \check{\otimes} K$ and $Y \check{\otimes} H$. We find a general form of such isometries provided K and H have strictly convex duals, and using this characterization we investigate the following problems.

Problem 1. Under what conditions on K, H or on X, Y, K, H are the spaces $X \check{\otimes} K$ and $Y \check{\otimes} H$ isometric if and only if either the spaces X, Y and K, H or X, H and K, Y are isometric?

Problem 2. Under what conditions on X, K is every isometry from $X \check{\otimes} K$ onto itself canonical?

We call an isometry T from $X \check{\otimes} K$ onto $Y \check{\otimes} H$ canonical if one of the following possibilities holds:

(A) T is of the form

$$T(x \otimes k) = T_1(x) \otimes T_2(k) \quad \text{for all } x \in X \text{ and } k \in K, \tag{1}$$

where $T_1\colon X \to Y$, $T_2\colon K \to H$ are onto isometries.

(B) There is a Banach space Z such that X is isometric to $Z \check{\otimes} H$ and Y is isometric to $Z \check{\otimes} K$, and under this identification T is of the form

$$T(z \otimes h \otimes k) = z \otimes k \otimes h \quad \text{for all } z \in Z,\ h \in H,\ k \in K. \tag{2}$$

Notice that, in general, the implications in the problems do not hold. For example, take four compact Hausdorff spaces $S_1, \ldots, S_4$ which are pairwise non-homeomorphic but $S_1 \times S_2$ and $S_3 \times S_4$ are homeomorphic, and put $X_i = C(S_i)$ for $i = 1, \ldots, 4$, where $C(S)$ is the space of all continuous functions on S with sup-norm. We have

$$X_1 \check{\otimes} X_2 \simeq C(S_1 \times S_2) \simeq C(S_2 \times S_4) \simeq X_3 \check{\otimes} X_4,$$

but any two of the spaces X_i, $i = 1, \ldots, 4$, are not isometric.

The special case of the above problems—$X = C(S)$, $Y = C(S')$—has been studied by many authors, the most important of which, for our purposes, is the monograph by Behrends ([**1**]), who proved, among other things, that if the centralizers $Z(K)$ and $Z(H)$ of Banach spaces K and H, respectively, are both trivial and if the spaces $C(S) \check{\otimes} K$ and $C(S') \check{\otimes} H$ are isometric, then K and H, and also $C(S)$ and $C(S')$, are isometric.

All fundamental results on centralizers and function module representations that we use are found in [**1**]. We use standard Banach space terminology. The set of extreme points of a convex set C is denoted by $\operatorname{ext} C$. For a Banach space Z, $B(Z)$ denotes the closed unit ball of Z. For Banach spaces U, V, we denote by $L(U, V)$ ($K(U, V)$) the Banach space of all continuous (compact) linear operators from U into V, and by $U \simeq V$ we mean that U and V are isometric. Throughout the paper we frequently view a Banach space V as a subspace of $C(\operatorname{ext} B(V^*))$ or $C(B(V^*))$, where $\operatorname{ext} B(V^*) \subset B(V^*)$ are equipped with the weak-$*$-topology. The space $X \check{\otimes} K$ is regarded as a subspace of $C(\operatorname{ext} B(X^*) \times \operatorname{ext} B(K^*))$.

1. The results.

THEOREM 1. *Let X, Y, H, K be real Banach spaces and assume H^* and K^* are strictly convex. If T is an isometry from $X \check{\otimes} K$ onto $Y \check{\otimes} H$, then there are Banach spaces Z and X_2 such that*

$$X \simeq (Z \check{\otimes} H) \oplus_\infty X_2 \quad \text{and} \quad Y \simeq (Z \check{\otimes} K) \oplus_\infty X_2,$$

and, up to the above isometries, the operator T is of the form

$$T(z \otimes h \otimes k, x_2 \otimes k_2) = (z \otimes k \otimes h, T_2(x_2 \otimes k_2)),$$

where

$$T_2(x_2 \otimes k_2)(x^* \otimes h^*) = x^*(x_2) \cdot h^*(\Phi(x^*)(k_2)), \tag{3}$$

and Φ: $\operatorname{ext} B(X_2^*) \to L(K, H)$ is an operator from $\operatorname{ext} B(X_2^*)$ into the set of isometries from K onto H.

Before proving Theorem 1 we formulate the following two theorems as corollaries.

THEOREM 2. *Let X, Y, K be real Banach spaces and assume K^* is strictly convex. Then*

(a) *$X \check{\otimes} K \simeq Y \check{\otimes} K$ if and only if $X \simeq Y$. If K has the approximation property, then*

(b) *$K(X, K) \simeq K(Y, K)$ if and only if $X^* \simeq Y^*$.*

Proof. Point (a) is an immediate consequence of Theorem 1, and to get (b) it is sufficient to notice that if K has the approximation property, then $K(X, K) \simeq X^* \check{\otimes} K$.

The next theorem characterizes Banach spaces with a trivial centralizer.

THEOREM 3. *For any real Banach space X the following are equivalent:*

(i) $\dim Z(X) = 1$;

(ii) *for any real Banach space K with K^* strictly convex, every isometry from $X \check{\otimes} K$ onto itself is canonical;*

(iii) *for any real Hilbert space H every isometry from $X \check{\otimes} H$ onto itself is canonical;*

(iv) *for the two-dimensional real Hilbert space H_2 every isometry from $X \check{\otimes} H_2$ onto itself is canonical.*

Proof. To prove (i) $\Rightarrow$ (ii), assume $\dim Z(X) = 1$, let K be a real Banach space with a strictly convex dual, and let T be an isometry from $X \check{\otimes} K$ onto itself. Notice that if X were a direct sum, with the sup-norm, of two Banach spaces, then the orthogonal projections onto both components would be in the centralizer of X. Thus by Theorem 1 and our assumption we have two possibilities.

(1) There is a Banach space Z such that $X \simeq Z \check{\otimes} K$, and up to this isometry T is of the form (2), so it is canonical.

(2) There is a linear isometry T_1 from X onto itself and an operator T_2: $\operatorname{ext} B(X^*) \to \operatorname{Isom}(K, K)$ such that T is of the form

$$T(x \otimes k)(x^* \otimes k^*) = T_1(x)(x^*)\big(T_2(x^*)(k)\big)(k^*)$$

for any k^* in K^* and any k in K. Then the operator $S_{k^*,k}$: $X \to X$, defined by

$$S_{k^*,k}(x)(x^*) = T(x \otimes k)\Big(\big(T_1^{-1}\big)^*(x^*) \otimes k^*\Big)$$
$$= x(x^*)\Big(T_2\Big(\big(T_1^{-1}\big)^*(x^*)\Big)(k)\Big)(k^*),$$

is a multiplier on X. So by assumption $S_{k^*,k}$ is just the multiplication by a constant, for any $k^* \in K^*$, $k \in K$. Hence T_2 is a one-dimensional operator, and this means that T is canonical of the form (1).

The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are trivial.

To prove (iv) $\Rightarrow$ (i), assume $Z(X)$ is not one-dimensional. Then there is a continuous, non-constant function φ: $\operatorname{ext} B(X^*) \to [0, 2\pi)$ such that the operator M_φ: $C(\operatorname{ext} B(X^*)) \to C(\operatorname{ext} B(X^*))$: $M_\varphi(f) = \varphi \cdot f$ leaves X invariant. It is easy to check that the operator

$$T\colon X \otimes H_2 \to X \otimes H_2\colon T(x \otimes h)(x^* \otimes h^*) = x^*(x)\Phi(x^*)(h),$$

where $\Phi(x^*)$: $H_2 \to H_2$ is the operator of rotation through angle $\varphi(x^*)$, is a well-defined, non-canonical onto isometry.

2. Proof of Theorem 1. The theorem is trivial when one of the spaces K or H is one dimensional, so we assume $\dim K \geq 2$, $\dim H \geq 2$. We start the proof with two propositions. The first is a special case of the theorem of Ruess and Stegall (it can also be found in Tseitlin's paper ([3])), and the second is a very easy, strictly algebraic fact.

THEOREM ([2], [3]). *Let X and K be real Banach spaces. Then* $\operatorname{ext} B((X \check{\otimes} K)^*) = \operatorname{ext} B(X^*) \otimes \operatorname{ext} B(K^*)$.

PROPOSITION 1. *Let U and V be linear spaces and assume $u_1 \otimes v_1 + u_2 \otimes v_2 = u_3 \otimes v_3$, where $u_i \in U$, $v_i \in V$, $i = 1, 2, 3$. Then the vectors u_1, u_2, u_3 or v_1, v_2, v_3 are proportional.*

Proof. Let v^* be any linear functional on V. We have

$$u_1 v^*(v_1) + u_2 v^*(v_2) = u_3 v^*(v_3).$$

Hence, if $v^*(v_3) \neq 0$, then u_3 is a linear combination of u_1 and u_2; if u_1, u_2, u_3 were not proportional, then the coefficients of this linear combination would be uniquely determined, and this would mean

$$v^*(v_1) = \text{const}\, v^*(v_3), \quad v^*(v_2) = \text{const}\, v^*(v_3) \qquad \text{for any } v^* \in V^*.$$

Hence, $v_1 \| v_2$ and $v_2 \| v_3$.

Now let X, Y, H, K, T be as in Theorem 1. Fix $y_0^* \in \operatorname{ext} B(Y^*)$, and let h_1^*, h_2^*, h_3^* be any linearly independent elements of $\operatorname{ext} B(H^*)$. By the Ruess-Stegall theorem

$$T^*(\operatorname{ext} B(Y^*) \otimes \operatorname{ext} B(H^*)) = \operatorname{ext} B(X^*) \otimes \operatorname{ext} B(K^*), \tag{4}$$

so there are $x_1^*, x_2^*, x_3^* \in \operatorname{ext} B(X^*)$ and $k_1^*, k_2^*, k_3^* \in \operatorname{ext} B(K^*)$ such that

$$T^*(y_0^* \otimes h_i^*) = x_i^* \otimes k_i^* \quad \text{for } i = 1, 2, 3.$$

Since H^* is strictly convex, it follows that $(h_1^* + h_2^*)/\|h_1^* + h_2^*\|$ is an extreme point of $B(H^*)$, so

$$x_1^* \otimes k_1^* + x_2^* \otimes k_2^* = T^*\big(y_0^* \otimes \big(h_1^* + h_2^*\big)\big) = \big\|h_1^* + h_2^*\big\| x_4^* \otimes k_4^*$$

for some $x_4^* \in \operatorname{ext} B(X^*)$ and $k_4^* \in \operatorname{ext} B(K^*)$. Hence by Proposition 1 we have $x_1^* \| x_2^*$ or $k_1^* \| k_2^*$. The same arguments show that $x_1^* \| x_3^*$ or $k_1^* \| k_3^*$, and $x_2^* \| x_3^*$ or $k_2^* \| k_3^*$, and this proves that $x_1^* \| x_2^* \| x_3^*$ or $k_1^* \| k_2^* \| k_3^*$. The strict convexity of H^* together with (4) now implies that, for any $y_0^* \in \operatorname{ext} B(Y^*)$, one of the following occurs:

(1) There is an element k_1^* in $\operatorname{ext} B(K^*)$ and a linear, weak-*-continuous into isometry Φ: $H^* \to X^*$ such that

$$T^*\big(y_0^* \otimes h^*\big) = \Phi(h^*) \otimes k_1^* \quad \text{for any } h^* \text{ in } H^*.$$

(2) There is an element x_2^* of $\operatorname{ext} B(X^*)$ and a linear, weak-*-continuous isometry Ψ: $H^* \to K^*$ such that

$$T^*\big(y_0^* \otimes h^*\big) = x_2^* \otimes \Psi(h^*) \quad \text{for any } h^* \text{ in } H^*.$$

Let us denote by S_1 the subset of $\operatorname{ext} B(Y^*)$ consisting of all points y^* for which the first possibility holds, and by S_2 the subset of $\operatorname{ext} B(Y^*)$ for which the second holds. We have four functions:

$$\Phi_1\colon S_1 \times H^* \to X^*, \qquad \Psi_1\colon S_1 \to \operatorname{ext} B(K^*),$$
$$\Phi_2\colon S_2 \to \operatorname{ext} B(X^*), \qquad \Psi_2\colon S_2 \times H^* \to K^*$$

such that for any $y^* \in S_1$

$$(5) \qquad T^*(y^* \otimes h^*) = \Phi_1(y^*, h^*) \otimes \Psi_1(y^*) \quad \text{for all } h^* \in H^*,$$

and for any $y^* \in S_2$,

$$(6) \qquad T^*(y^* \otimes h^*) = \Phi_2(y^*) \otimes \Psi_2(y^*, h^*) \quad \text{for all } h^* \in H^*.$$

Using the same arguments for T^{-1}, we get that $\operatorname{ext} B(X^*)$ is the sum of two disjoint subsets $\tilde{S}_1$ and $\tilde{S}_2$, and there are four functions:

$$\tilde{\Phi}_1\colon \tilde{S}_1 \times K^* \to Y^*, \qquad \tilde{\Psi}_1\colon \tilde{S}_1 \to \operatorname{ext} B(H^*),$$
$$\tilde{\Phi}_2\colon \tilde{S}_2 \to \operatorname{ext} B(Y^*), \qquad \tilde{\Psi}_2\colon \tilde{S}_2 \times K^* \to H^*$$

such that for any $x^* \in \tilde{S}_1$

$$(7) \quad (T^{-1})^*(x^* \otimes k^*) = \tilde{\Phi}_1(x^*, k^*) \otimes \tilde{\Psi}_1(x^*) \quad \text{for all } k^* \in K^*,$$

and for any $x^* \in \tilde{S}_2$

$$(8) \quad (T^{-1})^*(x^* \otimes k^*) = \tilde{\Phi}_2(x^*) \otimes \tilde{\Psi}_2(x^*, k^*) \quad \text{for all } k^* \in K^*.$$

It is easy to see that for any $y^* \in S_2$ the operator $\Psi_2(y^*, \cdot)$ is a weak-$*$-continuous isometry from H^* onto K^*. Let $0 \neq y_0^* \in \bar{S}_1$ (the bar always denotes closure in the weak-$*$-topology), and let $(y_\alpha^*)_{\alpha \in \Gamma} \subset S_1$ be a net convergent to y_0^*; we can assume that the net $(\Psi_1(y_\alpha^*))_{\alpha \in \Gamma}$ tends to $k_0^* \in K^*$. By (5) we have $k_0^* \neq 0$, and we get that the net $(\Phi_1(y_\alpha^*, h^*))_{\alpha \in \Gamma}$ is convergent for any $h^* \in H^*$. Moreover,

$$T^*(y_0^* \otimes h^*) = \lim_\alpha \Phi_1(y_\alpha^*, h^*) \otimes \lim_\alpha \psi_1(y_\alpha^*) \quad \text{for all } h^* \in H^*.$$

By the same arguments applied to formulas (6)–(8), we get that the functions $\Phi_i, \tilde{\Phi}_i, \Psi_i, \tilde{\Psi}_i$, $i = 1, 2$, can be extended to the weak-$*$-closures of their domains, and formulas (5)–(8) remain valid for the extended functions. They will be denoted by the same letters. These functions are not uniquely determined by the formulas (5)–(8), and we will show that we can assume $\Phi_2^{-1} = \tilde{\Phi}_2$. To show this, let us notice that by applying (6) to the extended functions we get

$$\begin{aligned} \|y^*\| \, \|h^*\| &= \|y^* \otimes h^*\| = \|T^*(y^* \otimes h^*)\| \\ &= \|\Phi_2(y^*)\| \, \|\Psi_2(y^*, h^*)\| \end{aligned}$$

for all $h^* \in H^*$, for any $y^* \in \bar{S}_2$.

Hence, for any $y^* \in \bar{S}_2 - \{0\}$ there is a $\lambda \in \mathbf{R} - \{0\}$ such that $\lambda\Psi_2(y^*, \cdot)$ is an isometry. We can define an equivalence relation on $\bar{S}_2 - \{0\}$ by

$$y_1^* \sim y_2^* \quad \text{if} \quad \Psi_2(y_1^*, \cdot) = \lambda\Psi_2(y_2^*, \cdot) \quad \text{for some } \lambda \in \mathbf{R}.$$

Multiplying the function $y^* \mapsto \Phi_2(y^*)$ by a scalar function, and the function $y^* \mapsto \Psi_2(y^*, \cdot)$ by its reciprocal, we can assume that both functions are constant in each equivalence class and, for $y^* \in \bar{S}_2 - \{0\}$,

$$\|\Phi_2(y^*)\| = \|y^*\| \quad \text{and} \quad \Psi_2(y^*, \cdot) \text{ is an isometry.}$$

By the same arguments $\tilde{\Phi}_2$ and $\tilde{\Psi}_2$ may have the same properties.

From (6) and (8) we get

$$(9) \qquad y^* \otimes h^* = \tilde{\Phi}_2(\Phi_2(y^*)) \otimes \tilde{\Psi}_2(\Phi_2(y^*), \Psi_2(y^*, h^*))$$

for any $y^* \in \bar{S}_2$ and all $h^* \in H^*$.

Hence $y^* \| \tilde{\Phi}_2(\Phi_2(y^*))$, so we get $y^* = \varepsilon\tilde{\Phi}_2(\Phi_2(y^*))$, with $|\varepsilon| = 1$. Similarly, for any $\lambda \in \mathbf{R}$, if $\lambda y^* \in \bar{S}_2$ we get

$$\lambda y^* = \varepsilon_\lambda \tilde{\Phi}_2(\Phi_2(\lambda y^*)) \quad \text{with } |\varepsilon_\lambda| = 1.$$

Hence, since the functions $x^* \mapsto \tilde{\Psi}_2(x^*, \cdot)^\cdot$ and $y^* \mapsto \Psi_2(y^*, \cdot)$ are constant in each equivalence class, we get

$$\tilde{\Psi}_2(\Phi_2(y^*), \Psi_2(y^*, \cdot)) = \tilde{\Psi}_2(\Phi_2(\lambda y^*), \Psi_2(\lambda y^*, \cdot)),$$

so (9) gives $\varepsilon_\lambda = \varepsilon$ for all $\lambda \in \mathbf{R}$.

The above proves that by multiplying Φ_2, in any point of its domain, by $+1$ or -1, depending on whether $\varepsilon = 1$ or $\varepsilon = -1$, we get $\tilde{\Phi}_2 \circ \Phi_2 = \mathrm{Id}_{\tilde{S}_2}$ and, by symmetry, $\Phi_2^{-1} = \tilde{\Phi}_2$.

We now put

$$X_i = \left\{ f \Big|_{\tilde{S}_i} : f \in X \subset C(\tilde{S}_1 \cup \tilde{S}_2) \right\},$$

$$Y_i = \left\{ g \Big|_{S_i} : g \in Y \subset C(S_1 \cup S_2) \right\} \quad \text{for } i = 1, 2.$$

We show that X_i, Y_i, $i = 1, 2$, are Banach spaces, and $X = X_1 \oplus_\infty X_2$, $Y = Y_1 \oplus_\infty Y_2$. First we study the spaces X_1 and Y_1.

LEMMA 1.

(i) *For any* $y_1^*, y_2^* \in S_1$ *we have*

$$\Phi_1(y_1^*, \cdot)(H^*) = \Phi_1(y_2^*, \cdot)(H^*)$$

or

$$\Phi_1(y_1^*, \cdot)(H^*) \cap \Phi_1(y_2^*, \cdot)(H^*) = \{0\}.$$

(ii) *If* $\Phi_1(y_1^*, \cdot)(H^*) = \Phi_1(y_2^*, \cdot)(H^*)$, *then*

$$\Phi_1(y_1^*, \cdot) = \lambda \Phi_1(y_2^*, \cdot), \quad \textit{where } \lambda = +1 \textit{ or } \lambda = -1.$$

Proof. We first prove the following implication:

(10) If $\Phi_1(y_1^*, h_1^*) = \Phi_1(y_2^*, h_2^*)$, then $y_1^* \| y_2^*$ or $h_1^* \| h_2^*$.

For this purpose notice that, since T^* is onto and $\Phi_2(S_2) = \tilde{S}_2$, there are $y_3^* \in S_1$, $h_3^* \in H^*$ such that

$$T^*(y_3^* \otimes h_3^*) = \Phi_1(y_1^*, h_1^*) \otimes (\Psi_1(y_1^*) + \Psi_1(y_2^*)).$$

We have

$$\begin{aligned} T^*(y_1^* \otimes h_1^* + y_2^* \otimes h_2^*) &= \Phi_1(y_1^*, h_1^*) \otimes \Psi_1(y_1^*) + \Phi_1(y_2^*, h_2^*) \otimes \Psi_1(y_2^*) \\ &= T^*(y_3^* \otimes h_3^*). \end{aligned}$$

Hence by Proposition 1 we get $h_1^* \| h_2^*$ or $y_2^* \| y_3^*$.

Now assume that $y_1^*, y_2^* \in S_1$, and $h_0^*, h_1^*, h_2^* \in H^* - \{0\}$ are such that $\Phi_1(y_1^*, h_1^*) = \Phi_1(y_2^*, h_2^*)$, but $\Phi_1(y_1^*, h_0^*) \notin \Phi_1(y_2^*, H^*)$. Let $y_4^*, y_5^* \in S_1$; $h_4^*, h_5^* \in H^*$ are such that

$$(11) \qquad \begin{cases} T^*(y_4^* \otimes h_4^*) = \Phi_1(y_1^*, h_0^*) \otimes \Psi_1(y_2^*), \\ T^*(y_5^* \otimes h_5^*) = \Phi_1(y_1^*, h_1^* + h_0^*) \otimes \Psi_1(y_2^*). \end{cases}$$

We have

$$\begin{aligned} T^*\left(y_2^* \otimes h_2^* + y_4^* \otimes h_4^*\right) &\\ &= \Phi_1\left(y_2^*, h_2^*\right) \otimes \Psi_1\left(y_2^*\right) + \Phi_1\left(y_1^*, h_0^*\right) \otimes \Psi_1\left(y_2^*\right) \\ &= \Phi_1\left(y_1^*, h_1^* + h_0^*\right) \otimes \Psi_1\left(y_2^*\right) = T\left(y_5^* \otimes h_5^*\right). \end{aligned}$$

Hence by Proposition 1 we have $y_2^* \| y_4^*$ or $h_2^* \| h_4^*$. By (10) and (11) we also have $h_1^* \| h_2^*$, and $h_4^* \| h_0^*$ or $y_4^* \| y_1^*$. Hence we have the following possibilities:

1°. $h_0^* \| h_1^*$;

2°. $y_2^* \| y_4^*$;

3°. $y_4^* \| y_1^*$.

If $h_0^* \| h_1^*$, then

$$\Phi_1\left(y_1^*, h_0^*\right) \| \Phi_1\left(y_1^*, h_1^*\right) = \Phi_1\left(y_2^*, h_2^*\right) \in \Phi_1\left(y_2^*, H^*\right),$$

which contradicts our assumption. If $y_2^* \| y_4^*$, then $\Phi_1(y_2^*, H^*) = \Phi_1(y_4^*, H^*)$, and by the assumption, $\Phi_1(y_1^*, h_0^*) \notin \Phi_1(y_4^*, H^*)$, which contradicts (11). If $y_4^* \| y_1^*$, then, since (11) implies $\Psi_1(y_2^*) \| \Psi_1(y_4^*)$, we get $\Psi_1(y_1^*) \| \Psi_1(y_2^*)$, and hence the vectors

$$T^*\left(y_1^* \otimes h_1^*\right) = \Phi_1\left(y_1^*, h_1^*\right) \otimes \Psi_1\left(y_1^*\right) \quad \text{and}$$
$$T^*\left(y_2^* \otimes h_2^*\right) = \Phi_1\left(y_2^*, h_2^*\right) \otimes \Psi_1\left(y_2^*\right)$$

are proportional, so T^* being injective gives $y_1^* \| y_2^*$, which is impossible.

So we have proved (i). To end the proof of (ii), let us notice that if the images of the isometric embeddings $\Phi_1(y_1^*, \cdot)$ and $\Phi_1(y_2^*, \cdot)$ coincide, then, by (10), for any $h^* \in H^*$ there is a number λ_{h^*} of modulus one such that

$$\Phi_1\left(y_1^*, h^*\right) = \lambda_{h^*} \Phi_1\left(y_2^*, h^*\right).$$

So if we compose $\Phi_1(y_1^*, \cdot)$ with the inverse map to $\Phi_1(y_2^*, \cdot)$ (restricted to its image), we get the isometry I from H^* onto itself with the property that every element of H^* is its eigenvector; hence $I = \text{const} \cdot \text{Id}_{H^*}$, and this means that the function $h^* \mapsto \lambda_{h^*}$ is constant, and we get (ii).

For any $y^* \in S_1$, the operator $\Phi_1(y^*, \cdot)$ is weak-$*$-continuous, so $(\Phi_1(y^*, \cdot))^*$ maps X onto H. Let us denote the restriction of $(\Phi_1(y^*, \cdot))^*$ to X by $\Phi_1^*(y^*, \cdot)$, and let Ω denote the subset $\{\lambda \Phi_1^*(y^*, \cdot)\colon y^* \in S_1, \lambda = \pm 1\}$ of the space $L(X, H)$ equipped with the topology given by the family of seminorms

$$\left\{L(X, H) \ni R \mapsto \left|h^*(Rx)\right| \colon x \in X,\ h^* \in H^*\right\}.$$

We define maps Q and Q_1:

$Q\colon S_1 \to \Omega \otimes \operatorname{ext} B(K^*)$, $\quad Q(y^*) = \Phi_1^*(y^*, \cdot) \otimes \Psi_1(y^*)$,

$Q_1 = Q \otimes \mathrm{Id}_{\operatorname{ext} B(H^*)}\colon S_1 \otimes \operatorname{ext} B(H^*) \to \Omega \otimes \operatorname{ext} B(K^*) \otimes \operatorname{ext} B(H^*)$,

$Q_1(x^* \otimes h^*) = Q(y^*) \otimes h^*$.

By (5) the operators Q and Q_1 are continuous and one-to-one, and by Lemma 1 they are onto.

To prove they are onto, it is sufficient to show that, for any $y_0^* \in S_1$ and $k_0^* \in \operatorname{ext} B(K^*)$, there is a $y^* \in S_1$ such that $\Phi_1(y_0^*, \cdot) = \varepsilon\Phi_1(y^*, \cdot)$ and $k_0^* = \varepsilon\Psi_1(y^*)$, where $|\varepsilon| = 1$. Let $h_0^* \in \operatorname{ext} B(H^*)$. Since T^* is onto, there is a $y^* \in S_1$ and $h^* \in \operatorname{ext} B(H^*)$ such that

$$T^*(y^* \otimes h^*) = \Phi_1(y_0^*, h_0^*) \otimes k_0^*. \tag{12}$$

On the other hand, from (5), we have

$$T^*(y^* \otimes h^*) = \Phi_1(y^*, h^*) \otimes \Psi_1(h^*). \tag{13}$$

Hence $\Phi_1(y_0^*, H^*) \cap \Phi_1(y^*, H^*) \neq \{0\}$, and by Lemma 1 there is an $\varepsilon \in \mathbf{R}$, $|\varepsilon| = 1$, such that $\Phi_1(y_0^*, \cdot) = \varepsilon\Phi_1(y^*, \cdot)$. So by (12), (13) we get $k_0^* = \varepsilon\Psi_1(h^*)$.

By definition we have

$$Q_1^{-1}(\omega \otimes k^* \otimes h^*) = (T^{-1})^*(\omega(h^*) \otimes k^*),$$

so Q^{-1} and Q_1^{-1} are also continuous.

Analogously we define $\tilde{\Phi}_1^*(x^*, \cdot) \in L(Y, K)$ for $x^* \in \tilde{S}_1$, the set $\tilde{\Omega} = \{\lambda\Phi_1^*(x^*, \cdot)\colon x^* \in \tilde{S}_1,\ \lambda = \pm 1\}$, and two homeomorphisms P and P_1:

$$P\colon \tilde{S}_1 \to \tilde{\Omega} \otimes \operatorname{ext} B(H^*), \quad P(x^*) = \tilde{\Phi}_1^*(x^*, \cdot) \otimes \tilde{\Psi}_1(x^*),$$
$$P_1 = P \otimes \mathrm{Id}_{\operatorname{ext} B(K^*)}.$$

The maps Q, Q_1, P, and P_1 are homeomorphisms, so they define the isometric embeddings:

$$Q^0\colon Y_1 \mapsto C(\Omega \otimes \operatorname{ext} B(K^*)),$$
$$Q_1^0\colon Y_1 \otimes H \mapsto C(\Omega \otimes \operatorname{ext} B(K^*) \otimes \operatorname{ext} B(H^*)),$$
$$P^0\colon X_1 \mapsto C(\tilde{\Omega} \otimes \operatorname{ext} B(H^*)),$$
$$P_1^0\colon X_1 \otimes K \mapsto C(\tilde{\Omega} \otimes \operatorname{ext} B(H^*) \otimes \operatorname{ext} B(K^*)).$$

By (5) and (7) the homeomorphism $P_1 \circ T^* \circ Q_1^{-1}$ is of the form

$$\Omega \otimes \operatorname{ext} B(K^*) \otimes \operatorname{ext} B(H^*) \ni \omega \otimes k^* \otimes h^*$$
$$\mapsto \varphi(\omega) \otimes h^* \otimes k^* \in \tilde{\Omega} \otimes \operatorname{ext} B(H^*) \otimes \operatorname{ext} B(K^*),$$

where φ is a homeomorphism from Ω onto $\tilde{\Omega}$. Hence, for any $h^* \in \operatorname{ext} B(H^*)$ and $k^* \in \operatorname{ext} B(K^*)$ we have

$$\operatorname{Im} Q^0\Big|_{\Omega\otimes\{k^*\}} = \operatorname{Im} Q_1^0\Big|_{\Omega\otimes\{k^*\}\otimes\{h^*\}} \simeq \operatorname{Im} P_1^0\Big|_{\tilde{\Omega}\otimes\{h^*\}\otimes\{k^*\}} = \operatorname{Im} P^0\Big|_{\tilde{\Omega}\otimes\{h^*\}},$$

so the space $\operatorname{Im} Q^0|_{\Omega\otimes\{k^*\}}$ does not depend on the choice of $k^* \in \operatorname{ext} B(K^*)$, and we denote it by Z.

For any h^* in $\operatorname{ext} B(H^*)$ we have

$$Y_1 \simeq \operatorname{Im} Q^0 \simeq \operatorname{Im} P_1^0\Big|_{\tilde{\Omega}\otimes\{h^*\}\otimes \operatorname{ext} B(K^*)} \subset Z \check{\otimes} K,$$

and by the same arguments, for any k^* in $\operatorname{ext} B(K^*)$,

$$X_1 \simeq \operatorname{Im} P^0 \simeq \operatorname{Im} Q_1^0\Big|_{\Omega\otimes\{k^*\}\otimes \operatorname{ext} B(H^*)} \subset Z \check{\otimes} H.$$

So Y_1 (X_1) is isometric to a subspace of $Z \check{\otimes} K$ ($Z \check{\otimes} H$) which contains any element of the form $z \otimes k$ ($z \otimes h$) for $z \in Z$, $k \in K$, $h \in H$, and therefore, to end the proofs of $Y_1 \simeq Z \check{\otimes} K$ and $X_1 \simeq Z \check{\otimes} H$, it is sufficient to show that Y_1, X_1, and, as a consequence, Z, are complete. For the sake of simplicity of notation, we will assume from now on, without loss of generality, that $S_1 = \Omega \otimes \operatorname{ext} B(K^*)$, $\tilde{S}_1 = \tilde{\Omega} \otimes \operatorname{ext} B(H^*)$, $\varphi = \mathrm{id}_\Omega$, $Z \otimes H \subset X_1 \subseteq Z \check{\otimes} H$, $Z \otimes K \subset Y_1 \subseteq Z \check{\otimes} K$, and, consequently,

$$(14)\qquad T^*(\omega \otimes k^* \otimes h^*) = \omega \otimes h^* \otimes k^* \quad \text{for any } \omega \otimes k^* \in S_1 \text{ and } h^* \in \operatorname{ext} B(H^*).$$

For any $h^* \in H^*$ and $k \in K$ we define a continuous, linear operator $S_{h^*,k}\colon X \to Y$:

$$y^*(S_{h^*,k}(x)) = y^* \otimes h^*(T(x \otimes k)) \quad \text{for any } y^* \in \operatorname{ext} B(Y^*).$$

Similarly, for any $k^* \in K^*$ and $h \in H$ we define a continuous, linear operator $\tilde{S}_{k^*,h}\colon Y \to X$:

$$x^*(\tilde{S}_{k^*,h}(y)) = x^* \otimes k^*(T^{-1}(y \otimes h)) \quad \text{for any } x^* \in \operatorname{ext} B(X^*).$$

By (14) and (6) we have

$$(15)\qquad y^*(S_{h^*,k}(x)) = \begin{cases} \omega \otimes h^*(x)k^*(k) & \text{for } y^* = \omega \otimes k^* \in S_1, \\ \Phi_2(y^*)(x)\Psi_2(y^*, h^*)(k) & \text{for } y^* \in \bar{S}_2, \end{cases}$$

and by (14) and (8) we have

$$(16)\qquad x^*(\tilde{S}_{k^*,h}(y)) = \begin{cases} \omega \otimes k^*(y)h^*(h) & \text{for } x^* = \omega \otimes h^* \in \tilde{S}_1, \\ \tilde{\Phi}_2(x^*)(y)\tilde{\Psi}_2(x^*, k^*)(h) & \text{for } x^* \in \bar{\tilde{S}}_2. \end{cases}$$

By the above equalities, for any $x_0^* = \omega_0 \otimes h_0^* \in \tilde{S}_1$ we have

$$(17)\qquad x_0^*(\tilde{S}_{k^*,h} \circ S_{h^*,k}(x)) = \omega_0 \otimes h^*(x)k^*(k)h_0^*(h),$$

and for any $x_0^* \in \bar{\tilde{S}}_2$ the equality $\Phi_2 \circ \tilde{\Phi}_2 = \mathrm{Id}_{\bar{\tilde{S}}_2}$ gives

$$(18)\quad x_0^*(\tilde{S}_{k^*,h} \circ S_{h^*,k}(x)) = x_0^*(x)\Psi_2(\tilde{\Phi}_2(x_0^*), h^*)(k)\tilde{\Psi}_2(x_0^*, k^*)(h).$$

To prove that X_1 is complete and $X = X_1 \oplus_\infty X_2$, we have to show that for any $x = (x_1, x_2) \in X$ we have $(x_1, 0) \in X$, and since the map $X \ni (x_1, x_2) \mapsto (x_1, 0) \in X$ is linear and continuous and $Z \otimes H$ (algebraic tensor product of Z and H) is a dense subset of X_1, it is sufficient to show that $(z_0 \otimes h_0, 0) \in X$ for any $z_0 \in Z$, $h_0 \in H$. For this purpose fix $x_0 = (z_0 \otimes h_0, x_2) \in X$ with $\|z_0\| = \|h_0\| = 1$. We show that for any $\varepsilon > 0$ there is a continuous operator $A\colon X \to X$ (which depends on x_0 and ε) such that $Ax_0 = (z_0 \otimes h_0, x_2')$ with $\|x_2'\| \le \varepsilon$, and, hence, by completeness of X we get $(z_0 \otimes h_0, 0) \in X$. To this end fix $x_0^* \in \bar{\tilde{S}}_0$ and let

$h_1^* \in \operatorname{ext} B(H^*)$ be such that $h_1^*(h_0) = 1$,
$k_1 \in K$ be such that $\|k_1\| = 1$ and
$\Psi_2(\tilde{\Phi}_2(x_0^*), h_1^*)(k_1) = 0$ (such k_1 exists provided that $\dim K \ge 2$),
$k_1^* \in \operatorname{ext} B(K^*)$ be such that $k_1^*(k_1) = 1$.

By (17) for any $x^* = \omega \otimes h^* \in \tilde{S}_1$ we have

$$\begin{aligned} \omega \otimes h^*\big(\tilde{S}_{k_1^*,h_0} \circ S_{h_1^*,k_1}(x_0)\big) &= \omega \otimes h_1^*(z_0 \otimes h_0, x_2)k_1^*(k_1)h^*(h_0) \\ &= \omega(z_0)h_1^*(h_0)h^*(h_0) = \omega(z_0)h^*(h_0) \\ &= \omega \otimes h^*(z_0 \otimes h_0) = \omega \otimes h^*(x_0); \end{aligned}$$

this means

$$\tilde{S}_{k_1^*,h_0} \circ S_{h_1^*,k_1}(x_0)\Big|_{\tilde{S}_1} = x_0,$$

and by (18) we get, by the same arguments,

$$\tilde{S}_{k_1^*,h_0} \circ S_{h_1^*,k_1}(x_0)\Big|_{\bar{\tilde{S}}_2} = f_{x_0^*} \cdot x_0,$$

where

$$f_{x_0^*}(x^*) = \Psi_2\big(\tilde{\Phi}_2(x^*), h_1^*\big)(k_1)\tilde{\Psi}_2\big(x^*, k_1^*\big)(h_0)$$

is a continuous function on $\bar{\tilde{S}}_2$, of norm not greater than one and such that $f_{x_0^*}(x_0^*) = 0$. Hence, by the compactness of $\bar{\tilde{S}}_2$, for any $\varepsilon > 0$, by iterating the action of the operators $\tilde{S}_{k_j^*,h_0} \circ S_{h_j^*,k_j}$ for suitable k_j^*, h_j^*, and k_j, we get $x' \in X$ such that

$$x'|_{\tilde{S}_1} = x_0|_{S_1} \quad \text{and} \quad |x^*(x')| \leq \varepsilon \qquad \text{for any } x^* \in \bar{\tilde{S}}_2.$$

So we have proven that X_1 and Y_1 are complete; we have actually proven even more—namely, that X_2, Y_2 are also complete and $X \simeq X_1 \oplus_\infty X_2$, $Y \simeq Y_1 \oplus_\infty Y_2$. Thus to end the proof it is sufficient if we restrict ourselves to investigating the isometry T between $X_2 \otimes K$ and $Y_2 \otimes H$. Without loss of generality we can assume that X_2 and Y_2 are subspaces of some function modules $\Pi_{\alpha \in \tilde{\Gamma}} X_\alpha$ and $\Pi_{\alpha \in \Gamma} Y_\alpha$, respectively, and that the identity embeddings π_X: $X_2 \mapsto \Pi_{\alpha \in \tilde{\Gamma}} X_\alpha$ and π_Y: $Y_2 \mapsto \Pi_{\alpha \in \Gamma} Y_\alpha$ give the maximal function module representation ([**2**]). Hence any $y^* \in S_2$ is of the form

$$\prod_{\alpha \in \Gamma} Y_\alpha \supset Y \ni y \overset{\delta_\alpha \otimes y_\alpha^*}{\mapsto} y_\alpha^*(y(\alpha))$$

$$\text{for some } y_\alpha^* \in \operatorname{ext} B(Y_\alpha^*) \text{ and } \alpha \in \Gamma.$$

Let $k \in K$, $h \in H$, $k^* \in K^*$, $h^* \in H^*$. By (18) the operator $\tilde{S}_{k^*,h} \circ S_{h^*,k}$: $X \to X$ is of the form

$$\delta_\alpha \otimes x_\alpha^*\left(\tilde{S}_{k^*,h} \circ S_{h^*,k}\left((x_\alpha)_{\alpha \in \Gamma}\right)\right) = f(\alpha \otimes x_\alpha^*) \cdot x_\alpha^*(x_\alpha),$$

where

$$f(\alpha \otimes x_\alpha^*) = \Psi_2\left(\tilde{\Phi}_2(\delta_\alpha \otimes x_\alpha^*), h^*\right)(k)\tilde{\Psi}_2(\delta_\alpha \otimes x_\alpha^*, k^*)(h),$$

so it is just multiplication by a function f: $\tilde{S}_2 \to \mathbf{R}$. Since we have the maximal function module representation, the function f does not depend on x_α^* but only on $\alpha \in \Gamma$, and consequently the functions

$$\tilde{S}_2 \ni \delta_\alpha \otimes x_\alpha^* \mapsto \tilde{\Phi}_2(\delta_2 \otimes x_\alpha^*)$$

and

$$\tilde{S}_2 \ni \delta_\alpha \otimes x_\alpha^* \mapsto \tilde{\Psi}_2(\delta_\alpha \otimes x_\alpha^*, \cdot)$$

also do not depend on x_α^* but only on $\alpha \in \Gamma$. Hence by (8) the operator $(T^{-1})^*$ is of the form

$$\begin{aligned}(T^{-1})^*(\delta_\alpha \otimes x_\alpha^* \otimes k^*) &= \tilde{\Phi}_2(\delta_\alpha \otimes x_\alpha^*) \otimes \tilde{\Psi}_2(\delta_\alpha \otimes x_\alpha^*, k^*)\\ &= \delta_{\varphi(\alpha)} \otimes \tilde{\Phi}_\alpha(x_\alpha^*) \otimes \tilde{\Psi}_\alpha(k^*),\end{aligned}$$

where φ: $\Gamma \to \tilde{\Gamma}$ and $\tilde{\Phi}_\alpha$: $X_\alpha^* \to Y_\alpha^*$, $\tilde{\Psi}_\alpha$: $K^* \to H^*$ are weak-$*$-continuous onto isometries.

Composing the above formula with an analogous formula for T^*, we get

$$\delta_\alpha \otimes x_\alpha^* \otimes k^* = \delta_{\psi\circ\varphi(\alpha)} \otimes \left(\Phi_{\varphi(\alpha)} \circ \tilde{\Phi}_\alpha(x_\alpha^*)\right) \otimes \left(\Psi_{\varphi(\alpha)} \circ \tilde{\Psi}_\alpha(k^*)\right).$$

Hence φ is a bijection between Γ and $\tilde{\Gamma}$, and we can assume $\Gamma = \tilde{\Gamma}$, $\varphi = \mathrm{id}_\Gamma$, and T^* is of the form

$$T^*(\delta_\alpha \otimes y_\alpha^* \otimes h^*) = \delta_\alpha \otimes \phi_\alpha(y_\alpha^*) \otimes \Psi_\alpha(h^*), \tag{19}$$

where Φ_α: $Y_\alpha^* \to X_\alpha^*$, Ψ_α: $H^* \to K^*$ are weak-$*$-continuous onto isometries.

Put

$$A = \prod_{\alpha\in\Gamma} \Phi_\alpha^*\colon \prod_{\alpha\in\Gamma} X_\alpha \to \prod_{\alpha\in\Gamma} Y_\alpha.$$

The operator A is an onto isometry, and to conclude the proof we show $A(X) = Y$. By (19) and (16) for any $h^* \in \operatorname{ext} B(H^*)$, $k \in K$ the operator $\tilde{S}_{k^*,h} \circ A$: $A^{-1}(Y) \to X$ is of the form

$$x_\alpha^*(\tilde{S}_{k^*,h} \circ A(w)(\alpha)) = x_\alpha^*(w(\alpha)) \cdot \tilde{\Psi}_\alpha(k^*)(h).$$

Hence the function $\Gamma \ni \alpha \mapsto \tilde{\Psi}_\alpha(k^*)(h)$ is continuous, and, since $A^{-1}(Y) \subset \prod_{\alpha\in\Gamma} X_\alpha$ is a function module, we get

$$\tilde{S}_{k^*,h} \circ A(w) \in X \cap A^{-1}(Y)$$

for any $w \in A^{-1}(Y)$, $k^* \in \operatorname{ext} B(K^*)$, $h \in H$.

So to prove $A^{-1}(Y) \subset X$ and, by symmetry, $A^{-1}(Y) = X$, it is sufficient to show that the set

$$\operatorname{Lin}\{\tilde{S}_{k^*,h} \circ A(w) : w \in A^{-1}(Y),\ k^* \in \operatorname{ext} B(K^*),\ h \in H\}$$

is dense in $A^{-1}(Y)$, but this is an immediate consequence of the definition of $\tilde{S}_{k^*,h}$:

$$x^*(\tilde{S}_{k^*,h}(y)) = x^* \otimes k^*(T^{-1}(y \otimes h)).$$

Hence

$$x^*\Big(\sum_j \tilde{S}_{k^*,h_j}(y_j)\Big) = x^* \otimes k^*\Big(T^{-1}\Big(\sum_j y_j \otimes h_j\Big)\Big),$$

and the set $\{\sum_j y_j \otimes h_j\colon y_j \in Y,\ h_j \in H\}$ is dense in $Y \check{\otimes} H$, and T^{-1} is onto.

Remark. As proved by E. Behrends in the special case of Theorem 1 when $X = C(S)$ and $Y = C(S')$, the assumption about K, H can be weakened to effect $\dim Z(H) = 1 = \dim Z(K)$. It is worthwhile to mention that, in general, this strengthened form of Theorem 1 is not valid: to provide an example, let A be the disc algebra, i.e., the complex Banach algebra of all continuous functions defined on the unit disc on the complex plane which are analytic in the interior of the disc, and let A_R denote the Banach space A over the field of real numbers. Put A_R^j for the injective tensor product of j copies of A_R. We have $\dim Z(A_R^j) = 1$ for $j = 1, 2, \ldots$ and

$$A_R^2 \otimes A_R^3 \simeq A_R^1 \otimes A_R^4,$$

while

$$A_R^j \simeq A_R^i \quad \text{only if} \quad j = i.$$

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