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ONE-DIMENSIONAL ALGEBRAIC FORMAL GROUPS

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Let K be an algebraically closed field of characteristic zero. We shall call an element of $K[[x_1, \dots, x_n]]$ algebraic if it is algebraic over $K(x_1, \dots, x_n)$. Thus a one-dimensional algebraic formal group is an element $F \in K[[x_1, x_2]]$ such that F is a formal group and F is algebraic. As is well known, such formal groups arise from one-dimensional algebraic groups. Our intention is to show that this is the only way they arise. All formal groups mentioned in this note shall be one-parameter formal groups.

DEFINITION. Two algebraic formal groups $F, F' \in K[[x_1, x_2]]$ are said to be algebraically isomorphic if there exists an algebraic element $f \in xK[[x]]$ such that $f \neq 0$ and

$$f(F(x_1, x_2)) = F'(f(x_1), f(x_2)).$$

It is easy to see that there exists a unique element $f^* \in xK[[x]]$ such that $f \circ f^* = x$. It then follows that

$$f^*F'(x_1, x_2) = F(f^*(x_1), f^*(x_2))$$

and that f^* is algebraic.

Now suppose $(X, e, [+])$ is a one-dimensional algebraic group over K . Let $z \in K(X)$ be a local parameter at e . Let $\rho_1, \rho_2: X \times X \rightarrow X$ be the natural projections. Then $\{z \circ \rho_1, z \circ \rho_2\}$ is a set of local parameters at $e \times e$ in $X \times X$, and so there exists a unique power series $H(x, y) \in K[[x, y]]$ such that

$$H(z \circ \rho_1, z \circ \rho_2) = z(\rho_1[+] \rho_2)$$

as elements of the complete local ring at $e \times e$ on $X \times X$. It is easy to see that H is an algebraic formal group. We shall call such a formal group a *formal algebraic group*.

PROPOSITION A. *Every algebraic formal group is algebraically isomorphic to a formal algebraic group.*

We will prove a stronger statement than Proposition A. We call a differential $\omega \in K[[x]] dx$ algebraic if ω/dx is an algebraic element of $K[[x]]$. If $H(x, y)$ is a formal group and

$$g(x) = \frac{d}{dy} H(x, y) \Big|_{y=0},$$

then $g(0) = 1$, and

$$\omega = g dx$$

is the invariant differential of H . If H is an algebraic, then so is ω . We will prove

PROPOSITION B. *Let ω be an algebraic differential. Suppose that there exist nonzero algebraic elements f_1, f_2 of $xK[[x]]$ such that*

$$f_1^*(\omega) = af_2^*(\omega)$$

where $a \in \mathbb{C}^$, a is not a root of unity. Then there exist a formal algebraic group with invariant differential ω' and an algebraic element u of $K[[x]]$ such that*

$$eu^*(\omega') = \omega$$

where $e = \text{Res}_0(\omega/x)$.

To deduce Proposition A from Proposition B, let F be an algebraic formal group, ω its invariant differential, $f_2(x) = x$, $f_1(x) = F(x, x)$. Then

$$(0) \quad f_1^*(\omega) = 2\omega = 2f_2^*(\omega).$$

It follows that there exists a formal algebraic group H with invariant differential ω' and an algebraic element $g \in xK[[x]]$ such that

$$(1) \quad g^*(\omega') = \omega.$$

We claim

$$g(F(x, y)) = H(g(x), g(y)).$$

Indeed, if $\lambda, \lambda' \in xK[[x]]$, $d\lambda = \omega$, $d\lambda' = \omega'$, then (1) implies $\lambda' \circ g = \lambda$. On the other hand,

$$\lambda F(x, y) = \lambda(x) + \lambda(y)$$

$$\lambda' H(x, y) = \lambda'(x) + \lambda'(y),$$

so that

$$\begin{aligned} gF(x, y) &= \lambda'^{-1} \circ \lambda F(x, y) = \lambda'^{-1}(\lambda(x) + \lambda(y)) \\ &= H(\lambda'^{-1} \circ \lambda(x) + \lambda'^{-1} \circ \lambda(y)) = H(g(x), g(y)) \end{aligned}$$

as required.

Proof of Proposition B. Let \mathbf{P}^1 denote the projective line over K and regard x as the standard parameter on \mathbf{P}^1 . In doing this we will identify $K[[x]]$ with the formal completion of the ring of functions on \mathbf{P}^1 regular at 0, $\widehat{\mathcal{O}_{\mathbf{P}^1, 0}}$.

Let $f_0 = \omega/dx$. Then for $i = 0, 1, 2$ there exist complete pointed curves (X_i, e_i) over K together with morphisms

$$x_i, \tilde{f}_i: Y_i \rightarrow \mathbf{P}^1$$

such that x_i is a local uniformizing parameter at e_i and $x_i^* f_i$ is the formal expansion of \tilde{f}_i in x_i at e_i . In other words, $x_i^* f_i$ is the image of f_i in \mathcal{O}_{Y_i, e_i} .

Now set $\tilde{\omega} = \tilde{f}_0 dx_0 \in \Omega_{Y_0/K}^1$. Also note that $f_i(e_i) = 0$ as $f_i(0) = 0$, $i = 1, 2$. Let (Z_i, e_i) denote the fiber product of (Y_0, e_0) and (Y_i, e_i) over $(\mathbf{P}^1, 0)$ with respect to the morphisms x_0 and \tilde{f}_i , $i = 1, 2$. Thus (Z_i, e_i') fits into a commutative diagram

$$\begin{array}{ccc} (Z_i, e_i') & \xrightarrow{y_i} & (Y_i, e) \\ \tilde{f}_i \downarrow & & \downarrow \tilde{f}_i \\ (Y_0, e_0) & \xrightarrow{x_0} & (\mathbf{P}^1, 0). \end{array}$$

Moreover, $(x_i \circ y_i)^* f_i^* \omega$ is the formal expansion of $\tilde{f}_i^* \tilde{\omega}$ at e_i' in $x_i \circ y_i$. Now let (W, e) denote the fiber product of (Z_1, e_1') and (Z_2, e_2') with respect to the morphisms $x_1 \circ y_1$ and $x_2 \circ y_2$. Thus we have a commutative diagram

$$\begin{array}{ccc} (W, e) & \xrightarrow{z_2} & (Z_2, e_2') \\ z_1 \downarrow & & \downarrow x_2 \circ y_2 \\ (Z_1, e_1) & \xrightarrow{x_1 \circ y_1} & (\mathbf{P}^1, 0). \end{array}$$

Let (W^c, e) denote the connected component of (W, e) passing through e . Let

$$\tilde{f}_i: (W^c, e) \rightarrow (Y_0, e_0)$$

denote the restriction of $\tilde{f}_i \circ z_i$ to W^c . Then

$$(x_i \circ y_i \circ z_i)^* f_i^* \omega$$

is the formal expansion of $\bar{f}_i^* \tilde{\omega}$ at e in $x_i \circ y_i \circ z_i$. Since $x_1 \circ y_1 \circ z_1 = x_2 \circ y_2 \circ z_2$, it follows from the hypothesis that

$$\bar{f}_1^* \tilde{\omega} = a \bar{f}_2^* \tilde{\omega}.$$

Taking $X_1 = X_0$, $X_2 = W^c$ and $\omega_1 = \tilde{\omega}$ we see that Proposition B follows from:

PROPOSITION C. *Let X_1, X_2 be two curves. Let ω_1 be a nonzero differential on X_1 and f_1, f_2 two nonconstant morphisms from X_2 to X_1 such that*

$$(2) \quad f_1^*(\omega_1) = a f_2^*(\omega_1)$$

for some $a \in K^$, a not a root of unity. Then there exists a one-dimensional algebraic group G with invariant differential ω , and a morphism $f: X_1 \rightarrow G$ such that*

$$f^*(\omega) = \omega_1.$$

Proof. For a curve C let \bar{C} denote its complete nonsingular model. Let $\omega_2 = f_2^*(\omega_1)$. Let S_i denote the set of poles of ω_i on \bar{X}_i . Clearly, $|S_1| \leq |S_2|$, $|S_i|$ denotes the order of S_i . We also claim:

$$g(X_1) < g(X_2) \quad \text{or} \quad g(X_2) \leq 1$$

where $g(X_i)$ denotes the genus of X_i . Indeed, if this is not the case, then by the Hurwitz genus formula we see that $g(X_1) = g(X_2) > 1$ and $1 = \deg(f_1) = \deg(f_2)$, but then $\bar{f}_i: \bar{X}_2 \rightarrow \bar{X}_1$ is biregular (\bar{f}_i is the “lifting” of f_i), so that $\alpha = \bar{f}_2^{-1} \circ \bar{f}_1$ is an automorphism of X_2 . But α is of finite order since $g(X_2) > 1$. On the other hand, the hypotheses of the lemma imply

$$\alpha^*(\omega_2) = a \omega_2.$$

Since a is not a root of unity, we obtain a contradiction, so we have our claim.

We also claim that there exists a curve X_0 with a differential ω_0 and two morphisms $g_1, g_2: X_1 \rightarrow X_0$ such that $g_2^*(\omega_0) = \omega_1$ and $g_1^*(\omega_0) = a g_2^*(\omega_0)$. Thus (X_0, ω_0) satisfies the same hypotheses as (X_1, ω_1) , so once we establish this claim, we will be able to use induction to suppose that $|S_1| = |S_2|$ and $g(X_2) \leq 1$.

For the results on generalized Jacobians used below, see [S].

Proof of Claim. Without loss of generality X_i is nonsingular, ω_i has no poles on X_i , and $f_i X_2 = X_1$, for $i = 1, 2$.

Let $i = 1$ or 2 in the following: Let M_i denote the polar divisor of ω_i . Let J_i denote the generalized Jacobian of X_i corresponding to M_i . There exists a unique invariant differential ν_i on J_i and an embedding of X_i in J_i (as $\omega_i \neq 0$) well defined up to translation such that ω_i is the pullback of ν_i to X_i . Henceforth we will view X_i as a subvariety of J_i . From the functoriality of generalized Jacobians there exists a canonical affine transformation

$$f'_i: J_2 \rightarrow J_1$$

whose restriction to X_2 is f_i . Let T_i denote translation on J_2 by $[-]f'_i(0)$ where $[-]$ denotes inversion on J_1 . Set $f''_i = T_i \circ f'_i$. Then f''_i is a homomorphism from J_2 to J_1 . It follows that

$$(f''_1)^* \nu_1 = a(f''_2)^* \nu_1 = a\nu_2.$$

There also exists a homomorphism $h: J_1 \rightarrow J_2$ such that

$$f''_2 \circ h = [d]$$

where d denotes the degree of f_2 and $[d]$ denotes multiplication by d on J_1 . Let

$$e = (f''_1 \circ h \circ f''_2 - [d] \circ f''_1): J_2 \rightarrow J_1.$$

Then e is a homomorphism and

$$\begin{aligned} e^* \nu_1 &= (f''_2)^* h^* (f''_1)^* \nu_1 - g_1^* [d]^* \nu_1 = a(f''_2)^* h^* \nu_2 - dg_1^* \nu_1 \\ &= a(f''_2)^* h^* f_2^* \nu_1 - da\nu_2 = a(f''_2)^* [d]^* \nu_1 - da\nu_2 = 0. \end{aligned}$$

Let A denote the quotient of J_1 by $e(J_2)$ and $\rho: J_1 \rightarrow A$ the quotient morphism. Since $e^* \nu_1 = 0$, it follows that there exists an invariant differential ν_0 on A such that $\rho^* \nu_0 = \nu_1$. Let

$$X_0 = (\rho \circ [d] \circ T_1)(X_1) \subseteq A.$$

As $\rho \circ e = 0$ we have $\rho \circ [d] \circ f''_1 = \rho \circ f''_1 \circ h \circ f''_2$. Hence as $f'_1(X_2) = f'_2(X_2) = X_1$,

$$\begin{aligned} X_0 &= (\rho \circ [d] \circ T_1 \circ f'_1)(X_2) \\ &= (\rho \circ [d] \circ f''_1)(X_2) = (\rho \circ f''_1 \circ h \circ f''_2)(X_2) \\ &= (\rho \circ f''_1 \circ h \circ T_2)(X_1). \end{aligned}$$

Now let $g_1, g_2: X_1 \rightarrow X_0$ denote the restrictions of

$$\rho \circ f''_1 \circ h \circ T_2 \quad \text{and} \quad \rho \circ [d] \circ T_1$$

respectively to X_1 . Also let ω_0 denote the restriction of ν_0/d to X_0 . Since $(\rho \circ f_1'' \circ h)^*\nu_0 = (f_1'' \circ h)^*\nu_1 = \text{ad } \nu_1 = a(\rho \circ [d])^*\nu_0$ it follows that

$$(2) \quad g_1^*\omega_0 = ag_2^*\omega_0 = a\omega_1$$

and so we have our claim. Thus by induction we may suppose

$$g(X_1) = g(X_2) \leq 1 \quad \text{and} \quad |S_2| = |S_1|.$$

We also have $f_i^{-1}(S_1) = S_2$, so that f_i induces a bijection from S_2 onto S_1 .

Case 1. $g(X_i) = 1$. Then \bar{X}_i has a unique group structure with origin at some point P_i . It follows that f_2 and $T_R \circ f_1$ are affine transformations from X_2 to X_1 . Now since $f_i|_{S_2}: S_2 \rightarrow S_1$ is a bijection and $f_i^{-1}(S_1) = S_2$, it follows that either

$$S_2 = S_1 = \emptyset$$

or degree $f_i = 1$, $i = 1, 2$, because f_i is étale. In the second case, f_2^{-1} exists and $\alpha = f_2^{-1} \circ f_1$ is an automorphism of X_2 such that $\alpha S_2 = S_2$. But if $S_2 \neq \emptyset$, α is of finite order. This contradicts

$$\alpha^*\omega_2 = a\omega_2.$$

Thus $S_1 = S_2 = \emptyset$, and ω_1 is an invariant differential on X_1 as required.

Case 2. $g(X_i) = 0$. Then $|S_i| \geq 1$. Let

$$A = \begin{cases} \{\infty\} & \text{if } |S_1| = 1, \\ \{\infty, 0\} & \text{if } |S_1| = 2, \\ \{\infty, 0, 1\} & \text{if } |S_1| \geq 3. \end{cases}$$

After composing with linear fractional transformations, we may suppose $A \subseteq S_2$ and $A \subseteq S_1$.

If $|S| = 1$, then $\omega_1 = b dx$ for some $b \in K^*$, and so is an invariant differential on G_a . Now suppose $|S_2| \geq 1$. Let h_i be a linear fractional transformation such that

$$h_i \circ f_i(p) = p, \quad p \in A.$$

Because $(h_i \circ f_i)^{-1}(p) = \{p\}$, $p \in A$, it follows that $h_i \circ f_i$ takes the value p with multiplicity n_i where n_i is the degree of f_i . As $\{0, \infty\} \subseteq A$ we must have

$$h_i \circ f_i = c_i x^{n_i}$$

where $c_i \in K^*$. If $|S_2| > 2$, then $1 \in A$. It follows that $c_i = 1$, and since $(h_i \circ f_i)^{-1}(1) = 1$, that $n_i = 1$. That is, $f_i = h_i^{-1}$. But then $\alpha = h_2^{-1} \circ h_1$ takes

S_2 onto itself, and $\alpha^*\omega_2 = a\omega_2$. As the group of linear fractional transformations preserving S_2 is finite this contradicts the hypothesis that a is not a root of unity. Thus $S_2 = S_1 = \{0, \infty\}$,

$$f_i = r_i x^{m_i} \quad \text{and} \quad \omega_1 = s dx + t \frac{dx}{x}$$

for some $r_i, t \in K^*$, $m_i \in \mathbf{Z}$, $m_i \neq 0$ and $s \in K$. So,

$$f_i^*(\omega_1) = sr_i m_i x^{m_i-1} dx + tm_i \frac{dx}{x}.$$

Since $a \neq 1$, the hypothesis $f_1^*(\omega) = af_2^*(\omega_1)$ implies $s = 0$. Thus ω_1 is an invariant differential on \mathbf{G}_m as required.

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Michael James Cambern , Near isometries of Bochner L^1 and L^∞ spaces	1
Kun Soo Chang, Gerald William Johnson and David Lee Skoug , The Feynman integral of quadratic potentials depending on two time variables	11
Robert Coleman , One-dimensional algebraic formal groups	35
Alberto Collino , The Abel-Jacobi isomorphism for the cubic fivefold	43
N. J. Dev and S. S. Khare , Finite group action and vanishing of $N_*^G[F]$	57
Harold George Diamond and Jeffrey D. Vaaler , Estimates for partial sums of continued fraction partial quotients	73
Kenneth R. Goodearl , Patch-continuity of normalized ranks of modules over one-sided Noetherian rings	83
Dean Robert Hickerson and Sherman K. Stein , Abelian groups and packing by semicrosses	95
Karsten Johnsen and Harmut Laue , Fitting structures	111
Darren Long , Discs in compression bodies	129
Joseph B. Miles , On the growth of meromorphic functions with radially distributed zeros and poles	147
Walter Volodymyr Petryshyn , Solvability of various boundary value problems for the equation $x'' = f(t, x, x', x'') - y$	169
Elżbieta Pol , The Baire-category method in some compact extension problems	197
Masami Sakai , A new class of isocompact spaces and related results	211
Thomas Richard Shemanske , Representations of ternary quadratic forms and the class number of imaginary quadratic fields	223
Tsuyoshi Uehara , On class numbers of cyclic quartic fields	251