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FINITE GROUP ACTION AND VANISHING OF $N_*^G[F]$

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Let G be a finite group (not necessarily abelian). The object of this paper is to describe a G-bordism theory which vanishes. We construct a family F of G slice types, such that the N_*-module $N_*^G[F]$ is zero. Kosniowski has proved a similar result earlier for a finite abelian group. The present work is a generalisation of his result by using basically the same technique. A recent result of Khare is obtained as a corollary to the vanishing of $N_*^G[F]$.

1. Preliminaries and statement of the main theorem. Let G be a finite group with centre $C(G)$ and G_2 be the subgroup generated by the elements of order 2 in $C(G)$. We also assume that G_2 is nontrivial. By a G-manifold M we mean throughout a closed differentiable manifold on which G acts smoothly. G_x denotes the isotropy subgroup at $x \in M$. For every $x \in M$, there exists a G_x-module $\overline{V}_x$ which is equivariantly diffeomorphic to a G_x-invariant neighbourhood of x. $\overline{V}_x$ has a submodule V_x' in which G_x acts trivially and a complementary submodule V_x in which no nonzero element is fixed by all of G_x. By the G-slice type of x we mean the pair $[G_x;\ V_x]$. By a G-slice type we mean a pair $[H;\ U]$ where U is a H-module in which no nonzero element is fixed by all of H (equivalently U contains no trivial H-submodule). A family F of G-slice types is a collection of G-slice types such that if $[H;\ U] \in F$ then for every $x \in G \times_H U$ the G-slice type $[G_x;\ V_x] \in F$. A G-manifold is said to be of type F if for all $x \in M$, $[G_x, V_x] \in F$. Bordism relation is defined in the usual way. Two n-dimensional closed G-manifolds M_1, M_2 of type F are said to be F-bordant if there exists an $(n+1)$-dimensional compact differentiable G-manifold W of type (F, F) such that the disjoint union of M_1 and M_2 is G equivariantly diffeomorphic to ∂W. It is clearly an equivalence relation on the set of G-manifolds of type F and gives rise to a bordism theory $N_*^G[F]$. We note that $N_*^G[F]$ is a graded N_*-module, N_* being the unoriented bordism ring.

Kosniowski has described a family $\tilde{F}(\hat{G})$ in [4] for an abelian group G such that $N_*^G[\tilde{F}(\hat{G})] = \hat{0}$, $\hat{G}$ being a subgroup of G containing G_2. As a consequence he proved that if M is a G-manifold (G abelian) in which G_2 acts without fixed points then M is a G-boundary—a result obtained earlier by Khare using a different technique [1]. The main theorem of this paper is a generalisation of Kosniowski's theorem in [4] for an arbitrary finite group. Once again another result of Khare [2] is obtained as a corollary of this theorem.

The subgroup G_2 consisting of the identity and elements of order two in the centre of G is isomorphic to $\mathbf{Z}_2^k$ for some $k > 0$. Kosniowski has studied $\mathbf{Z}_2^k$-bordism in [3] and the techniques used here are generalized from his techniques. We choose once for all a basis $g_1, g_2, \dots, g_k$ of G_2 and order the elements by

$$g_1 < g_2 < \cdots < g_k < g_1 g_2 < \cdots < g_1 g_k < \cdots < g_1 g_2 \cdots g_k.$$

Now let $[G_x;\ V_x]$ be the G-slice type of a point x in a G-manifold M and $G(x)$ be the orbit of x. Then $G(x)$ is a closed and compact submanifold of M. Consider the normal bundle $\nu(i)$ of the canonical embedding of $G(x)$ in M. This is a G-vector bundle and its disc bundle is a closed G-invariant tubular neighborhood of $G(x)$. Further G acts as a group of bundle maps on the normal bundle and the fibre over x is G_x-invariant and contains no G_x-trivial subspace. It is precisely V_x the G_x-module present in the G-slice type $[G_x;\ V_x]$ of x. Let g_* be the map on the total space $E(\nu(i))$ induced by the action of g on the base space $G(x)$. The G-slice type of $gx \in G(x)$ is $[gG_x\bar{g}^1;\ g_* V_x]$. The underlying vector space of V_x and $g_* V_x$ are same and the action of $gh\bar{g}^1$, $h \in G_x$ on $v \in g_* V_x$ is same as the action of h on $v \in V_x$. Again if F be a family of G-slice types and $[H;\ V] \in F$ then from the definition of family the G-slice type $[G_x;\ V_x]$ of every point $x \in G \times_H V$ belongs to F. Now the G-slice type of $[e, 0] \in G \times_H V$ is $[H;\ V]$ and the G slice type of $[g, 0] \in G \times_H V$ is $[gH\bar{g}^1;\ g_* V]$. The G-slice type $[H;\ V]$ will be denoted by ρ and the collection

$$\left\{ \left[gH\bar{g}^1;\ g_* V\right] \middle| g \in G \right\}$$

termed as a conjugate class of G-slice types will be denoted by $\bar{\rho}$ or $[H;\ V]^g$.

Suppose that K is a subgroup of H. We write $K \subset_2 H$ if $H = (x) \times K$ where $x \in G_2$. Quite a number of elements of G_2 may yield H when a direct product of above type is formed. We take the minimal element x according to the total order fixed at the beginning of this article. We now have a homomorphism

$$p = p_{H,K} \colon H \to K.$$

which is the projection onto the second factor. This is termed as the distinguished projection. It enables us to obtain an H-module p^*U from a K-module U. The modules p^*U and U have the same underlying vector space and H acts on p^*U via the map p. Corresponding to a G slice type $[K;\ U]$ such that $K \subset_2 H$ we have an extension function $e = e_{K,H}$ given by

$$e_{K,H}[K;\ U] = [H;\ V(K) \oplus p^*U]$$

where $V(K)$ is one dimensional real representation of H in which $h \in H$ acts by multiplication with 1 if $h \in K$ and multiplication with -1 if $h \notin K$. Since $gHg^{\bar{1}} = (x) \times gKg^{\bar{1}}$ when $H = (x) \times K$, we have

$$\begin{aligned} e[gKg^{\bar{1}};\ g_*U] &= [gHg^{\bar{1}};\ V(gKg^{\bar{1}}) \oplus p^*(g_*U)] \\ &= [gHg^{\bar{1}};\ g_*(V(K) \oplus p^*U)]. \end{aligned}$$

Thus $e_{K,H}$ induces a map $e^g = e^g_{K,H}$ on the collection of conjugate classes of G slice types $[K;\ U]^g$ and

$$e^g_{k,H}[K;\ U]^g = [H;\ V(K) \oplus p^*U]^g.$$

Corresponding to a subgroup $\hat{G}$ of G containing G_2 we have three families of G slice types.

$$F(\hat{G}) = \{[gHg^{\bar{1}};\ g_*V] \mid [H, V] \text{ is a } G \text{ slice type with } H \text{ contained in } \hat{G},\ g \in G\}$$

$$F'(\hat{G}) = \{[K;\ U] \in F(\hat{G}) \mid K \cap G_2 \neq G_2\}$$

and

$$\tilde{F}(\hat{G}) = F'(\hat{G}) \cup \{e_{K,H}[K;\ U] \mid [K;\ U] \mid \in F'(\hat{G}) \text{ and } K \subset_2 H \text{ with } H \cap G_2 = G_2\}.$$

That each collection is a family is clear. Now we are in a position to state the main theorem of this paper.

THEOREM 1. *If G be a finite group and $\hat{G}$ be a subgroup of G which contains G_2 then $N_*^G[\tilde{F}(\hat{G})] = 0$.*

COROLLARY (*Khare* [2]). *Suppose that G is a finite group. If M is a G-manifold on which G_2 acts without fixed points then M is a G-boundary.*

The corollary follows because if G_2 acts without fixed points then an isotropy subgroup H of a point in M satisfies the condition $H \cap G_2 \neq G_2$ so that M is of the type $F'(G)$ and consequently of the type $\tilde{F}(G)$.

The proof of the theorem will be given in §7. In §2, §3, §4 and §5, we develop the necessary tools and results.

2. Vector bundles of type $\bar{\rho}$. Let $F' \subseteq F$ be two families of G slice types with $F = F' \cup \bar{\rho}$ where $\bar{\rho}$ is a class of conjugate G slice types. By a G-vector bundle of type $\bar{\rho}$ we mean a G-vector bundle ξ: $E(\xi) \xrightarrow{p} B(\xi)$ where the set of points of $E(\xi)$ having G slice type in $\bar{\rho}$ is precisely the zero section. We have the bundle bordism groups $N_n^G[\bar{\rho}]$ obtained by defining a bordism relation on the set of all G vector bundles of type $\bar{\rho}$ having total dimension n.

Let M^n be a G-manifold of type F and $F_{\bar{\rho}}$ be the set of all points in M^n with slice type in $\bar{\rho}$. Then the normal bundle over $F_{\bar{\rho}}$ is a G vector bundle of type $\bar{\rho}$. This assignment of the normal bundle over $F_{\bar{\rho}}$ in M^n leads to a N_*-homomorphism

$$\nu_{\bar{\rho}}\colon N_n^G[F] \to N_n^G[\bar{\rho}].$$

We have the following proposition and lemmas involving the bundle bordism groups.

PROPOSITION 2. *There exists a long exact sequence*

$$\cdots \to N_n^G[F'] \to N_n^G[F] \xrightarrow{\nu} N_n^G[\bar{\rho}] \xrightarrow{\partial} N_{n-1}^G[F'] \to \cdots$$

where $F' \subseteq F$ are families of G slice types such that $F - F' = \bar{\rho}$.

For proof we refer to 1.4.2 of [**3**].

LEMMA 3. *Suppose that $K \subset_2 H$ and $\bar{\rho} = [H; V]^g$, $\bar{\rho}' = [K; U]^g$ be two classes of conjugate G slice types such that $e^g(\bar{\rho}') = \bar{\rho}$. Then there exists an N_*-isomorphism*

$$N_n^G[\bar{\rho}] \to N_{n-1}^G[\bar{\rho}']$$

given by $[\xi] \to [\nu_{\bar{\rho}'}S(\xi)]$, *where $S(\xi)$ is the sphere bundle of ξ.*

The proof of this lemma is similar to that given for Lemma 4.5.8 of [**3**].

LEMMA 4. *Let $F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots$ be a sequence of families of G-slice types with*

(i) $F_0 = \bar{\rho}_0 = \{[e; \mathbf{R}^0]\}$
(ii) $F_i = F_{i-1} \cup \bar{\rho}_i$ *for all* $i \geq 1$
(iii) $\bigcup_{i\geq 0} F_i = F$

and

(iv) $e^g(\bar{\rho}_{2i}) = \bar{\rho}_{2i+1}$ *for all* $i \geq 0$. *Then* $N_*^G[F] = 0$.

Proof. Using Proposition 2 and Lemma 3 we get

$$N_*^G[F_{2i}] = N_*^G[\bar{\rho}_{2i}]$$

and $N_*^G[F_{2i+1}] = 0$.

Taking direct limit

$$N_*^G[F] = \varinjlim N_*^G[F_i] = 0.$$

The rest of the paper is aimed to show that the family $\tilde{F}(\hat{G})$ satisfies the conditions laid down in Lemma 4. The G slice types of $\tilde{F}(\hat{G})$ are to be ordered suitably now in order to get the families $F_0 \subset F_1 \subset \cdots$.

3. Ordering the conjugate classes of G slice types. We define three distinct relations $<$ on the collection $\tilde{A}$ of all subgroups conjugates to subgroups of $\hat{G}$, on the collection of all H-modules, $H \in \tilde{A}$ and finally on the collection of all conjugate classes of G slice types of the family $\tilde{F}(\hat{G})$ and extend each of these relations into a total order on the respective collection. We note that the elements of G_2 are totally ordered by

$$g_1 < g_2 < \cdots < g_k < g_1 g_2 < \cdots < g_1 g_k < \cdots < g_1 g_2 \cdots g_k$$

and a subgroup H_2 of G_2 has a distinguished base $h_1 < h_2 < \cdots < h_m$ such that h_1 ($\neq$ identity) is the least element in H and for $i > 1$, h_i is the least element in H which is not present in $(h_1, h_2, \ldots, h_{i-1})$, the subgroup generated by $h_1, h_2, \ldots, h_{i-1}$. The subgroups of G_2 are now totally ordered first by the order of the subgroup and then lexicographically on the distinguished base:

$$(e) < (g_1) < (g_2) < \cdots < (g_1 g_2 \cdots g_k) < (g_1, g_2) < \cdots.$$

Rule A. Let H and K belong to $\tilde{A}$. We define $\leq$ by:

(i) if $|H| \leq |K|$ Then $H \leq K$,

(ii) if $|H| = |K|$ and $|K_2| \leq |H_2|$. Then $H \leq K$ where $K_2 = K \cap G_2$ and $H_2 = H \cap G_2$,

(iii) if $|H| = |K|$, $|K_2| = |H_2|$ but $H_2 \leq K_2$ then $H \leq K$ and

(iv) if $|H| = |K|$, $H_2 = K_2$ then we order them arbitrarily so as to make the relation $\leq$ a total ordering on $\tilde{A}$.

Next a relation $\leq$ is introduced on the collection of all nontrivial irreducible H-modules $H \in \tilde{A}$. We write $U \leq V$ if $U = V$ or else there exists $K \subset_2 H$ such that $U = p^* i^* V$ where i: $K \to H$ is the natural inclusion and p: $H \to K$ is the distinguished projection. We now have the following lemma whose proof is similar to Lemma 8 of [4].

LEMMA 5. *The relation $\leq$ is a partial order on the collection of all nontrivial irreducible H-modules.*

We now choose a total ordering on the set of all nontrivial irreducible H-modules having the same dimension compatible with the partial ordering introduced. The total ordering is now extended to all irreducible

H-modules by writing $U \leq V$ if and only if $\dim U \leq \dim V$. Since any H-module can be expressed uniquely as the sum of irreducible H-modules, we can extend this total ordering on all H-modules by lexicography. The following rule expresses the whole rule coincisely.

Rule B. Let U and V be two H-modules.
(i) If $\dim U \leq \dim V$ then $U \leq V$
(ii) If $\dim U = \dim V$ and V follows U lexicographically then $U \leq V$.

Finally Rule C given as below defines the order $\leq$ on the collection of all classes of conjugate G slice types of the family $\tilde{F}(\hat{G})$.

Rule C. Let $\bar{\rho} = [H; U]^g$, $\bar{\rho}' = [K; V]^g$ be two classes of conjugate G slice types of $\tilde{F}(\hat{G})$
(i) If $\dim U \leq \dim V$ then $\bar{\rho} \leq \bar{\rho}'$.
(ii) If $\dim U = \dim V$ and $H \leq K$ then $\bar{\rho} \leq \bar{\rho}'$.
(iii) If $\dim U = \dim V$, $H = K$ and $U \leq V$ then $\bar{\rho} \leq \bar{\rho}'$.

We now proceed to prove some algebraic results relating to the extension map e.

4. Algebraic lemmas and extension map. The following lemmas are generalisations of propositions of $\mathbf{Z}_2^k$ bordism given in 4.5 of [3]

LEMMA 6. *Let* $(e) \subset K \subset_2 H \subset G$ *and*

$$g_1 < g_2 < \cdots < g_k,$$
$$h_1 < h_2 < \cdots < h_m,$$

and

$$k_1 < k_2 < \cdots < k_{m-1}$$

be the distinguished bases of G_2, H_2 *and* K_2 *respectively and* r *be the greatest integer for which* $k_i = h_i$ *for all* $i < r$. *Then* K *is not contained in a predecessor of* H *if and only if* $h_i = g_i$ *for all* $i < r$. (*By a predecessor of* H *we mean a subgroup* $H' \simeq H$ *such that* $H_2' < H_2$.)

Proof. We have $(e) \subset K_2 \subset_2 H_2 \subset G_2$. If $K \subset_2 H'$, a predecessor of H then by definition $K_2 \subset_2 H_2'$, a predecessor of H_2. Further if $K_2 \subset_2 H_2'$, a predecessor of H_2, then $H_2' = (x) \times K_2$ (x being chosen minimally) and $K \subset_2 (x) \times K$, a predecessor of H.

Thus K is not contained in a predecessor of H if and only if K_2 is not contained in a predecessor of H_2. The latter statement implies and is implied by $h_i = g_i$ for all $i < r$ and this follows from 4.5.12 of [3].

LEMMA 7. *Let $K \subset_2 H$, $K' \subset_2 H$ with K and K' not contained in a predecessor of H. If*

$$H = (x) \times K = (x') \times K'$$

where x and x' are chosen minimally, $x \in K'$, $x' \in K$ and K precedes K' then $K \cap K'$ is not contained in a predecessor of K.

Proof. We have

$$H_2 = (x) \times K_2 = (x') \times K_2'.$$

and K_2 precedes K_2'. By the Proposition 4.5.13 of [**3**], $K_2 \cap K_2'$ is not contained in a predecessor of K_2 and this in turn implies that $K \cap K'$ is not contained in a predecessor of K.

In order to proceed further we need the following constructions and lemmas.

$S(H)$ = collection of all conjugate classes of G slice types with isotropy subgroup H. For any $K \subset_2 H$ we have the extension function

$$e^g = e^g_{K,H}\colon S(K) \to S(H)$$

and consequently a function

$$E^g\colon \bigcup_{\substack{K \subset_2 H \\ K \not\subset_2 P(H)}} S(K) \to S(H).$$

where by $P(H)$ one means a predecessor of H. Let

$$\bar{S}(K) = S(K) - \text{image}\left\{E^g\colon \bigcup_{\substack{L \subset_2 K \\ L \not\subset_2 P(K)}} S(L) \to S(K)\right\}.$$

The function

$$\bar{E}^g\colon \bigcup_{\substack{K \subset_2 H \\ K \not\subset_2 P(H)}} \bar{S}(K) \to S(H)$$

is the restriction of E^g.

LEMMA 8. Image $\bar{E}^g$ = image E^g.

Proof. Clearly image $\bar{E}^g \subseteq$ image E^g.

Let $\bar{\rho} \in \operatorname{im} E^g$ i.e. $\bar{\rho} = e^g(\bar{\rho}')$ for some $\bar{\rho}' \in S(K)$ where $K \subset_2 H$ and $K \not\subset_2 P(H)$.

If $\bar{\rho}' \notin \bar{S}(K)$ then $\bar{\rho}' = e^g(\bar{\rho}'')$ for some $\bar{\rho}'' \in S(L)$ where $L \subset_2 K$ and $L \not\subset_2 P(K)$. By Lemma 6 we have the following distinguished bases of H_2, K_2 and L_2

$$
\begin{aligned}
&L_2\colon g_1 < g_2 < \cdots < g_{s-1} < l_s < \cdots\\
&K_2\colon g_1 < g_2 < \cdots < g_{s-1} < g_s < \cdots < g_{r-1} < k_r < \cdots\\
&H_2\colon g_1 < g_2 < \cdots < g_{r-1} < g_r < h_{r+1} < \cdots .
\end{aligned}
$$

We note that $l_s \neq g_s$ and $k_r \neq g_r$. So

$$H = (g_r) \times K \quad \text{and} \quad K = (g_s) \times L.$$

Writing $\bar{\rho}'' = [L; U]^g$ we get

$$\bar{\rho}' = e^g(\bar{\rho}'') = [K; V(L) \oplus q^*U]^g,$$

and

$$
\begin{aligned}
\bar{\rho} = e^g(\bar{\rho}') &= [H; V(K) \oplus p^*(V(L) \oplus q^*U)]^g\\
&= [H; V(K) \oplus V((g_r) \times L) \oplus p^*q^*U]^g
\end{aligned}
$$

$q\colon K \to L$ and $p\colon H \to K$ are the distinguished projections.

Taking $K' = (g_r) \times L$ we note that $K' \subset_2 H$ and K precedes K'. Moreover $K' \not\subset_2 P(H)$. Extending $\bar{\rho}''$ through K' we get

$$\bar{\rho}''' = e^g_{L,K'}(\bar{\rho}'') = [K'; V(L) \oplus q'^*U]^g \in S(K')$$

and

$$e^g_{K',H}(\bar{\rho}''') = [H; V(K') \oplus V((g_s) \times L) \oplus p'^*q'^*U]^g$$

where $p'\colon H \to K'$ and $q'\colon K' \to L$ are the distinguished projections. Since $qp = q'p'$, we have

$$e^g_{K',H}(\bar{\rho}''') = [H; V(K') \oplus V(K) \oplus p^*q^*U]^g = \bar{\rho}.$$

If $\bar{\rho}''' \in \bar{S}(K')$ then $\bar{\rho} \in \operatorname{image} \bar{E}^g$. If not then by arguing as before we get a conjugate class of G slice type $\bar{\rho}^{(v)} \in S(K'')$ such that $\bar{\rho} = e^g(\bar{\rho}^{(v)})$ where $K'' \subset_2 H$ and $K < K' < K'' \not\subset_2 P(H)$.

Continuing this way we exhaust all the finite number of possibilities and find some $\bar{\rho}^{(2n+1)} \in \bar{S}(K^{(n)})$ such that $K^{(n)} \subset_2 H$, $K^{(n)} \not\subset_2 P(H)$ and $\bar{\rho} = e^g(\bar{\rho}^{(2n+1)})$ i.e. $\bar{\rho} \in \operatorname{image} \bar{E}^g$.

LEMMA 9. *The function*

$$\bar{E}^g\colon \bigcup_{\substack{K \subset_2 H\\ K \not\subset_2 P(H)}} S(K) \to S(H)$$

is injective.

Proof. Suppose that

$$\bar{\rho} = [K; U]^g, \bar{\rho}' = [K'; U']^g$$

where K and $K' \subset_2 H$, K and $K' \not\subset_2 P(H)$, K precedes K' and

$$e^g(\bar{\rho}) = e^g(\bar{\rho}') = [H; V]^g.$$

From Lemma 6 we get

$$H = (g_r) \times K = (g_s) \times K'$$

where g_r and g_s are the minimal possible choices and $s < r$. We have

$$[H; V(K) \oplus p^*U]^g = [H; V]^g = [H; V(K') \oplus p'^*U']^g$$

where p: $H \to K$, p': $H \to K'$ are the distinguished projections. Writing $U = \sum_i n_i U_i$ and $U' = \sum n'_j U'_j$ where U_i and U'_j are nontrivial irreducible K and K' modules respectively we get

$$V(K) \oplus \sum_i n_i p^* U_i = V(K') \oplus \sum_j n'_j p'^* U'_j.$$

Since $K \neq K'$, $V(K) = p'^*U'_t$ for some t and $n'_t = 1$. The underlying vector space of these modules is **R**.

We write $g_s = g_r^{\alpha_1}k$, $\alpha_1 \in \{0,1\}$ and $k \in K$ and consider its action on $x \in V(K) = p'^*U'_t$. We get $g_s x = x$ i.e. $(-1)^{\alpha_1}x = x$ i.e. $\alpha_1 = 0$. So $g_s \in K$. Similarly $g_r \in K'$. By Lemma 7, $L = K \cap K' \not\subset_2 P(K)$ and $K = (g_s) \times L$ (L is the intersection of two normal subgroups of H). We have also the restriction function

$$r^g = r^g_{H,K}\colon S(H) \to S(K)$$

such that $r^g[H; V]^g = [K; I^*V]^q$ where I^*V is the nontrivial part of i^*V, i: $K \hookrightarrow H$ being the natural inclusion. Note that

$$\begin{aligned} r^g_{H,K} e^g_{K,H}[K; U]^g &= r^g_{H,K}[H; V(K) \oplus p^*U]^g \\ &= [K; I^*(V(K) \oplus p^*U)]^g \\ &= [K; I^*p^*U]^g = [K; i^*p^*U]^g = [K; U]^g \end{aligned}$$

i.e. $r^g_{H,K} e^g_{K,H} =$ identity.

Therefore

$$\begin{aligned} \bar{\rho} = [K, U]^g = r^g_{H,K} e^g_{K,H}[K; U]^g &= r^g_{H,K} e^g_{K',H}[K'; U']^g \\ &= r^g_{H,K}[H; V(K') \oplus p'^*U']^g \\ &= [K; V(K' \cap K) \oplus I^*p'^*U']^g \\ &= [K; V(L) \oplus NTq^*j'^*U']^g \end{aligned}$$

where $i\colon K \hookrightarrow H$, $i'\colon K' \hookrightarrow H$, $j\colon L \hookrightarrow K$, $j'\colon L \hookrightarrow K'$ are the natural inclusions and $p\colon H \to K$, $p'\colon H \to K'$, $q\colon K \to L$, $q'\colon K' \to L$ are the distinguished projections. We have $p'i = j'q$ and NT stands for the nontrivial part. Also

$$r^g_{K,L}(\bar{\rho}) = [L;\ NTj^*q^*j'^*U']^g = [L;\ NTj'^*U']^g$$

(since $qj = \mathrm{id}$). So

$$e^g_{L,K}r^g_{K,L}(\bar{\rho}) = [K;\ V(L) \oplus NTq^*j'^*U']^g = \bar{\rho}.$$

Thus $\bar{\rho} = e(\bar{\rho}'')$ for $\bar{\rho}'' = r_{K,L}(\bar{\rho}) \in S(L)$ and $L \subset_2 K$, $L \not\subset_2 P(K)$ i.e.

$$\bar{\rho} \in \mathrm{im}\left\{E^g\colon \bigcup_{\substack{L \subset_2 K \\ L \not\subset_2 P(K)}} S(L) \to S(K)\right\}$$

i.e. $\bar{\rho} \notin \bar{S}(K)$—a contradiction.

With this we come to an end of this section.

5. Decomposition of the collection of conjugate classes of G slice types of a family. If we now define the dimension of a conjugate class of G-slice types as dimension of the module present therein then it is clear that there are only a finite number of conjugate classes of G slice types of a given dimension. The classes of the family $\bar{F}(\hat{G})$ are totally ordered by the Rule C and we index them by nonnegative integers as

$$\bar{\rho}_0 < \bar{\rho}_1 < \bar{\rho}_2 <$$

where $\bar{\rho}_0 = \{[(e), \mathbf{R}^0]\}$. We define $F_j = \bigcup_{i \leq j} \bar{\rho}_i$. F_j is clearly a family of G slice types. Corresponding to the family F_j we form the collection $\bar{F}_j = \{\bar{\rho}_0, \bar{\rho}_1, \ldots, \bar{\rho}_j\}$ and define inductively three subcollections A_j, B_j and C_j of $\bar{F}_j$ such that $\bar{F}_j = A_j \cup B_j \cup C_j$. For $j = 0$, $\bar{F}_j = \{\bar{\rho}_0\}$ and we set

$$A_j = \{\bar{\rho}_0\}, \quad B_j = \varnothing, \quad C_j = \varnothing$$

Let A_{j-1}, B_{j-1}, C_{j-1} be defined for some $j \geq 1$. We have

$$\bar{F}_{j-1} = A_{j-1} \cup B_{j-1} \cup C_{j-1}$$

and

$$\bar{F}_j = \bar{F}_{j-1} \cup \{\bar{\rho}_j\}.$$

There are two possibilities:

(i) either $\bar{\rho}_j = e^g(\bar{\rho})$ for some $\bar{\rho} \in A_{j-1}$ or

(ii) $\bar{\rho}_j \neq e^g(\bar{\rho})$ for any $\bar{\rho} \in A_{j-1}$.

In case of (i) We define

$$A_j = A_{j-1} - \{\bar{\rho}\}, \quad B_j = B_{j-1} \cup \{\bar{\rho}_j\}, \quad C_j = C_{j-1} \cup \{\bar{\rho}\}$$

and in case of (ii)

$$A_j = A_{j-1} \cup \{\bar{\rho}_j\}, \quad B_j = B_{j-1}, \quad C_j = C_{j-1}.$$

We now establish an analogue of Lemma 9 of **[4]**.

LEMMA 10. *There is at most one conjugate class of G slice types* $\bar{\rho} \in A_{j-1}$ *such that* $e^g(\bar{\rho}) = \bar{\rho}_j$.

Proof. The proof of this lemma is given by induction. Clearly the lemma holds for $j = 1$. Let it be true for all $i < j$.

Let $\bar{\rho}_j = e^g(\bar{\rho}_m)$ and take $\bar{\rho}_j \in S(H)$ and $\bar{\rho}_m \in S(K)$ with $K \subset_2 H$. We claim that $K \not\subset_2 P(H)$. If $K \subset_2 P(H)$ then we choose J to be the least of all predecessors of H. We get $K \subset_2 J$ and

$$\bar{\rho}_t = e^g_{K,J}(\bar{\rho}_m) < \bar{\rho}_j = e^g_{K,H}(\bar{\rho}_m).$$

By the induction hypothesis there exists at most one such $\bar{\rho}_m$ such that $\bar{\rho}_t = e^g_{K,J}(\bar{\rho}_m)$. Consequently neither $\bar{\rho}_m$ nor $\bar{\rho}_t$ belongs to A_{j-1}. So

$$K \not\subset_2 P(H) \quad \text{and} \quad \rho_m \in \bigcup_{\substack{K \subset_2 H \\ K \not\subset_2 P(H)}} S(K).$$

By Lemma 8, this implies

$$\bar{\rho}_j \in \operatorname{image} E^g = \operatorname{image} \bar{E}^g.$$

If now

$$\rho_m \in \operatorname{image}\left\{E^g\colon \bigcup_{\substack{L \subset_2 K \\ L \not\subset_2 P(K)}} S(L) \to S(K)\right\}$$

then $\bar{\rho}_m = e^g(\bar{\rho}')$ for $\bar{\rho}' \in S(L)$, $L \subset_2 K$ and $L \not\subset_2 P(K)$. From the construction of the families A_j it follows that $\bar{\rho}_m \notin A_{j-1}$. So

$$\bar{\rho}_m \in \bar{S}(K) = S(K) - \operatorname{image}\left\{E^g\colon \bigcup_{\substack{L \subset_2 K \\ L \not\subset_2 P(K)}} S(L) \to S(K)\right\}.$$

By Lemma 9, $\bar{E}^g$ is injective and this establishes our lemma.

The next theorem further characterises the families A_j.

LEMMA 11. *If N is sufficiently large compared to n then* A_N *consists of conjugate classes of G slice types of dimension greater than n.*

Proof. Let F_i be the family which contains all conjugate G slice types of dimension $\leq n$ and

$$A_i = \{\bar{\rho}_{i_1}, \bar{\rho}_{i_2}, \dots, \bar{\rho}_{i_k}\}$$

with $\bar{\rho}_{i_t} = \{K_t;\ U_t\}^g$, $1 \le t \le k$. Then $K_t \cap G_2 \ne G_2$ because $K_t \cap G_2 = G_2 \Rightarrow \bar{\rho}_{i_t} = e^g(\bar{\rho}')$ for some $\bar{\rho}'$. We take

$$\rho_{j_t} = e^g(\rho_{i_t})$$

If $N \ge \max\{j_1, \dots, j_k\}$ then clearly A_N does not contain any conjugate class of G slice types of dimension $\le n$.

The next theorem reveals the necessity of ordering the conjugate classes of G slice types.

THEOREM 12. *If $[H;\ U]$ is a G slice type and $\bar{\rho} \in A_j$ is a conjugate class of G slice types of an orbit of a point of $G \times_H U$, then either $\bar{\rho} = [H;\ U]^g$ or $[H;\ U]^g \notin \bar{F}_j$.*

Proof. Let $\bar{\rho} \ne [H;\ U]^g$. Then $\bar{\rho}$ is not the conjugate class of G slice types of the orbit of $[e, 0] \in G \times_H U$. So $\bar{\rho}$ is a conjugate class of G slice types of the orbit of a point $[e, u] \in G \times_H U$, $0 \ne u \in U$. The isotropy subgroup of $[e, u]$ is a proper subgroup K of H. We can write $\bar{\rho} = [K;\ I^*U]^g$ where I^*U is the nontrivial part of i^*U, i: $K \hookrightarrow H$ being the natural inclusion. Clearly $\dim I^*U \le \dim i^*U = \dim U$. We now discuss the two possible cases separately.

Case I. $K \subset_2 H$ i.e. $H = (x) \times K$.

We have

$$e_{K,H}(\bar{\rho}) = [H;\ V(K) \oplus p^*I^*U]^g$$

where p: $H \to K$ is the distinguished projection.

Since K fixes $u \in U$, K has trivial action on the one dimensional subspace $L(u)$ spanned by u. Also H has nontrivial action on $L(u)$. So (x) acts on $L(u)$ nontrivially and we get $V(K) = L(u) \subset U$. If

$$\dim(V(K) \oplus p^*I^*U) < \dim U$$

then

$$\bar{\rho} < \bar{\rho}_k = e^g_{K,H}(\bar{\rho}) \le [H;\ U]^g = \bar{\rho}_t.$$

If

$$\dim(V(K) \oplus p^*I^*U) = \dim U$$

then $\dim I^*U$ is just one less than $\dim U$ and by writing $U = V(K) \oplus U'$ we get $I^*U = i^*U'$. So $p^*I^*U = p^*i^*U' \le U'$ by the ordering of irreducible H-modules and its extension by lexicography i.e. $V(K) \oplus p^*I^*U \le V(K) \oplus U' = U$. Again we have

$$\bar{\rho} < \bar{\rho}_k = e^g_{K,H}(\bar{\rho}) \le [H;\ U]^g = \rho_t.$$

Case II. Let $K \not\subset_2 H$ i.e. $K < H$ but $H \neq (x) \times K$ for any $x \in G_2$. If $K_2 = G_2$ then the class $\bar{\rho}$ is the e^g-image of some conjugate class of G slice types occurring earlier according to the order so constructed. But this means $\bar{\rho} \notin A_j$—a contradiction. So $K_2 \subsetneq G_2$ and there exists an element $x \in G_2$ such that $(x) \times K$ can be formed. Since K is a proper subgroup of H, $|(x) \times K| \leq |H|$. If $|(x) \times K| < |H|$ then by (i) of Rule A

$$\bar{\rho} < \bar{\rho}_k < \bar{\rho}_t.$$

If $|(x) \times K| = |H|$ then $|H\colon K| =$ index of K in $H = 2$. Since $K \not\subset_2 H$, $x \notin H$. Also there does not exist $y \in G_2$ such that $y \in H$ but $y \notin K$.

Hence $K_2 = H_2$ and $|(x) \times K_2| > |H_2|$. By (ii) of Rule A, $(x) \times K < H$ and

$$\bar{\rho} < \bar{\rho}_k < \bar{\rho}_t.$$

Now

$$\begin{aligned} \bar{\rho}_t = [H;\, U]^g \in \bar{F}_j &\Rightarrow \bar{\rho}_t < \bar{\rho}_j \\ &\Rightarrow \bar{\rho} < e^g(\bar{\rho}) = \bar{\rho}_k < \bar{\rho}_t < \bar{\rho}_j \\ &\Rightarrow \bar{\rho} \in A_{k-1} \quad \text{and} \quad \bar{\rho} \notin A_k \qquad \text{(Lemma 10)} \\ &\Rightarrow \bar{\rho} \notin A_j\text{—a contradiction.} \end{aligned}$$

A consequence of this theorem is:

COROLLARY 13. *The union of all conjugate classes of G slice types of B_j and C_j is a family.*

Proof. Let $[H;\, U]^g \in B_j \cup C_j \subseteq F_j$ and $\bar{\rho}$ is a conjugate class of G-slice types of an orbit of a point of $G \times_H U$. Clearly $\bar{\rho} \subset F_j$. If $\bar{\rho} \notin B_j \cup C_j$ then $\bar{\rho} \in A_j$ and this contradicts Theorem 12.

6. Proof of the main theorem. We denote the elements of C_j by $\bar{\sigma}_0$, $\bar{\sigma}_2, \ldots, \bar{\sigma}_{2k}$ where $k = |C_j|$ and $\bar{\sigma}_{2t} \leq \bar{\sigma}_{2m}$ if and only if $t \leq m$. We have $B_j = \{ e^g(\bar{\sigma}_{2i}) | 0 \leq i \leq k \}$ and write $e^g(\bar{\sigma}_{2i}) = \bar{\sigma}_{2i+1}$.

By Corollary 13, $\tilde{F}_k = \cup_{i=0}^{k} \bar{\sigma}_i$ is a family when k is odd. When k is even $\tilde{F}_k$ is again a family because the G slice types of $\bar{\sigma}_k$ are 'maximal' in $\tilde{F}_k$. By Lemma 11 we see that $\tilde{F}(\hat{G})$ satisfies all the conditions of Lemma 4 and so

$$N_*^G[\tilde{F}(\hat{G})] = 0.$$

An alternative proof of Theorem 1 can be given by generalising Theorem 4.5.11 of [3].

THEOREM 14. *There is an isomorphism*

$$\oplus\, \nu_i\colon N_*^G[F_j] \to \bigoplus_{\bar{\rho}_i \in A_j} N_*^G[\bar{\rho}_i].$$

Proof. We prove the result by induction. Clearly the result is true for $j = 0$. Now suppose it is true for $j-1$ i.e.

$$\oplus\, \nu_i\colon N_*^G[F_{j-1}] \to \bigoplus_{\bar{\rho}_i \in A_{j-1}} N_*^G[\bar{\rho}_i].$$

From the long exact sequence of Proposition 2 we have the composite

$$\nu_i \partial_j\colon N_n^G[\bar{\rho}_j] \to N_{n-1}^G[\bar{\rho}_i].$$

If $\nu_i\partial_j \neq 0$ then $\bar{\rho}_i$ is a conjugate class of G slice types of $G \times_H V$ where $[H, V]^g = \bar{\rho}_j$ and by Theorem 12 $\rho_i \notin A_j$.

Now for the class $\bar{\rho}_j$ there exists almost one conjugate class of G slice types $\bar{\rho}_i$ such that $e^g(\bar{\rho}_i) = \bar{\rho}_j$. If there does not exist any such $\bar{\rho}_i \in A_{j-1}$ then for any $\bar{\rho}_i \in A_{j-1}$ both $\bar{\rho}_i$ and $\bar{\rho}_j$ belong to A_j and $\nu_i\partial_j = 0$ for every $\bar{\rho}_i \in A_{j-1}$. Thus $(\oplus_{\bar{\rho}_i \in A_{j-1}} \nu_i)\partial_j = 0$ and consequently $\partial_j = 0$. We have a short exact sequence

$$0 \to N_n^G[F_{j-1}] \to N_n^G[F_j] \xrightarrow{\nu_j} N_n^G[\bar{\rho}_j] \to 0.$$

If again for $\bar{\rho}_j$ we have $\bar{\rho}_i \in A_{j-1}$ s.t. $\bar{\rho}_j = e^g(\bar{\rho}_i)$ then neither $\bar{\rho}_j$ nor $\bar{\rho}_i$ belong to A_j and by Lemma 3

$$\nu_i\partial_j\colon N_n^G[\bar{\rho}_j] \to N_{n-1}^G[\bar{\rho}_i]$$

is an isomorphism and we have again a short exact sequence

$$0 \to N_n^G[\bar{\rho}_i] \to N_n^G[F_{j-1}] \to N_n^G[F_j] \to 0.$$

Both the short exact sequences split as the modules involved are vector spaces over $\mathbf{Z}_2$. So

$$N_n^G[F_j] \simeq \bigoplus_{\rho_i \in A_j} N_n^G[\bar{\rho}_i].$$

COROLLARY 15. $N_*^G[\tilde{F}(\hat{G})] = 0$.

Proof. Corresponding to the positive integer n we take all conjugate classes of G slice types of dimension $\leq n+1$. If F_N be the union of all these classes then

$$N_n^G[\tilde{F}(\hat{G})] = N_n^G[F_N] \simeq \bigoplus_{\bar{\rho}_i \in A_N} N_n^G[\bar{\rho}_i].$$

If now N is made sufficiently large compared to n then by Lemma 11 A_N consists of all conjugate classes of G slice types of dimension $> n$ and hence the isomorphism $\oplus\, \nu_i$ is zero.

COROLLARY 16.

$$N_*^G[F'(\hat{G})] \simeq N_{*+1}^{\ G}[\tilde{F}(\hat{G}), F'(\hat{G})].$$

This follows from the main theorem and the long exact sequence for the pair $F'(\hat{G}) \subset \tilde{F}(\hat{G})$ of families of G-slice types.

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