

# Pacific Journal of Mathematics

**ON CLASS NUMBERS OF CYCLIC QUARTIC FIELDS**

TSUYOSHI UEHARA

# ON CLASS NUMBERS OF CYCLIC QUARTIC FIELDS

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Let  $n$  be a given natural number and  $F$  a quadratic field contained in a cyclic quartic field. In this paper we shall construct infinitely many imaginary cyclic quartic fields containing  $F$  whose relative class numbers are divisible by  $n$ .

**1. Introduction.** Let  $K$  be an imaginary abelian number field,  $K^+$  its maximal real subfield, and let  $h$  and  $h^+$  be the respective class numbers. It is known that  $h^+$  divides  $h$ . The quotient  $h^- = h/h^+$  is called the relative class number of  $K$ . The purpose of this paper is to give a complement of a result in our previous paper [3]. Namely we shall prove the following

**THEOREM.** *Let  $F$  be a quadratic field contained in a cyclic quartic field. Then there exist infinitely many imaginary cyclic quartic fields containing  $F$  each with relative class number divisible by a given integer.*

It is seen from Lemma 2 in the next section that for a square free rational integer  $m$  the quadratic field generated by  $m^{1/2}$  is contained in a cyclic quartic field if and only if  $m = s^2 + t^2$  for some rational integers  $s, t$ .

**2. Lemmas.** By  $Z, Q$  we denote the ring of rational integers, the field of rational numbers respectively. For any number field  $L$  let  $C(L)$  be the ideal class group of  $L$ .

**LEMMA 1** (for instance, cf. [2], Ch. 3, Theorem 4.3). *Let  $K$  be an imaginary abelian number field, and  $K^+$  its maximal real subfield. Let  $\phi$  be the norm mapping from  $C(K)$  to  $C(K^+)$  and put  $C^-(K) = \text{Ker } \phi$ . Then  $\phi$  is surjective, and the relative class number of  $K$  is equal to the order of  $C^-(K)$ .*

**LEMMA 2** (cf. [1]). *Let  $m \neq 1$  be a square free rational integer and  $a, b$  rational numbers. Put  $\eta = a + bm^{1/2}$ . Then  $Q(\eta^{1/2})$  is a cyclic quartic field if and only if  $a^2 - b^2m = c^2m$  for some  $c$  in  $Q$ .*

We now take rational integers  $s, t$  for which  $m = s^2 + t^2$  is square free and put

$$\eta = f(m + tm^{1/2}), \quad \theta = \eta^{1/2},$$

$f$  being a square free rational integer. By Lemma 2,  $K = Q(\theta)$  is a cyclic quartic field. Let  $\sigma$  be a generator of the Galois group  $\text{Gal}(K/Q)$ . Then  $(\theta^\sigma)^2 = f(m - tm^{1/2})$ . We put  $\omega = m^{1/2}$  if  $m$  is even and  $\omega = (1 + m^{1/2})/2$  if  $m$  is odd. Note that  $\theta^{\sigma^2} = -\theta$  and  $\omega^{\sigma^2} = \omega$ .

LEMMA 3. *Let the notation be as above. If  $p$  is an odd prime dividing  $f$ , then for any integer  $\alpha$  in  $K$  and any  $k > 0$  in  $Z$  there is an integer  $\beta$  in  $Q(m^{1/2})$  such that*

$$\alpha^{p^k} \equiv \beta \pmod{p^k}.$$

*If  $m \equiv 0 \pmod{2}$  or  $f \equiv t \pmod{2}$ , the above assertion is also valid for  $p = 2$ .*

*Proof.* First we remark that if  $\alpha^p \equiv \beta \pmod{p}$  is true for some  $\beta$  in  $Z[\omega]$  then the assertion is easily shown by induction on  $k$ .

Let  $p$  be an odd prime dividing  $f$ . We can find integers  $a, b, c, d$  in  $Z$  such that

$$\alpha \equiv (a + b\omega + c\theta + d\theta^\sigma)/p^e \pmod{p}, \quad e \geq 0.$$

Since  $\alpha + \alpha^{\sigma^2} \equiv 2(a + b\omega)/p^e \pmod{p}$  we have  $a \equiv b \equiv 0 \pmod{p^e}$ . Hence  $\pi = (c\theta + d\theta^\sigma)/p^e$  is an integer and  $\alpha - \pi \equiv \beta_1 \pmod{p}$  holds for some  $\beta_1$  in  $Z[\omega]$ . Observing  $c\pi - d\pi^\sigma = (c^2 + d^2)\theta/p^e$  we get  $c^2 + d^2 \equiv 0 \pmod{p^e}$ . We compute

$$p^{2e}\pi^2 = (c^2 + d^2)fm + \{(c^2 - d^2)t \pm 2cds\}fm^{1/2}.$$

If  $e = 0$  then  $\pi^2 \equiv 0 \pmod{p}$ . When  $e > 0$  we may assume  $(c, p) = (d, p) = 1$ . We derive  $ct \pm ds \equiv 0 \pmod{p^e}$ . This implies that  $s \equiv \pm lc \pmod{p^e}$ ,  $t \equiv -ld \pmod{p^e}$  for some  $l$  in  $Z$ . Hence  $m \equiv l^2(c^2 + d^2) \equiv 0 \pmod{p^e}$  and so  $e = 1$ . Notice that  $p \geq 5$  and  $\pi^4 \equiv 0 \pmod{p}$  in this case. Thus  $\pi^p \equiv 0 \pmod{p}$  and  $\alpha^p \equiv \beta_1^p \pmod{p}$  in all cases.

To verify the assertion in the case  $p = 2$  we put  $\xi = (\theta + \theta^\sigma)/2$ ,  $\xi' = (\theta - \theta^\sigma)/2$  and suppose that for some  $u, v, x, y$  in  $Z$ ,  $\zeta = (u + v\omega + x\xi + y\xi')/2$  is an integer. We shall show that  $u, v, x$  and  $y$  are all even. We write  $4\xi\xi'^{\sigma^2} \equiv M + N\omega$  with  $M, N$  in  $Z$ . Clearly  $M \equiv N \equiv 0 \pmod{4}$ . In the case that  $m$  is even, one sees

$$\begin{cases} M = u^2 + v^2m - (x^2 + y^2)fm/2, \\ N = 2uv - \{xyt \pm (x^2 - y^2)s/2\}f. \end{cases}$$

Since  $s$  and  $t$  are both odd and  $f \not\equiv 0 \pmod{4}$ , we get  $x \equiv y \pmod{2}$ . This implies  $u \equiv 0 \pmod{2}$  and hence  $N \equiv -xyt \equiv 0 \pmod{2}$ . The last congruence shows  $x \equiv y \equiv 0 \pmod{2}$ . Thus  $v$  is also even. Next let  $m$  be odd. Then

$$\begin{cases} M = u^2 + v^2(m-1)/4 - \{(x^2 + y^2)m \pm (x^2 - y^2)s - 2xyt\}f/2, \\ N = (2u + v)v + \{\pm(x^2 - y^2)s - 2xyt\}f. \end{cases}$$

If  $f$  and  $t$  are both even, we have  $v \equiv 0 \pmod{2}$  and  $x \equiv y \pmod{2}$  because  $(2u + v)v \pm (x^2 - y^2)fs \equiv 0 \pmod{4}$  and  $s$  is odd. Hence it follows from  $M \equiv 0 \pmod{4}$  that  $u \equiv x \equiv y \equiv 0 \pmod{2}$ . If  $f$  and  $t$  are both odd, since  $s$  is even, we first see  $v \equiv 0 \pmod{2}$ . Observing  $2M \equiv (x^2 + y^2)fm \equiv 0 \pmod{2}$  we have  $x \equiv y \pmod{2}$ . From  $N \equiv -2xyt \equiv 0 \pmod{4}$  one can derive  $x \equiv y \equiv 0 \pmod{2}$ . Thus  $u$  is also even. The above argument shows that under the assumption  $\alpha \equiv \beta_1 + c\xi + d\xi' \pmod{2}$  holds with  $\beta_1$  in  $Z[\omega]$  and  $c, d$  in  $Z$ . Here  $\xi_1 = c\xi + d\xi'$  is an integer and  $\xi_1^2$  is in  $Z[\omega]$ . This yields that  $\alpha^2 \equiv \beta \pmod{2}$  with  $\beta$  in  $Z[\omega]$ . Hence the proof is complete.

**3. Proof of the theorem.** In the following, for any prime  $p$  and any rational integer  $g$ ,  $\text{ord}_p g$  means the exponent of the exact power of  $p$  dividing  $g$ . Let  $m = s^2 + t^2$  be a square free rational integer with  $s, t > 0$  in  $Z$ . For a given natural number  $n$  we put

$$n' = \begin{cases} 2^3 n^2 & \text{if } n \text{ is even and } mt \text{ is odd,} \\ 2^2 n^2 & \text{if } n \text{ and } mt \text{ are both even,} \\ n^2 & \text{if } n \text{ is odd.} \end{cases}$$

**PROPOSITION.** *Let the notation be as above. Take rational integers  $a, b > 0$  satisfying*

- (i)  $(a, bt) = 1$ ,
- (ii)  $(a^2 - b^2 t^2 m, 2ms) = 1$ ,
- (iii)  $\text{ord}_p b = 1$  for every prime  $p$  dividing  $n$ ,
- (iv)  $A - Bm > 0$ ,

where  $A + Btm^{1/2} = (a + btm^{1/2})^{n'}$  with  $A, B$  in  $Z$ . Moreover put

$$\eta = (2Bm - A + Btm^{1/2})^2 - (A + Btm^{1/2})^2.$$

Then  $K = Q(\eta^{1/2})$  is an imaginary cyclic quartic field, and the relative class number of  $K$  is divisible by  $n$  unless  $K$  is the fifth cyclotomic field.

*Proof.* Computing  $\eta = 4B(Bm - A)(m + tm^{1/2})$  we obtain the first assertion from Lemma 2 and (iv). We put

$$\alpha = a + btm^{1/2}, \quad \beta = 2Bm - A + Btm^{1/2}, \quad \theta = \eta^{1/2}.$$

Then  $(\beta + \theta)(\beta - \theta) = \alpha^{2n'}$ . Suppose that there is a prime ideal  $P$  of  $K$  dividing both the integers  $\beta \pm \theta$ . Let  $p$  be the prime lying below  $P$ . Then  $p$  divides  $a^2 - b^2 t^2 m$  because  $\alpha$  is in  $P$ . By (ii) we have  $(p, 2ms) = 1$ . The congruence  $a \equiv -btm^{1/2} \pmod{P}$  implies  $Btm^{1/2} \equiv -2^{n'-1}a^{n'} \pmod{P}$ . Hence from (i) we can derive  $(p, B) = 1$ . On the other hand  $2\alpha^{n'} + 2\beta = 4B(m + tm^{1/2})$  is contained in  $P$  and hence  $p$  must divide  $2Bms$ . This gives a contradiction. Thus  $(\beta + \theta, \beta - \theta) = 1$  and  $(\beta + \theta) = I^{2n'}$  holds for some ideal  $I$  of  $K$ . The ideal class represented by  $I$  belongs to  $C^-(K)$ , which was defined in Lemma 1, because  $II^\tau = (\alpha)$ , where  $\tau$  is the generator of the Galois group  $\text{Gal}(K/Q(m^{1/2}))$ .

Let  $p$  be any prime dividing  $n$ . From (iii) it is easy to see  $\text{ord}_p({}_i^{n'})b' > 1 + \text{ord}_p n'$  for any odd integer  $i$ ,  $3 \leq i \leq n'$ . By (i) we get  $(a, p) = 1$ . Hence it follows that  $(A, p) = 1$  and  $\text{ord}_p B = 1 + \text{ord}_p n'$ . We write  $4B(Bm - A) = r^2 f$ , where  $r, f$  are in  $Z$  and  $f$  is square free. Let  $l = \text{ord}_p n$ . Then we obtain

$$\text{ord}_p r = \begin{cases} l + 3 & \text{if } p = 2 \text{ and } mt \text{ is odd,} \\ l + 2 & \text{if } p = 2 \text{ and } mt \text{ is even,} \\ l & \text{if } p > 2. \end{cases}$$

Moreover  $f$  is divisible by every odd prime dividing  $n$ , and  $f \equiv t \pmod{2}$  is valid if  $n$  is even and  $m$  is odd.

We now assume  $\text{ord}_p C^-(K) < l$ . We put  $k = \text{ord}_p 2n'$  and consider the ideal  $J = I^{2n'/p^k}$ . Then  $J^{p^{l-1}} = (\zeta)$  for some integer  $\zeta$  in  $K$ . Hence  $\beta + \theta = \varepsilon \zeta p^{k-l+1}$  holds,  $\varepsilon$  being a unit of  $K$ . We know that  $\varepsilon_1 = \varepsilon/\varepsilon^\tau$  is a root of unity. Since  $Q(\varepsilon_1) \subset K$ , it is seen from Lemma 2 that  $\varepsilon_1 = \pm 1$  if  $K$  is not equal to the fifth cyclotomic field. By means of Lemma 3 we have

$$\zeta p^{k-l+1} \equiv (\zeta^\tau)^{p^{k-l+1}} \equiv \xi \pmod{p^{k-l+1}}$$

for some  $\xi$  in  $Z[\omega]$ . Thus  $\beta + \theta \equiv \pm(\beta - \theta) \pmod{p^{k-l+1}}$ . Since  $\beta \equiv -A \not\equiv 0 \pmod{p}$ , it holds that

$$2\theta = \pm 2r(f\eta')^{1/2} \equiv 0 \pmod{p^{k-l+1}}$$

with  $\eta' = m + tm^{1/2}$ . On the other hand  $\text{ord}_p 2r < k - l + 1$ . Therefore  $f\eta'$  must be divisible by  $p^2$ . But this is impossible. Hence the order of  $C^-(K)$  is a multiple of  $p^l$ . This proves the second assertion.

*Proof of Theorem.* Let  $K_i$  ( $i = 1, \dots, g$ ) be a finite number of quartic fields each generated by  $(f_i \eta')^{1/2}$  with  $f_i$  in  $Z$  and  $\eta' = m + tm^{1/2}$ . To prove the theorem it suffices to find an imaginary cyclic quartic field different from any  $K_i$  such that  $h^- \equiv 0 \pmod{n}$ . Take a prime  $q$  not dividing  $10f_1 \cdots f_g mn$  and choose a positive rational integer  $b$  satisfying

the condition (iii) and  $\text{ord}_q b = 1$ . The condition (iv) is equivalent to the inequality

$$(m^{1/2} + t)(a - btm^{1/2})^{n'} > (m^{1/2} - t)(a + btm^{1/2})^{n'}.$$

By simple computation we see that if  $t > s$  and  $a > 3bn'tm^{1/2}$  then (iv) is valid. Hence we can find an integer  $a > 0$  in  $\mathbb{Z}$  satisfying (i), (ii) and (iv). Let  $K$  be the field generated by  $(f'\eta)^{1/2}$  over  $Q$  with  $f' = 4B(Bm - A)$ , where  $A, B$  are defined as in Proposition. It is clear that  $\text{ord}_q f' = 1$  and  $K$  is not equal to the fifth cyclotomic field. Further  $K \neq K_i$  for any  $i$ ,  $1 \leq i \leq g$ . Indeed, if  $K = K_i$  for some  $i$ , then  $(f'/f_i)^{1/2}$  is contained in the quadratic field  $Q(m^{1/2})$ . This contradicts the choice of  $q$ . Hence  $K$  is a desired field, and the proof is complete.

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Received November 15, 1984.

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